

# Feasible-Cooperation Distributed Model Predictive Control Scheme Based on Game Theory<sup>\*</sup>

Felipe Valencia<sup>\*</sup> Jairo J. Espinosa<sup>\*</sup> Bart De Schutter<sup>\*\*</sup>  
Kateřina Staňková<sup>\*\*</sup>

<sup>\*</sup> Faculty of Mines, Universidad Nacional de Colombia (e-mail:  
fvalenc@unal.edu.co jairo.espinosa@ieee.org).

<sup>\*\*</sup> Delft Center for Systems and Control, The Netherlands (e-mail:  
b.deschutter@tudelft.nl k.stankova@tudelft.nl).

---

## Abstract:

This work deals with the formulation of a distributed model predictive control scheme as a decision problem in which the decisions of each subsystem affect the decisions of the other subsystems and the performance of the whole system. This decision problem is formulated as a bargaining game. This formulation allows each subsystem to decide whether to cooperate or not depending on the benefits that the subsystem can gain from the cooperation. A solution based on game theory is proposed. The convexity and feasibility properties of the proposed solution are also presented. The proposed control scheme is tested on a case study with a chain of two continuous stirred tank reactors followed by a nonadiabatic flash separator.

## Keywords:

Distributed model predictive control, game theory, bargaining game, negotiation model.

---

## 1. INTRODUCTION

Large-scale systems are systems composed of several interacting components. Their operation is based on controllers that face different situations according to their own interests. For the control of large-scale systems, distributed and hierarchical control schemes based on model predictive controllers have been proposed, due to their ability to handle complex systems with hard input and state constraints (Camponogara et al., 2002; Di Palma and Magni, 2004; Dunbar and Desa, 2007; Necoara et al., 2008; Negenborn et al., 2009), and for their ability to obtain a good performance starting from rather intuitive design principles and simple models (Di Palma and Magni, 2004).

Some approaches of hierarchical and distributed model predictive control are proposed in (Camponogara and Talukdar, 2007; Doan et al., 2008; Dokucu et al., 2008; Dunbar and Murray, 2006; Hennes, 2003; Jia and Krogh, 2002; Laabidi et al., 2008; Stephanopoulos et al., 2000; Talukdar et al., 2005; Tatjewski, 2008; Vargas-Villamil and Rivera, 2000; Venkat et al., 2008; Wang and Cameron, 2007; Wang et al., 2008; Wisniewski and Doyle, 1998; Wu et al., 2009). However, the hierarchical approaches do not guarantee the lack of competition among control layers because each layer may take its own decisions without taking lower layers into account. The distributed approaches may force the subsystems to cooperate, regardless of the

benefits produced by the cooperative behavior, and might steer the subsystems to operating points in where they do not perceive any benefit.

In order to deal with these drawbacks of the distributed control schemes, it is possible to assume that the local controllers may “bargain” among themselves, and that in that way an agreement may be achieved. With such assumptions, the distributed model predictive control (MPC) problem can be reformulated as an  $n$ -persons cooperative game. The  $n$ -person cooperative game involves  $n$  individuals that can collaborate for mutual benefit. The individuals communicate with each other in order to (jointly) decide which strategy is the best for each individual, based on the profit received for each of them by the cooperative behavior (Nash, 1950a).

In this work, based on the Nash theories about the bargaining problem (Nash, 1950a) and two-persons cooperative games (Nash, 1953) distributed model predictive control is analyzed as a game. The convexity and the feasibility of the proposed control scheme are presented. In order to test the proposed control scheme, a chain of two reactors and one adiabatic flash separator is used as simulation testbed.

This paper is composed as follows. In Section 2 the distributed model predictive control framework is presented. In Section 3 a game-theoretical formulation of distributed model predictive control is introduced, and the negotiation model is explained in Section 4. In Section 5 the case study is introduced and the results of the simulations are explained. Section 6 presents the conclusions.

---

<sup>\*</sup> This research has been supported by the European 7th Framework Network of Excellence “Highly-complex and networked control systems (HYCON2)”, and by the European 7th framework STREP project “Hierarchical and distributed model predictive control (HD-MPC)”, contract number INFSO-ICT-223854.

## 2. DISTRIBUTED MODEL PREDICTIVE CONTROL

Consider the nonlinear system given by

$$\begin{aligned}\dot{x}(t) &= f_x(x(t), u(t)) \\ y(t) &= f_y(x(t), u(t))\end{aligned}\quad (1)$$

where  $f_x(\cdot)$ ,  $f_y(\cdot)$  are smooth  $C^1$  functions, and  $x \in R^n$ ,  $u \in R^m$ , and  $y \in R^z$  denote the state, input, and output vector of the dynamical system (1).

Assume that at each time step  $k$  the system (1) can be approximated by a discrete-time linear time-invariant system

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k)\end{aligned}\quad (2)$$

In the following, we focus on the trajectories of the states  $x$  and their constraints. However, the method described in this work can easily be extended to the cases in which the output  $y$  is considered in the cost function and in the constraints.

