Traveling waves propagation on networks of dynamical systems

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Abstract: Conditions for traveling wave propagation in networks of dynamical systems are investigated by suitably defining a family of ordinary differential equations (ODEs) whose solutions approximate the wave itself with increasing degree of accuracy. A reduced order ODE is considered for computing reference solutions, which are then exploited to prove via implicit function theorem the existence of similar solutions in the original network. A numerical example is included to illustrate the effectiveness of the proposed approach.

1. INTRODUCTION

Analysis of networks of dynamical systems has gained the attention of modern research due to the widespread diffusion of these systems in recent technological advancements. For example, networks of dynamical systems have been introduced in computer science as a paradigm for analog dynamic processors arrays, such as cellular neural networks (see, e.g., Roska and Chua [1993]), that have been proved to be effective for image analysis as in Grassi and Grieco [2003], pattern generation as in Arena et al. [2005], Nishio and Nishio [2008] and self-emerging behaviour capabilities (see Arena et al. [1998]). Similarly, dynamical systems networks are commonly exploited by living organisms for generating spatiotemporal patterns in the nervous systems as in Kopell et al. [1991], Horikawa and Kitajima [2009] and for providing locomotory patterns such as peristaltic waves during crawling, as described in Alexander [2006]. A paradigmatic example of complex networks composed by nodes with internal dynamics is represented by the brain, where informations are transferred between different areas via propagation of electric and chemical waves across neurons, see Chen et al. [2008]. In physics, chains of masses linked by linear or nonlinear springs have been used to model propagation of information across networks, as in Ermentrout and Kopell [1994] as well as in material science as in Calleja and Sire [2009]. Dynamics of interconnected dynamical systems has also recently gained the attention in social and medical sciences thanks to the capability of modeling phenomena such as opinion dynamics (see, e.g., Gabbay [2007], Han et al. [2010]) and epidemic spreading, as in Christensen et al. [2010].

From this brief list of examples, it is clear that understanding the dynamics of such networks would be highly beneficial in various scientific fields. However, the interplay between the local node dynamics and the overall network behaviour is not fully understood for generic dynamics and interconnection features. Several effort have been reported in the literature for addressing this problem, such as contraction analysis in Wang and Slotine [2005] and harmonic balance techniques in Gilli [1997]. Note that the dimension of the state space of these systems tends to be very large due to the large number of nodes which are present in the network. This fact, together with the presence of nonlinearities, make the analysis (and control) of such systems very challenging.

An alternative approach, aimed at a drastic reduction of the model dimension, is considering the network systems as the sampling of a continuous distribution of nodes with the same dynamics. In other words, one tries to define a partial differential equation (PDE), whose solutions evaluated at the node positions are close to the ones showed by the original network (see Sarlette and Sepulchre [2009] for an example in the linear setting). Here, the term close assumes different meanings according to the purpose of the analysis one is carrying on. For example, sometimes one desires ‘pointwise equivalence’, i.e. the solution of the PDE evaluated at each node location must be exactly equal to the node output. On the other hand, it is often sufficient to require that the difference between the PDE and the network solution is bounded in a given norm. Moreover, it is known that the dynamics of networked dynamical systems is a broader class than that, which can be captured by PDEs. This is due to the presence of phenomena, such as propagation failure, that are peculiar of spatially discrete systems as described in Gilli et al. [2002]. Then, approaches based on partial differential equations do not result suitable for describing such degenerate situations.

Note, however, that in most of the above mentioned examples the behaviour of the network can be easily described by the propagation of a traveling waves across the nodes present in the network. Thus, building a reduced order model capable of describing these phenomena can provide new insights on the interplay between local dynamics and
interconnection topology. Understanding this interplay opens new possibilities for novel design and control strategies for dynamical systems networks capable of showing a desired behavior with increased efficiency and robustness. To this end, here we propose a method for studying wave propagation across networks of evenly spaced identical dynamical systems with the goal of describing the dynamics via a reduced order model and of deriving conditions under which the approximated solution is close to the one shown by the full-scale system. For the sake of simplicity, we limit the presentation to the case of chains, i.e. one-dimensional networks, of nonlinear systems with linear interconnections between them. Extensions of this approach to higher dimensional networks and more general interconnections are quite straightforward as briefly discussed in the text.

