Virtual Decomposition Control for
Modular Robot Manipulators

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Abstract: Virtual decomposition control (VDC) is one of the most efficient approaches toward precision control of complex robots. The VDC approach uses subsystem (such as links and joints) dynamics with parametric uncertainty to construct model-based feedforward compensation terms and guarantees the stability of the entire complex robot with mathematical certainty, leading to precise motion/force tracking control with the control bandwidth being independent of the feedback control gains. The VDC approach can be naturally applied to modular robot manipulators to overcome their long-standing problem of lacking control precision in coordinated motion. In this paper, with respect to a typical module comprised of two links and one joint, the control equations of VDC are given and the virtual stability, a necessary and sufficient condition to ensure the stability of the entire robot, is proven. The experiment result yields unprecedented motion tracking precision for a one-module robot using a harmonic drive with significant friction. The ratio of the maximum position tracking error to the maximum velocity reaches 0.00024 (s).

1. INTRODUCTION

Control of complex systems (such as a robot) has long been a challenging topic for the past five decades. The difficulties in control are primarily attributed to the complexity of system dynamics demanding heavy computation that can easily exceed the capability of a model-based controller. Solutions in the past were either to use reduced system models or even to use model-free intelligent control methods, resulting in compromised control performances.

Virtual decomposition control (VDC) provides a rigorous solution to precision control of complex robots, a category of complex systems, see Zhu et al. (1997) and Zhu (2010). The VDC approach bases control on subsystem (such as links and joints) dynamics, while rigorously guaranteeing the $L_2/L_\infty$ stability of the complete system without jeopardizing control performances. This approach holds a great promise for extension to other systems such as electrical systems and even those (systems) described by distributed-parameter models.

Modular robot manipulators have a long-standing problem of lacking control precision due to their attractiveness. Despite their undeniable advantages of having reconfigurability, reformability, ease for repair, and flexibility in operation, see Yim et al. (2007). Applying VDC to modular robot manipulators, therefore, is becoming a natural solution for maintaining the advantages while overcoming their drawbacks.

Harmonic drives have been widely used in robotic applications due to their attractive properties such as high reduction ratio, compact size, low mass and coaxial assembly, see Tuttle and Seering (1996). A typical harmonic drive consists of a wave-generator that is connected to the motor, a circular spline that is connected to the joint, and a flexspline that is placed in-between. In spite of their attractive features, harmonic drives are subject to many factors that imperil their performances. These factors include, but are not limited to, friction, dynamics of the flexspline, nonlinearity, kinematic error and hysteresis. The friction and the dynamics of the flexspline were identified as the two main factors that substantially affect control performances, see Zhu et al. (2007), and the kinematic error was addressed by Ghorbel et al. (2001). Recently, inroads have been made by addressing these difficulties. For instance, a field programmable gate array (FPGA) implementation of the LuGre (Canudas et al. (1995)) model-based adaptive friction compensation gives rise to a result with an unprecedented accuracy, see Zhu (2010a).

In this paper, the LuGre model-based adaptive friction compensation joint control developed by Zhu (2010a) is incorporated into the link control of a typical module. The virtual stability of the module combined with its joint/link control equations is proven, leading to the stability of the entire robot.

2. VIRTUAL DECOMPOSITION CONTROL

The VDC approach by Zhu (2010) allows us to convert the control problem of a modular robot manipulator into the control problem of individual modules provided that each module combined with its respective control equations qualifies to be virtually stable in the sense of Definition
2.17, leading to the stability of the entire robot according to Theorem 2.1 1. The rigorous mathematical foundation of VDC ensures that the ultimate control performances can not be jeopardized by solely using subsystem dynamics in control. Therefore, VDC can simplify the control problem to a great degree and allows us to concentrate on the dynamics and control problem of each individual module, rather than on the entire robot.

3. A TYPICAL MODULE: KINEMATICS AND DYNAMICS

A typical module comprised of two rigid links connected by a joint is illustrated in Fig. 1, see Zhu and Lamarche (2007). There are two connection interfaces located at both ends.