Assume that the state update equation of (2) can be decomposed into  $M$  subsystems such that the behavior of each subsystem can be expressed as

$$x_i(k+1) = \sum_{j=1}^M [A_{ij}x_j(k) + B_{ij}u_j(k)] \quad (3)$$

where  $x_i \in R^{n_i}$ , and  $u_i \in R^{m_i}$  denote the state and input vector of the subsystem  $i$ ,  $i = 1, \dots, M$ , and  $A_{ij}$ ,  $B_{ij}$  are submatrices of the entire linear model of the system, i.e., submatrices of  $A$ ,  $B$ . This model is also used in (Camponogara et al., 2002; Doan et al., 2008; Du et al., 2001; Necoara et al., 2008; Venkat et al., 2008).

Let  $\tilde{x}(k) = [x^T(k), \dots, x^T(k + N_p)]^T$  and  $\tilde{u}(k) = [u^T(k), \dots, u^T(k + N_u)]^T$ . From the system decomposition (3), the cost function  $L(\tilde{x}(k), \tilde{u}(k))$  can be expressed as (Venkat et al., 2006b, 2008)

$$\begin{aligned}L(\tilde{x}(k), \tilde{u}(k)) &= \sum_{i=1}^M \left( \sum_{t=0}^{N_p-1} x_i^T(k+t) Q_i x_i(k+t) \right. \\ &\quad \left. + \sum_{t=0}^{N_u} u_i^T(k+t) R_i u_i(k+t) + x_i^T(k+N_p) P_i x(k+N_p) \right)\end{aligned}\quad (4)$$

with  $P_i, Q_i, R_i \geq 0$ .

Substituting the equation (3) for  $x_i(k+t)$  into (4) yields

$$L(\tilde{x}(k), \tilde{u}(k)) = \sum_{i=1}^M \phi_i(\tilde{u}(k); x(k)) \quad (5)$$

where

$$\begin{aligned}\phi_i(\tilde{u}(k); x(k)) &= \tilde{u}^T(k) Q_{uui} \tilde{u}(k) + 2x^T(k) Q_{xui} \tilde{u}(k) \\ &\quad + \tilde{x}^T Q_{xxi} \tilde{x}(k)\end{aligned}\quad (6)$$

where  $Q_{uui} \geq 0$ , for  $i = 1, \dots, M$ . In the following, the term  $\tilde{x}^T Q_{xxi} \tilde{x}(k)$  will not be included into the cost function because the value of this term is independent of the value of the control inputs.

In (6) the arguments for  $\phi_i(\tilde{u}(k); x(k))$  indicate that the argument of  $\phi_i$  is  $\tilde{u}(k)$  and  $x(k)$  is parameter of  $\phi_i$ . Clearly,  $\phi_i$  is a positive-definite quadratic function of  $\tilde{u}(k)$  and thus it is convex in  $\tilde{u}(k)$ .

Let  $\Omega_i = \prod_{j=0}^{N_u} \Lambda_i$  be the set of feasible control actions for  $\tilde{u}_i(k)$ , where  $\tilde{u}_i(k) = [u_i^T(k), \dots, u_i^T(k + N_u)]^T$ , and let  $\Lambda_i$  be the feasible set for the control action  $u_i(k+j)$ , for  $j = 0, \dots, N_u$  determined by the physical and operational limits of subsystem  $i$ . Assume that  $0 \in \Lambda_i$ , and that  $\Lambda_i$  is convex and independent of  $k$  for  $i = 1, \dots, M$ . Note that  $\Omega = \prod_{i=1}^M \Omega_i$ . Then, the MPC problem can be written as (Li et al., 2005; Necoara et al., 2008; Venkat et al., 2006b, 2008)<sup>1</sup>

$$\min_{\tilde{u}(k)} \sum_{i=1}^M \phi_i(\tilde{u}(k)) \quad (7)$$

subject to:  $\tilde{u}_i(k) \in \Omega_i$ , for  $i = 1, \dots, M$

Let  $\sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)) = \phi_r(\tilde{u}(k))$ , where

$$\tilde{u}_{-i}(k) = [\tilde{u}_1^T(k), \dots, \tilde{u}_{i-1}^T(k), \tilde{u}_{i+1}^T(k), \dots, \tilde{u}_M^T(k)]^T$$

for  $\tilde{u}_i(k) \in \Omega_i$ , and  $\tilde{u}_{-i}(k) \in \Omega_{-i}$ , with  $\Omega_{-i} = \Omega_1 \times \dots \times \Omega_{i-1} \times \Omega_{i+1} \times \dots \times \Omega_M$ . In order to solve the optimization problem (7) in a distributed fashion, two main approaches have been proposed in the literature (Venkat et al., 2008, 2006a,b): the communication-based approach and the feasible-cooperation approach. In this paper we focus on the feasible-cooperation approach, because there exists evidence (e.g. the cases presented by Venkat et al. (2008)) that the communication-based MPC may lead to unacceptable closed-loop performance or closed-loop instability.