The paper is organised as follows. In Section 2 the problem is formally stated and a family of PDEs is associated to a given network. We then obtain a corresponding family of ODEs that describes the propagation of traveling waves across the network. An exact model for the network dynamics would potentially require an infinite order ODE, thus a reference ODE of finite order is defined by truncating higher order terms. Conditions for ‘equivalence’ or ‘closeness’ between the reference ODE and the network solutions are derived in Section 3. In Section 4 a numerical example is described to illustrate the effectiveness of the proposed approach. Finally, in Section 5 some final remarks are reported.

**Notation.** In the following we will denote as \( \mathbb{R} \) the set of real numbers and as \( \mathbb{R}^+ \) and \( \mathbb{R}_0^+ \) the set, respectively, of positive and non-negative real numbers. The set of complex numbers is referred to as \( \mathbb{C} \) and the set of the natural numbers as \( \mathbb{N} \). Given a function \( f : [0, T] \to \mathcal{C} \) we define its \( p \)-norm as

\[
\|f\|_p = \left( \int_0^T |f(t)|^p \, dt \right)^{\frac{1}{p}}.
\]

We indicate with \( L^2([0, T]) \) the space of functions with \( \|f\|_2 < \infty \). Moreover, given two Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \), we denote by \( \mathcal{X} \oplus \mathcal{Y} \) the space \( \mathcal{X} \times \mathcal{Y} \) equipped with the direct sum norm

\[
\|x \oplus y\| = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}.
\]

Note that \( \mathcal{X} \oplus \mathcal{Y} \) is a Banach space as well (see Megginson [1998]). Given an operator \( \Pi : (\alpha, \beta, \gamma) \in U \times V \times W \to \vartheta, x, \gamma \in Z \), we refer to its restriction to \( U \times V \) parametrized by \( \gamma \in W \) as \( \Pi|_{\gamma} : (\alpha, \beta) \in U \times V \to \vartheta \in Z \). Finally, we denote equivalence relationships by the ‘\( \equiv \)’ operator.

2. NETWORKS OF DYNAMICAL SYSTEMS

In this section we introduce a procedure for investigating traveling waves propagation in networks of evenly distributed identical dynamical systems. For the sake of simplicity, hereafter we will consider dynamical systems settled in one-dimensional boundary conditions (chains), with periodic boundary conditions and linear connections. The main idea consists in the definition of a suitable ODE, whose periodic solutions are related to the patterns showed by the original network. Some preliminary observations are in order. Since the studied solutions are traveling waves, the outputs of the systems along the chain represent a sampling of their temporal evolutions. Therefore, since the number of systems equals the number of spatial samples, the network can only sustain real traveling waves with suitable shapes according to the Shannon theory (see Gasquet et al. [1999]). In particular, the spatially discrete system can not show traveling wave solutions having spatial frequencies greater than one-half of the characteristic frequency \( z^{-1} \), \( z \) being the distance between two nodes of the network.

Let us now consider \( N \) identical dynamical systems placed along a chain of total length \( l \) with pace \( z = l/N \). Assume that they admit the following model in their variables \( \xi_j(t) \in \mathbb{R}, j = 1, \ldots, N \):

\[
L_0(D_t^n, \ldots, D_t, 1) \xi_j(t) =
= F_0(D_t^{n-1}t \xi_j(t), \ldots, D_t \xi_j(t), \xi_j(t))
+ \sum_{r=-\alpha}^{\beta} h_r \xi_{j+r}(t) \; ,
\]

where \( D_t \) is the time derivative, \( L_0 \) expresses a linear combination of its arguments, \( F_0 : \mathbb{R}^n \to \mathbb{R} \) is a scalar nonlinear function of \( \xi_j \) and its derivatives, \( \alpha, \beta \in \mathbb{N} \) are integers such that \( 0 < \alpha, \beta < N \) and \( h_l \in \mathbb{R} \) are constant, with \( h_0 = 0 \). Systems of the form (2) are an extension of the well known Lur’e model [Khalil, 2002], which has been largely studied in the literature as central element of dynamical networks (see for example Gilli [1997], Gilli et al. [2004] and references within). In order to have a clear separation between the state space of the networked systems and their position along the chain, we introduce the spatial coordinate \( x_j \in [0, l) \). We also assume periodic boundary conditions, so that the equivalence relationship \( x + zN \equiv x \) holds. Moreover, let us denote by \( x_j \) the \( j \)-th node position. It follows that \( x_{j \pm r} \equiv (x_j \pm rz) \bmod l \), being \( \cdot \bmod \cdot \) the modulo operator.