3.1 Coordinate Frames

Four coordinate frames are attached to each module. Frames \{A_{k}\} and \{C_{k}\} share the same origin and are attached to Link a and Link b, respectively. The \(z\) axes of the two frames coincide with the joint axis. Frames \{T_{k}\} and \{B_{k}\} are attached to the two connection interfaces of the \(k\)th module, respectively, with their \(x\) axes aligning with the two link axes.

3.2 Velocity Transformation

In view of Definition 2.8 in Zhu (2010) (page 29), the linear/angular velocity vectors of the four frames can be expressed as

\[
T_{k}V = A_{k} U_{k}^{T} A_{k} V \quad (1)
\]

\[
A_{k} V = C_{k} U_{k}^{T} C_{k} V + z_{r} \dot{q}_{k} \quad (2)
\]

\[
C_{k} V = B_{k} U_{k}^{T} B_{k} V \quad (3)
\]

with \(z_{r} = [0, 0, 0, 0, 0, 1]^{T} \in \mathbb{R}^{6}\), where \(\dot{q}_{k} \in \mathbb{R}\) denotes the joint velocity and \(U_{\alpha} \in \mathbb{R}^{6 \times 6}\) denotes a force/moment transformation matrix defined by (2.65) in Zhu (2010).

Note that (1) and (3) are based on the fact that frames \{A_{k}\} and \{T_{k}\} are fixed to a common link a and frames \{B_{k}\} and \{C_{k}\} are fixed to a common link b, see Fig. 1.

1 Theorem 2.1 in Zhu (2010) states that the entire robot is \(L_{2}/L_{\infty}\) stable as long as every module combined with its respective control equations qualifies to be **virtually stable**.

3.3 Link Dynamics

In view of (2.74) in Zhu (2010), the dynamics of the two links can be written as

\[
A_{k} F^{*} = M_{A_{k}} \frac{d}{dt} A_{k} V + C_{A_{k}} (A_{k} \omega) A_{k} V + G_{A_{k}} \quad (4)
\]

\[
C_{k} F^{*} = M_{C_{k}} \frac{d}{dt} C_{k} V + C_{C_{k}} (C_{k} \omega) C_{k} V + G_{C_{k}} \quad (5)
\]

where \(A_{k} F^{*} \in \mathbb{R}^{6}\) and \(C_{k} F^{*} \in \mathbb{R}^{6}\) denote the net force/moment vectors of Link a and Link b, respectively. The detailed expressions for \(M_{A_{k}} \in \mathbb{R}^{6 \times 6}\), \(M_{C_{k}} \in \mathbb{R}^{6 \times 6}\), \(C_{A_{k}} \in \mathbb{R}^{6 \times 6}\), \(C_{C_{k}} \in \mathbb{R}^{6 \times 6}\), \(G_{A_{k}} \in \mathbb{R}^{6}\), and \(G_{C_{k}} \in \mathbb{R}^{6}\) are given by (2.75)-(2.77) in Zhu (2010).

Let \(B_{k} F \in \mathbb{R}^{6}\) be the force/moment vector applied from the lower \((k-1)\)th module to the \(k\)th module, expressed in frame \{B_{k}\}, see Fig. 2. Also, let \(T_{k} F \in \mathbb{R}^{6}\) be the force/moment vector applied from the \(k\)th module to the upper \((k+1)\)th module, expressed in frame \{T_{k}\}. Then, the two net force/moment vectors are related to the two exerting force/moment vectors at the two connection interfaces by the following equation

\[
B_{k} F = B_{k} U_{C_{k}} C_{k} F^{*} + B_{k} U_{A_{k}} A_{k} F^{*} + B_{k} U_{T_{k}} T_{k} F. \quad (6)
\]

**Remark 1:** Equation (6) is a force/moment transformation equation that transforms \(T_{k} F \in \mathbb{R}^{6}\) to \(B_{k} F \in \mathbb{R}^{6}\).

3.4 Joint Dynamics

**Assumption 1:** Due to the small size of the manipulator, the flexibility of the harmonic drive is neglected throughout this paper.