**Definition 1.** A function \( \xi(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is an interpolation of the network state, if \( \xi(x_j,t) = \xi_j(t) \), \( \forall j = 1, \ldots, N \), \( \forall t \in \mathbb{R} \).

The interpolation property define a class of equivalence. Among all the functions which satisfy this condition, we point out the subclass of the ones with the following property.

**Definition 2.** An interpolation \( \xi(x,t) \) of the network state is said a regular approximation of the system solution, if it admits the power series expansion of radius \( R > z \cdot \max(\alpha, \beta) \) along the coordinate \( x \):

\[
\xi(x_{j \pm r}, t) = \sum_{i=0}^{\infty} \frac{1}{i!} \partial_x^i \xi(x_j, t)(\pm rz)^i, \quad \forall j = 1, \ldots, N, \quad \forall t \in \mathbb{R}, \quad R > r \in \mathbb{N}.
\]

Observe that in terms of regular approximations, the dynamics of \( \xi(x, t) \) can be represented by a PDE of infinite order in the space variable \( x \). Indeed, denoting by \( D_t \) and \( D_x \) the partial derivatives with respect to time and space respectively, the above model can be formulated as:

\[
L_0(D_t^n, \ldots, D_t, 1) \xi(x_j, t) =
= F_0(D_t^{n-1}t \xi_j(x, t), \ldots, D_t \xi_j(x, t), \xi_j(x, t)) + \sum_{r=-\alpha}^{\beta} h_r \xi_j(x_{j+r}, t) \; .
\]
In the following, we focus on a particular class of solutions, namely traveling waves. To this aim, let us consider regular approximations of the form

$$\xi(x, t) = \psi(kx - ct) = \psi(\tau),$$

where $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is a periodic function of period $T$ in $\tau$. It is worth observing that, with respect to $\psi$, equation (3) must hold at every point $x$. Denoting as $D_\tau$ the derivative of $\psi$ with respect to $\tau$, it follows that

$$D_\tau \xi(x, t) = -cD_\tau \psi(kx - ct) = \psi(\tau - krz) = \psi(\tau) + rzkD_\tau \psi(\tau) + \frac{1}{2}(rzk)^2D^2\psi(\tau) + \ldots + \frac{1}{(n - 1)!}(rzk)^nD^n\psi(\tau).$$

Let us define the spatial shift operator parametrized by $z$,

$$S[z]\psi(\tau) = \sum_{r=-\alpha}^{\beta} h_r \psi(\tau + rz),$$

the linear parametric operator

$$L_1[z](D^{n-1}, \ldots, D, 1)\psi(\tau) = \sum_{r=-\alpha}^{\beta} h_r \psi(\tau + rzkD + \frac{1}{2}(rzk)^2D^2 + \ldots + \frac{1}{(n - 1)!}(rzk)^nD^n)$$

and their composition

$$G(D^{n-1}, \ldots, D, 1, S[z])\psi(\tau) = z^n(S[z] - L_1[z](D^{n-1}, \ldots, D, 1))\psi(\tau) = \sum_{r=-\alpha}^{\beta} \sum_{k=0}^{m} h_r z^{-n}(rk)^{n}\psi(\tau).$$

Then, equation (3) reads

$$L_{\alpha}[(-c)^nD^n, \ldots, -cD, 1]\psi(\tau) = F_0[(-c)^nD^n\psi(\tau), \ldots, -cD\psi(\tau), \psi(\tau)] + L_1[z](D^{n-1}, \ldots, D, 1)\psi(\tau) + z^nG(D^{n-1}, \ldots, D, 1, S[z])\psi(\tau),$$

which, for the sake of simplicity, can be rewritten in compact form as

$$L[c, z](D)\psi(\tau) = F_0[c](D, \psi(\tau)) + z^nG(z, S[z])\psi(\tau)$$

where

$$L[c, z](D)\psi(\tau) = (L_0[c](D) - L_1[z](D))\psi(\tau).$$

The analysis of traveling waves propagation in the original network (2) has boiled down to the investigation of the periodic solutions of the associated ODE (6), which is derived from the general interpolating PDE (4).

**Definition 3.** The $n$-th order reference ODE describing the propagation of a traveling wave in the original network is defined as

$$L[c, z](D)\psi(\tau) = F_0[c](D, \psi(\tau)).$$

Observe that $L_1$ has been defined in (5) by truncating the expansion to the $n$-th term, where $n$ represents also the order of the local dynamics. In principle one may cut-off at any value of $n'$, but when $n' > n$ model (7) may undergo structural modifications when $z = 0$, which in turn will result in a crucial case for our analysis. On the other hand, we are interested in putting in (7) the larger number of terms deriving from the above expansion, so to reduce the residual order. Hence, the natural choice falls just on $n$.  

**Remark 4.** Notice that the residual term $z^nG(z, S[z])\psi(\tau)$ is of order $O(z^n)$. Indeed, $G(D^{n-1}, \ldots, D, 1, S[z])\psi(\tau)$ has a leading order of $\psi(\tau)$ and $\psi(\tau)$ has a leading order of $\psi(\tau)$ as $z \rightarrow 0$.  

**Remark 5.** Notice that (7) is still a nonlinear ODE that, in general, is hard to solve. However, it has the advantage of being defined on a small dimensional state space where analysis techniques developed for nonlinear ODEs such as harmonic balance and contraction analysis can be readily exploited.

In the following section we will introduce a procedure to study the existence of periodic solutions of (6), when the dynamical system represented by (7) admits limit cycles.

### 3. Existence of Periodic Solutions

Let us consider a periodic function $\psi(\tau): \mathbb{R} \rightarrow \mathbb{R}$ with period $T = 2\pi/\omega$. Observe that it has a unique representation in terms of the functional basis of $L^2[0, T]$ given by the powers of $e^{i\omega t}$, i.e. in terms of its Fourier series development. Moreover, notice that, by introducing the time transformation $\tau \rightarrow \tau/T = \tau/T$, $\psi(\tau)$ narrows to a new periodic function $\psi(T\tau) = \psi(T\tau + T) \in L^2[0, T]$. The knowledge of both $u$ and $\omega = 2\pi/T$ allows one to obtain the original function $\psi$.

**Proposition 6.** Any periodic non-constant function can be represented as a pair $(u, \omega) \in \mathbb{Q} = L^2[0, T] \times \mathbb{R}^+$, where $u$ specifies its shape and $\omega = 2\pi/T$ takes into account its original period. Any non-constant periodic function admits infinite isolated representations in $\mathbb{Q}$.

**Proof.** The only non trivial point is given by the infinite isolated representations.

Let us consider a non-constant periodic function $\psi(\tau)$ of period $T = 2\pi/\omega$ and assume that $T$ is a proper period, i.e. it is the minimum positive value such that $\psi(\tau + T) = \psi(\tau)$. It follows that $\psi(\tau + mT) = \psi(\tau) \forall m \in \mathbb{N}$. However, any restriction of $\psi(\tau)$ to the period $[0, mT], m > 1$, can be regarded as the concatenation of $m$ basic cases $m = 1$. Therefore, let $u_0$ and $\omega_0$ be the function’s profile and frequency associated to the case $m = 1$ respectively. Then, observe that all the other equivalent descriptions of $\psi(\tau)$ can only be originated via multiple concatenations of the above basic pattern $u_0$, associating to it as related angular velocity the value $\omega_0/m$. This implies the existence of infinite descriptions, which are isolated along the $\omega$ coordinate, because $m \in \mathbb{N}$. To complete the proof, it is sufficient to notice that all the considered operators are time invariant and thus time shifts can be neglected.  

**Remark 7.** The space $\mathbb{Q} = L^2[0, T] \times \mathbb{R}^+$ provided with the direct sum norm (1) is a Banach space.