The joint dynamics, after neglecting the flexibility of the flexspline, can be written as

\[
I_{mk} \ddot{\theta}_{k} + f_{k} + d_{k} = \tau_{k} - z_{r}^{T} (A_{k} F^{*} + A_{k} U_{T_{k}} T_{k} F) \quad (7)
\]

where \(I_{mk} \in \mathbb{R}\) represents the moment of inertia; \(d_{k} \in \mathbb{R}\) denotes a constant disturbance representing the bias of the Coulomb friction in different directions; \(\tau_{k} \in \mathbb{R}\) is the control torque; and \(f_{k} \in \mathbb{R}\) represents the equivalent joint friction torque governed by the LuGre model

\[
\dot{z}_{k} = \dot{q}_{k} - \frac{\sigma_{0} |\dot{q}_{k}|}{g(\dot{q}_{k})} z_{k} \quad (8)
\]

\[
f_{k} = \sigma_{0} z_{k} + \sigma_{1} \dot{z}_{k} + \sigma_{v}(\dot{q}_{k}) \quad (9)
\]
where $z_k$ is an internal variable representing the average deflection of the bristles (Canudas et al. (1995)); $\sigma_0 > 0$, $\sigma_1 > 0$, and $\sigma_1 > 0$ are three constants; $g(q_k)$ specifies the profiles of the Coulomb and Striebeck effects; and finally $\sigma_v(q_k)$ specifies the profile of the viscous friction.

Note that the last term in the right hand side of (7) represents the projection of the link dynamics onto the joint axis. This term can be interpreted as the payload of having $z_k r$, the feedforward friction compensation term is designed as

$$f_k = \hat{\sigma} [\sigma_0 z_k + \sigma_1 \dot{z}_k] + \sigma_v(q_k)$$  \hfill (24)

### 4. CONTROL EQUATIONS

The control equations are model-based. They appear in the same form as the kinematics and dynamics of the $k$th module presented in the last section.

#### 4.1 Required Velocity Transformation

As stated in Zhu (2010) (page 50), the terminology of required velocity gives rise to an important concept in the VDC approach. A required velocity is different from a desired velocity which usually serves as the reference trajectory of a velocity with respect to time. The meaning of a required velocity is that if the true velocity tracks the required velocity, then the control objectives including position control and force control can be achieved. In other words, a required velocity is the velocity that is required to accomplish the control objectives.

In view of (1)-(3), the required linear/angular velocity vectors of the four frames are subject to

$$\begin{align*}
T_{x_i} V_r &= A_i U_{T_i}^T A_i V_r \quad (10) \\
A_k V_r &= C_k U_{A_k}^T C_k V_r + z_r \dot{q}_{kr} \quad (11) \\
C_k V_r &= B_k U_{C_k}^T B_k V_r \quad (12)
\end{align*}$$

where variables with subscript “r” refer to the corresponding required variables.

Equations (10)-(12) are recursively used to compute all required velocities from the robot base to the robot tip.

If joint space position control is used, the variable $\dot{q}_{kr}$ in (11) can be further designed as

$$\dot{q}_{kr} = \dot{q}_{kd} + \lambda (g_{kd} - q_k)$$  \hfill (13)

where $q_{kd} \in \mathbb{R}$ represents the desired joint trajectory and $\lambda > 0$ is a constant.

#### 4.2 Link Control Equations

Analogous to the required velocity, a required force/momentum is the force/momentum that is required to accomplish the control objectives.

The required net force/momentum vectors for the two rigid links are

$$A_k F_r = Y_{A_k} \dot{\theta}_{A_k} + K_{A_k} \left( \lambda \dot{V}_r - \dot{A}_k V_r \right)$$  \hfill (14)

$$C_k F_r = Y_{C_k} \dot{\theta}_{C_k} + K_{C_k} \left( \lambda \dot{V}_r - \dot{C}_k V_r \right)$$  \hfill (15)

where $K_{A_k} \in \mathbb{R}^{6 \times 6}$ and $K_{C_k} \in \mathbb{R}^{6 \times 6}$ are two positive-definite gain matrices characterizing velocity feedback control; $Y_{A_k} \dot{\theta}_{A_k}$ and $Y_{C_k} \dot{\theta}_{C_k}$ denote the model based feedforward compensation terms defined by

$$Y_{A_k} \dot{\theta}_{A_k} = M_{A_k} \frac{d}{dt} (A_k V_r) + C_{A_k} \left( \lambda \dot{\omega} \right) A_k V_r + G_{A_k} \dot{\theta}_{A_k}$$  \hfill (16)

$$Y_{C_k} \dot{\theta}_{C_k} = M_{C_k} \frac{d}{dt} (C_k V_r) + C_{C_k} \left( \lambda \dot{\omega} \right) C_k V_r + G_{C_k} \dot{\theta}_{C_k}$$  \hfill (17)