Let us consider the periodic function $\psi(\tau) \equiv (u, \omega) \in \mathbb{Q}$ and define the $c$-parametrized operator $F_0[c]: \mathbb{Q} \rightarrow \mathbb{Q}$ so that...
that
\[ F_0[c](u, \omega) = (F_0^n[c](u, \omega), \omega)^T \equiv F_0[c](\mathcal{D}, \psi) , \]
where the derivative operator \( \mathcal{D} \) has to be intended in the
sense of Frechét differentiability and the second element
takes into account that \( F_0[c] \) does not change the period of
\( \psi(\tau) \). Moreover, let us also introduce the \( \omega \)-parametrized
operator \( \Delta[\omega]: L^2[0, 1] \rightarrow L^2[0, 1]^n \), defined as
\[ \Delta[\omega]u = (\omega^{-1}D^0 u, \ldots, \omega D, 1)^T u . \quad (8) \]
It is worth underlining that \( \Delta[\omega] \) is linear in \( u \). Observe that we can write
\[ F_0[c](u, \omega) \equiv F_0[c](\Delta[\omega]u) , \]
that is, we can regard the operator \( F_0[c] \) as the composition
\( F_0[c] \circ \Delta[\omega] \) of the original nonlinearity and the above
operator \( (8) \).

Let us now consider the parametric operator \( L[c, z](\mathcal{D}) \).
For a fixed pair \((c, z)\), it produces a linear combination of
the input function and its derivatives. Then, if we apply
such an operator to periodic functions, the corresponding
kernel turns out being characterized only by the period of
the input function and, in particular, there may exist at
most \( n \) different frequencies which are roots of its
characteristic polynomial. We denote by \( \mathcal{K}_L[z] = \{ \omega_1, \ldots, \omega_n \} \)
the set of these frequencies, i.e. \( L[c, z](\omega) u \equiv 0, \forall u \in L^2[0, 1] \)
and \( \forall \omega \in \mathcal{K}_L[z] \). It is worth highlighting that
the restriction of \( L[c, z] \) to \( \mathcal{Q} \) is a linear combination of
\( \omega_n D^n \) and the elements of \( \Delta[\omega] \). Moreover, \( L[c, z] \) is linear
in \( u \). Therefore, let us define the parametric operator
\( \mathcal{L}[c, z]: \mathcal{Q} \rightarrow \mathcal{Q} \)
\[ \mathcal{L}[c, z](u)(\omega) = (L^u[c, z](\omega) u, \omega)^T \equiv L[c, z](\Delta[\omega]u) , \]
where \( \Delta[\omega] = (\omega_n D^n, \Delta[\omega])^T \).

Analogous considerations hold for \( G[z](\mathcal{D}) \), leading us to define a related operator
\( G[z]: \mathcal{Q} \rightarrow \mathcal{Q} \), which is linear in \( u \).

Since \( L[c, z](\mathcal{D}) \) does not change the period and it is linear
in \( u \), it is also locally invertible, when dealing with periodic
functions which do not belong to \( \mathcal{K}_L[z] \). In particular, we
denote such an inverse operator as \( L^{-1}[c, z](\mathcal{D}) \). Similarly,
under the above assumptions, we denote as \( L^{-1}[c, z] \)
the inverse operator of the corresponding \( \mathcal{L}[c, z] \).

Let us now finally introduce the parametric operator
\( \mathcal{H}[c, \varepsilon]: \mathcal{Q} \rightarrow \mathcal{Q} \)
\[ \mathcal{H}[c, \varepsilon](u, \omega) = (L^{-1}[c, z] \{ \varepsilon \}[c, \omega] u, e^\varepsilon H[\varepsilon](\omega) u) , \]
where we have introduced the continuously varying parame-
ter \( \varepsilon \in \mathbb{R} \) in place of \( z \), which only admits a discrete
set of values depending on \( N \), and we have defined:
\[ H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2} . \]

Hereafter, we will study for a given \( c \) how the fixed points
\((u(\varepsilon), \omega(\varepsilon)) \in \mathcal{Q} \) of \( \mathcal{H}[c, \varepsilon] \) are affected by \( \varepsilon \), i.e., how the solutions
\[ \mathcal{H}[c, \varepsilon](u, \omega) = (u(\varepsilon), \omega(\varepsilon)) \quad (9) \]

vary depending on such parameter.