The estimated parameter vectors $\hat{\theta}_{A_k} \in \mathbb{R}^{13}$ and $\hat{\theta}_{C_k} \in \mathbb{R}^{13}$ are to be updated. Define

$$s_{A_k} = Y_{A_k}^T A_k V_r - A_k V_r$$  \hfill (18)

$$s_{C_k} = Y_{C_k}^T C_k V_r - C_k V_r$$  \hfill (19)

The $P$ function given by Definition 2.11 in Zhu (2010) (page 32) is used to update each element of $\hat{\theta}_{A_k} \in \mathbb{R}^{13}$ and $\hat{\theta}_{C_k} \in \mathbb{R}^{13}$ as

$$\begin{align*}
\hat{\theta}_{A_k, \gamma} &= P \left( s_{A_k, \gamma}, \rho_{A_k, \gamma}, \bar{s}_{A_k, \gamma}, \bar{\theta}_{A_k, \gamma}, t \right), \gamma \in \{1, 13\} \\
\hat{\theta}_{C_k, \gamma} &= P \left( s_{C_k, \gamma}, \rho_{C_k, \gamma}, \bar{s}_{C_k, \gamma}, \bar{\theta}_{C_k, \gamma}, t \right), \gamma \in \{1, 13\}
\end{align*}$$  \hfill (20)

(21)

where $\hat{\theta}_{A_k, \gamma}$ denotes the $\gamma$th element of $\hat{\theta}_{A_k} \in \mathbb{R}^{13}$ and $\hat{\theta}_{C_k, \gamma}$ denotes the $\gamma$th element of $\hat{\theta}_{C_k} \in \mathbb{R}^{13}$; $s_{A_k, \gamma}$ denotes the $\gamma$th element of $s_{A_k}$ defined by (18) and $s_{C_k, \gamma}$ denotes the $\gamma$th element of $s_{C_k}$ defined by (19); $\rho_{A_k, \gamma} > 0$ and $\rho_{C_k, \gamma} > 0$ are update gains; $\bar{s}_{A_k, \gamma}$ and $\bar{s}_{C_k, \gamma}$ denote the lower and upper bounds of $s_{A_k, \gamma}$ - the $\gamma$th element of $\hat{\theta}_{A_k} \in \mathbb{R}^{13}$; and $\bar{\theta}_{A_k, \gamma}$ and $\bar{\theta}_{C_k, \gamma}$ denote the lower and upper bounds of $\theta_{A_k, \gamma}$ - the $\gamma$th element of $\hat{\theta}_{C_k} \in \mathbb{R}^{13}$.

The force/moment transformation equation (6) also applies to the required variables as

$$B_k F_r = B_k U_{C_k} C_k F_r + B_k U_{A_k} A_k F_r + B_k U_{T_k} T_k F_r$$  \hfill (22)

Equation (22) is used recursively to compute all required force/moment vectors from the robot tip to the robot base.

#### 4.3 Joint Control

Let $z_{kr}$ be a (design) variable characterizing the required deflection of the bristles in the friction model. The following dynamics are designed

$$\dot{z}_{kr} = \dot{q}_k - \sigma_0 |\dot{q}_k| z_{kr} + \alpha(t) (\dot{q}_k - \dot{q}_k)$$  \hfill (23)

where $\alpha(t)$ is a time-variant control parameter to be specified later.

After having $z_{kr}$, the feedforward friction compensation term is designed as

$$f_k = \dot{\sigma} [\sigma_0 z_{kr} + \sigma_1 \dot{z}_{kr}] + \sigma_v(q_k)$$  \hfill (24)
where \( \hat{\sigma} \) denotes the estimate of \( \sigma \) in (9) and is updated by using the \( P \) function as
\[
\hat{\sigma} = P(s_\sigma, \rho_\sigma, \sigma, \sigma, m_k, t)
\]
with
\[
s_\sigma = (\dot{q}_k - \dot{q}_0) (\sigma_0 z_{kr} + \sigma_1 \dot{z}_{kr}).
\]
In (25), \( \rho_\sigma > 0 \) is an update gain and \( \sigma \geq 0 \) and \( \sigma > 0 \) denote the lower and upper bounds of \( \sigma \), respectively.