Theorem 8. Consider the linear operator \( L[c, \varepsilon] \) and
assume that \( \omega_0 \notin \mathcal{K}_L[0] \). Moreover, let \( \psi_0(\tau) \equiv (u_0, \omega_0) \)
be a periodic solution of \((7) \) for \( z = 0 \) and let the nonlin-
earity \( F_0[c] \) be differentiable. Then, if the linear operator
\( JF_0[c](\Delta[\omega_0]u_0) \Delta[\omega_0] : L^2[0, 1] \rightarrow L^2[0, 1] \) is invertible,
there exists a sufficiently large number \( N \) of systems, such
that the corresponding network admits a traveling wave
close to \( \psi_0(\tau) \) and with angular velocity approximated by \( c \).

Proof. Let us identify \( \psi_0(\tau) \) as \((u_0, \omega_0) \in \mathcal{Q} \). By hypoth-
thesis, observe that \((u_0, \omega_0) \) is a fixed point of \( \mathcal{H}[c, 0] \).

Observe that according to the composition rule, the con-
tinuity property of \( \mathcal{H}[c, \varepsilon] \) on \( \mathcal{Q} \) depends on the analogous
property of \( F_0[c] \) on \( \mathcal{C} \), being \( L[c, \varepsilon], G[c], \Delta \) and \( S[c] \)
continuous in their own domains. The same observation
holds for the Frechét differentiability property. In partic-
ular, since \( F_0[c] \) is Frechét differentiable by assumption,
we can compute the derivative of \( F_0[c] \) with respect to its
arguments by exploiting the composition rule, i.e.
\[ DF_0[c](u, \omega) = \left[ \frac{\partial F_0^n[c](u, \omega)}{JF_0^n[c](\Delta[\omega]u)} \frac{\partial \omega}{\Delta[\omega]u} \right] = \]
\[ = \left[ JF_0^n[c](\Delta[\omega]u), \frac{\partial \omega}{\Delta[\omega]u} \right] \Delta[\omega]u \]

where \( JF_0[c] \) represents the Jacobian row vector of \( F_0[c] \),
\( J \Delta \) is the Jacobian column vector of \( \Delta[\omega] \) with respect to \( \omega \)
and \( \partial \Delta \) is the Frechét derivative with respect to \( u \) of
\( \Delta[\omega]u \), which turns out equal to the operator \( \Delta[\omega] \) itself.
It is worth recalling that the continuity and differentiabil-
ity properties of \( \mathcal{H}[c, \varepsilon] \) do not imply anything about the
smoothness of the profile \( u(\varepsilon) \) associated to the solution of
(9). Moreover, it is straightforward to check that \( \mathcal{H}[c, \varepsilon] \)
is also continuous and differentiable in \( \varepsilon \).
of the Implicit Function Theorem in a small neighborhood around of \( (u_0, \omega_0) \). Hence, for a sufficiently small \( \varepsilon \), there exists a function \( \varepsilon \mapsto (u(\varepsilon), \omega(\varepsilon)) \) such that \( H[\varepsilon] = H[u(\varepsilon), \omega(\varepsilon), \varepsilon] = H[u(\varepsilon), \omega(\varepsilon)] \).

The statement directly follows by observing that \( \varepsilon \) represents the continuous extension of \( z \), which tends to zero as \( N \to +\infty \). \( \square \)

4. A NUMERICAL EXAMPLE

In this section a numerical example is presented for better describing the proposed approach. Let us then consider a chain of \( N \) oscillators with nonlinear damping and nearest neighbor antisymmetric interconnections

\[
\dot{\xi}_j + a \xi_j \dot{\xi}_j + d \dot{\xi}_j + \xi_j = \frac{b}{2} (\xi_{j+1} - \xi_{j-1}) . \tag{11}
\]

Comparing this dynamics with the general expression (2) yields, by setting \( n = 2 \) and \( \alpha = \beta = 1 \), \( L_0 (D^2_t, D_t, 1) = D^2_t + d D_t + 1 \), \( F_0 (D_t \dot{\xi}_j, \dot{\xi}_j) = a \xi_j^2 D_t \dot{\xi}_j \), \( h_{-1} = -b/2 \), \( h_1 = b/2 \). By introducing the interpolating function \( \varphi(x, t) \) such that \( \varphi(j, t) = \xi(t) \) the associated PDE (4) reads

\[
\frac{\partial^2 \varphi}{\partial t^2} + a \varphi \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t} + \varphi = \frac{b}{2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{b}{2} \frac{\partial^2 \varphi}{\partial x^2} \psi^3 + \ldots .
\]

A traveling wave solution \( \varphi(x, t) = \psi(kx - ct) \) is then imposed so that the first-order reference ODE (7) reads

\[
e^2 \psi'' + (-acv^2 - dc - bkz) \psi' + \psi = 0 . \tag{12}
\]

Note that if \( ac > 0 \) and \( dc + bkz > 0 \), then equation (12) describes a nonlinear oscillator with negative damping and therefore all the trajectories diverge. On the other hand, if \( ac < 0 \) and \( dc + bkz < 0 \), one obtains a nonlinear oscillator with positive damping and the trajectories converge toward the origin. A more interesting case arises when \( ac < 0 \) and \( dc + bkz > 0 \). Indeed, in this situation (12) resembles a van der Pol oscillator and, therefore, stable limit cycle solutions are expected.

Periodic boundary conditions impose that \( sT = kL \), \( s = 1, 2, \ldots \), where \( T \) is the period of the solution of (12). Hence, given a value for \( c \) (and \( N \)), one needs to find the value of \( k \) that satisfies that condition. Moreover, the discrete nature of the network imposes a natural cut-off frequency at \( 1/(2z) \), because higher spatial frequencies can not be represented by sampling on intervals of length \( z \). Thus, the solution of the distributed system is expected to be similar to the one predicted by the reference ODE, once the higher frequencies have been filtered out.

To give a quantitative measure of the agreement between the reference ODE solution and the one exhibited by the network, we simulated equation (11) and (12) for different values of \( N \). The following procedure has been used:

(1) given the parameters \( a, b, d \) and the desired \( c \), a set of reference ODEs (12) is simulated in order to find the right value \( k = k_0 \), that makes the periodic boundary conditions satisfied;
(2) we simulate in Matlab the reference ODE with the right wave vector \( k_0 \), obtaining the candidate solution for the distributed system;
(3) since for any given \( t_0 \), \( \psi(kx - ct_0) \) is the spatial profile of the periodic traveling wave, we initialize the distributed system by sampling \( \psi(t) \) every \( kz \);
(4) we numerically integrate the network dynamics (11) in the Matlab environment;

(5) we measure the discrepancies between the solution given by the reference ODE and the actual solution provided by the network.

In Figure 1 a comparison between the solution of the reference ODE and the one exhibited by the distributed system is shown. Observe that the shape of the two solutions is very close, although the second is slightly smoother due to the absence of high frequencies, which have been filtered out thanks the finite spatial sampling frequency. Note also that the periods of the two solutions are slightly different due to errors on the estimate of \( c \) as quantified in Figure 3. Finally, in Figure 2 the overall spatiotemporal evolution of the network is shown.

5. FINAL REMARKS

In this paper the propagation of traveling waves along one-dimensional networks of dynamical systems has been analysed by modeling the network behaviour via a partial differential equation. By imposing solutions in the form of traveling waves, a family of ODEs capable of approximating the overall dynamics has been defined and a reduced order ODE has been exploited to compute reference solutions. Implicit function theorem has been then exploited to show the existence of solution of the networked systems, which are close to the ones provided by the reference
Fig. 3. Prediction error measures as a function of the number $N$ of systems.

ODE. In particular, the number $N$ of the systems along a one-dimensional chain of length $l$ has turned out being a crucial element in order to derive the existence of traveling waves from the periodic solution of the reference ODE. An illustrative numerical example has been included to show the effectiveness of the proposed approach.

Although the presentation has focused on one-dimensional networks with linear interconnections, extensions to higher dimensions and nonlinear interconnections are straightforward. Indeed, increasing the dimension of the space, where the nodes are placed, simply implies considering PDE defined in the appropriate domain and, similarly, nonlinear interconnections simply induce additional terms, when one evaluates the spatial derivatives in the definition of the PDE. Notice that, in both cases, the only step that differs from the present result is the definition of the PDE, but all the remaining results remain valid.

This paper results represent a step in the direction of a rigorous analysis of networks of dynamical systems via reduced order models. Enriching such theory, for example performing solution stability and robustness analysis, would allow for a better understanding of the network behaviour and, from an engineering perspective, for the development of novel control techniques capable of influencing the network global behaviour by local actions.

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