Finally, the joint control torque is designed as
\[
\tau_k = \hat{I}_{mk} \dot{q}_k + f_k + k_s (\dot{q}_k - \dot{q}_0) + \dot{d}_k
\]
where \( k_s > 0 \) is a control gain, and \( \hat{I}_{mk} \) and \( \dot{d}_k \) denoting the estimates of \( I_{mk} \) and \( d_k \), respectively, are being updated by using the \( P \) function as
\[
\hat{I}_{mk} = P(s_m, \rho_m, \hat{I}_{mk}, \overline{I}_{mk}, t)
\]
with
\[
s_m = (\dot{q}_k - \dot{q}_0) \ddot{q}_k
\]
\[
s_t = (\dot{q}_k - \dot{d}_k)
\]
where \( \rho_m > 0 \) and \( \rho_d > 0 \) are two update gains; \( \overline{I}_{mk} > 0 \) and \( \overline{I}_{mk} > 0 \) denote the lower and upper bounds of \( I_{mk} \); and \( \ddot{d}_k \) and \( \overline{d}_k \) denote the lower and upper bounds of \( d_k \).

5. STABILITY

The following lemma ensures the virtual stability of the \( k \)th module when being combined with its respective control equations, in the sense of Definition 2.17 in Zhu (2010).

**Lemma 1.** The \( k \)th module described by (1)-(9) and combined with its respective control equations (10)-(12) and (14)-(31), under the following condition
\[
\alpha(t) = \frac{\sigma_0 - \sigma f(t)}{\beta}
\]
with \( \beta > 0 \) being a constant, is virtually stable with its affiliated vectors \( A_k V_r - A_k V \) and \( C_k V_r - C_k V \) and variable \( \dot{q}_k - \dot{q}_0 \) being virtual functions in both \( L_2 \) and \( L_\infty \), in the sense of Definition 2.17 in Zhu (2010).

**Proof.** Chose a non-negative function
\[
\nu = \nu_a + \nu_b + \nu_f
\]
with
\[
\nu_a = \frac{1}{2} (A_k V_r - A_k V)^T M_{A_k} (A_k V_r - A_k V)
\]
\[
+ \frac{1}{2} \sum_{\gamma=1}^{13} (\theta_{A_k\gamma} - \hat{\theta}_{A_k\gamma})^2 / \rho_{A_k\gamma}
\]
\[
\nu_b = \frac{1}{2} (C_k V_r - C_k V)^T M_{C_k} (C_k V_r - C_k V)
\]
\[
+ \frac{1}{2} \sum_{\gamma=1}^{13} (\theta_{C_k\gamma} - \hat{\theta}_{C_k\gamma})^2 / \rho_{C_k\gamma}
\]
\[
\nu_f = \frac{I_m}{2} (\dot{q}_k - \dot{q}_0)^2 + \frac{\sigma}{2} (z_{kr} - z_k)^2
\]
\[
+ \frac{1}{2\rho_\sigma} (\sigma - \dot{\sigma})^2 + \frac{1}{2\rho_m} (I_{mk} - \dot{I}_{mk})^2
\]
\[
+ \frac{1}{2\rho_d} (\dot{d}_k - \ddot{d}_k)^2.
\]
Using the results of Appendix A yields
\[
\dot{\nu} \leq - (A_k V_r - A_k V)^T K_{A_k} (A_k V_r - A_k V)
\]
\[
- (C_k V_r - C_k V)^T K_{C_k} (C_k V_r - C_k V)
\]
\[
- [k_s + \sigma_\gamma \alpha(t)] (\dot{q}_k - \dot{q}_0)^2
\]
\[
- \sigma^2 \sigma_\gamma |\dot{q}_k| (z_{kr} - z_k)^2
\]
\[
+ p_{B_k} \tau_k - p_{T_k} \dot{q}_k.
\]

The two terms \( p_{B_k} \) and \( -p_{T_k} \) appearing in the right hand side of (37) are named virtual power flows by Definition 2.16 in Zhu (2010). They behave as a positive (plus sign) and a negative (minus sign) “stability connectors” of the \( k \)th module to the remaining modules. Having virtual power flows is a unique characteristic of the virtual stability. If every module (when being combined with its respective control equations) is virtually stable, then Theorem 2.1 in Zhu (2010) ensures that the complete system (system) is stable. This is because all the virtual power flows of the entire system cancel out with each other as if each positive “stability connector” is connected to its corresponding negative “stability connector.”

6. EXPERIMENT

A single module was tested with \( n = k = 1 \). With no force at frame \( \{T_r\} \) and no velocity at frame \( \{B_1\} \), it follows that \( p_{B_k} = 0 \) and \( p_{T_k} = 0 \) in (37).

The system setup has been well described in Zhu and Lamarche (2007); Lamarche and Zhu (2007).

The control equations (10)-(12) and (22) were computed in the master node with a sampling period of 1.0 (\( ms \)). The link control equations (14), (15), and (18)-(21) were then computed in an embedded FPGA device (Virtex II-1000 from Xilinx Inc.) by using block memories with a sampling period of 163.84 (\( \mu s \)). Finally, the joint control equations (23)-(31) were computed in the same FPGA device with a sampling period of 1.28 (\( \mu s \)), where \( \alpha(t) \) in (23) is replaced by a constant.

The desired motion trajectory was given by a fifth order polynomial \( 6(p_f - p_0) (t/t_f)^5 - 15(p_f - p_0) (t/t_f)^4 + 10(p_f - p_0) (t/t_f)^3 + p_0 \), where \( p_0 \) and \( p_f \) denote the initial and final positions, respectively, and \( t_f > 0 \) denotes the time duration. This trajectory guarantees zero velocity and zero acceleration at both the initial and final time instants.
Fig. 3. Position tracking.

Fig. 4. Position tracking error and control torque.

Let \( p_0 = 0 \) (rad), \( p_f = 0.3927 \) (rad), and \( t_f = 1 \) (sec). The maximum velocity reaches 0.7362 (rad/s) occurred at \( t = 0.5 \) (sec).

The position tracking result is illustrated in Fig. 3. The dashed red line represents the desired position and the solid black line represents the actual position. Accordingly, the position tracking error and the control torque are illustrated in Fig. 4. The maximum position tracking error is about \( 1.8 \times 10^{-4} \) (rad). Thus, the ratio of the maximum position error to the maximum velocity reaches 0.00024 (s) in SI Units. This number is lower than what has been reported in Zhu (2010a).

7. CONCLUSION

In this paper, the VDC approach has been applied to modular robot manipulators to overcome their long-standing problem of lacking precise control performances. With respect to a typical module comprised of two rigid links connected by a joint equipped with a harmonic drive and a brushless motor, model-based control equations have been thoroughly designed by incorporating a LuGre friction model. Parametric uncertainty is addressed by using independent parameter adaptation with bounds. The virtual stability of the addressed module combined with its respective control equations has been rigorously proven. The virtual stability of all modules leads to the stability of the entire robot with mathematical certainty. Experimental results on a single module robot revealed that the ratio of the maximum position tracking error to the maximum velocity reaches \( 0.00024 \) (s) in SI Units. This is by far the lowest number in the category of control of robots with harmonic drives.

REFERENCES


Appendix A. DERIVATION IN PROOF OF LEMMA 1

Using Lemma 2.9 in Zhu (2010) yields

\[
\left( \dot{\theta}_{\text{A}_k} - \dot{\theta}_{\text{A}_k} \right) \left( s_{\text{A}_k} - \dot{\gamma}_{\text{A}_k} / \rho_{\text{A}_k} \right) \leq 0 \quad (A.1) \\
\left( \dot{\theta}_{\text{C}_k} - \dot{\theta}_{\text{C}_k} \right) \left( s_{\text{C}_k} - \dot{\gamma}_{\text{C}_k} / \rho_{\text{C}_k} \right) \leq 0 \quad (A.2) \\
\left( \sigma - \dot{\sigma} \right) \left( s_{\sigma} - \dot{\gamma}_{\text{A}_k} / \rho_{\text{A}_k} \right) \leq 0 \quad (A.3) \\
\left( I_{mk} - \dot{I}_{mk} \right) \left( s_{m} - \dot{I}_{mk} / \rho_{m} \right) \leq 0 \quad (A.4) \\
\left( d_{k} - \dot{d}_{k} \right) \left( s_{d} - \dot{d}_{k} / \rho_{d} \right) \leq 0 \quad (A.5)
\]

Subtracting (4) and (5) from (14) and (15), respectively, and using (16) and (17) yields...
\[ A_k F^* - A_k F^* = M A_k \left( \frac{d}{dt}(A_k V_r) - \frac{d}{dt}(A_k V) \right) + C A_k (A_k V_r - A_k V) - Y A_k (\theta A_k - \dot{\theta} A_k) \]  
(A.6)

\[ C_k F^* - C_k F^* = M C_k \left( \frac{d}{dt}(C_k V_r) - \frac{d}{dt}(C_k V) \right) + C C_k (C_k V_r - C_k V) - Y C_k (\theta C_k - \dot{\theta} C_k) \]  
(A.7)

Note the fact that matrices \( M A_k \) and \( M C_k \) are constant and matrices \( C A_k \) and \( C C_k \) are skew-symmetric.

Using (18), (19), (A.1), (A.2), (A.6), and (A.7), the time-derivatives of \( \nu_a \) and \( \nu_b \) in (34) and (35) can be expressed as:

\[ \nu_a = (A_k V_r - A_k V)^T M A_k \frac{d}{dt}(A_k V_r - A_k V) \]
\[ - \sum_{\gamma=1}^{13} \left( \theta A_{k,\gamma} - \dot{\theta} A_{k,\gamma} \right) \frac{\dot{\theta} A_{k,\gamma}}{\rho A_{k,\gamma}} \]
\[ \leq - (A_k V_r - A_k V)^T K A_k (A_k V_r - A_k V) \]
\[ + (A_k V_r - A_k V)^T A_k (A_k V_r - A_k V^*) \]  
\[ \nu_b = (C_k V_r - C_k V)^T M C_k \frac{d}{dt}(C_k V_r - C_k V) \]
\[ - \sum_{\gamma=1}^{13} \left( \theta C_{k,\gamma} - \dot{\theta} C_{k,\gamma} \right) \frac{\dot{\theta} C_{k,\gamma}}{\rho C_{k,\gamma}} \]
\[ \leq - (C_k V_r - C_k V)^T K C_k (C_k V_r - C_k V) \]
\[ + (C_k V_r - C_k V)^T C_k (C_k V_r - C_k V^*) . \]  
(A.8)

Using (1)-(3), (6), (10)-(12), (22), (38), and (39) makes the summation of the last terms in the right hand sides of (A.8) and (A.9) as:

\[ (A_k V_r - A_k V)^T (A_k F^* - A_k F^*) + (C_k V_r - C_k V)^T (C_k F^* - C_k F^*) \]
\[ = \left[ (B_k V_r - B_k V)^T B_k U_k \right] \]
\[ + \left[ (B_k V_r - B_k V)^T B_k U_{C_k} \right] \left( C_k F^* - C_k F^* \right) \]
\[ = \left[ (B_k V_r - B_k V)^T (B_k F_r - B_k F) \right] \]
\[ + \left[ (B_k V_r - B_k V)^T B_k U_{T_k} \right] \left( T_k F_r - T_k F \right) \]
\[ + \left[ (q_{kr} - \dot{q}_k) \right] \left( A_k F^* - A_k F^* \right) \]
\[ = \left[ (B_k V_r - B_k V)^T \right] T_k \left( T_k F_r - T_k F \right) \]
\[ + \left[ (q_{kr} - \dot{q}_k) \right] \left( A_k F^* - A_k F^* \right) \]
\[ + \left[ (q_{kr} - \dot{q}_k) \right] A_k U_{T_k} \left( T_k F_r - T_k F \right) \]
\[ = p_{B_k} - p_{T_k} \]

Finally, substituting (A.8), (A.9) and (A.14) into the time-derivative of (33) and using (A.10) yields (37).