In the feasible-cooperation approach, the local cost function  $\phi_i(\cdot)$  is replaced by a cost function that measures the systemwide impact of the local control inputs. This is done with the purpose of avoiding competition and to increase the cooperation among subsystems in a DMPC scheme. Venkat et al. (2008, 2006b) propose to use a convex combination of the controller objectives:

$$\phi_i(\tilde{u}(k)) = \sum_{r=1}^M w_r \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)) \quad (8)$$

where  $w_r > 0$ ,  $\sum_{i=1}^M w_r = 1$ , as a cost function of each subsystem, because this is the simplest choice for such an objective function. Also, the subsystems must carry out a negotiation process in order to select the best control actions, with respect to the performance of the entire system. Let  $\tilde{u}_{i,q}(k) = [u_{i,q}^T(k), \dots, u_{i,q}^T(k + N_u)]^T$ , where  $q$  denotes the iteration number at time step  $k$  of the negotiation process ( $q = 1, \dots, q_{\max}$ , with  $q_{\max}$  the maximum number of iterations of the negotiation process). Let  $\tilde{u}_{-i,q}(k) = [\tilde{u}_{1,q}^T(k), \dots, \tilde{u}_{i-1,q}^T(k), \tilde{u}_{i+1,q}^T(k), \dots, \tilde{u}_{M,q}^T(k)]^T$ , where the superscript  $*$  denotes the optimal value of  $\tilde{u}_{j,q}(k)$ , for  $j = 1, \dots, i-1, i+1, \dots, M$ . Then, the optimization problem (7) can be solved in a cooperative way by computing the solution of the optimization problem (Venkat et al., 2008, 2006b)

$$\min_{\tilde{u}_{i,q}(k)} \sum_{r=1}^M w_r \sigma_r(\tilde{u}_{i,q}(k), \tilde{u}_{-i,q-1}(k)) \quad (9)$$

subject to:  $\tilde{u}_{i,q}(k) \in \Omega_i$

<sup>1</sup> For the sake of simplicity of notation we will not indicate the dependence of  $\phi_i$  on  $x(k)$  explicitly in the remainder of this paper and thus write  $\phi_i(u(k))$  instead  $\phi_i(u(k); x(k))$ .

for  $q = 1, \dots, q_{\max}$ , where after each iteration  $q$  the subsystems communicate their optimal solution to the other subsystems.

In the next section, theoretical concepts of game theory will be used in order to deal with the feasible-cooperation MPC (FC-MPC) problem as a decision problem in which the decisions of each subsystem affect the decisions of the other subsystems.

### 3. FC-MPC AS A GAME

A game is defined as the tuple  $(N, \{\Omega_i\}_{i \in N}, \{\phi_i\}_{i \in N})$ , where  $N = \{1, \dots, M\}$  is the set of players,  $\Omega_i$  is a finite set of possible actions of player  $i$ , and  $\phi_i : \Omega_1 \times \dots \times \Omega_M \rightarrow R$  is the payoff function of the  $i$ th player (Akira, 2005).

Based on the definition of a game, the FC-MPC problem can be defined as a tuple  $G = (N, \{\Omega_i\}_{i \in N}, \{\phi_i\}_{i \in N})$ , where  $N = \{1, \dots, M\}$  is the set of subsystems,  $\Omega_i$  is the non-empty set of feasible control actions for subsystem  $i$ , and  $\phi_i : \Omega_1 \times \dots \times \Omega_M \rightarrow R$ , is the cost function of the  $i$ th subsystem. From this point of view, FC-MPC is a game in which the players are the subsystems, the actions are the control inputs, and the payoff of each subsystem is given by the value of its cost function. Moreover, in FC-MPC the subsystems can cooperate in order to obtain a common benefit. So, FC-MPC can be analyzed as a cooperative game.

Following the cooperative game theory introduced in (Nash, 1950b, 1953; Peters, 1992), the formulation of FC-MPC as a game is completed by introducing the concept of disagreement point. The disagreement point of subsystem  $i$ ,  $d_i(k)$ , at time step  $k$ , is given by  $d_i(k) = \phi_i(\tilde{u}^d(k))$ , where  $\tilde{u}^d(k)$  are the control inputs solving the following optimization problem

$$\begin{aligned} & \min_{\tilde{u}_i(k)} \max_{\tilde{u}_{-i}(k)} \phi_i(\tilde{u}(k)) \\ \text{subject to: } & \tilde{u}_i(k) \in \Omega_i \\ & \tilde{u}_{-i}(k) \in \Omega_{-i} \end{aligned} \quad (10)$$

Note that the optimization problem (10) defines the worst case for subsystem  $i$ . Then,  $d_i(k)$  is the best benefit that the  $i$ th subsystem can achieve given the worst case.

According to (Nash, 1950a, 1953), the solution of the cooperative game associated with the DMPC problem can be computed as the solution of the following optimization problem:

$$\begin{aligned} & \max_{\tilde{u}(k)} \prod_{i=1}^M [d_i(k) - \phi_i(\tilde{u}(k))]^{w_i} \\ \text{subject to: } & d_i(k) > \phi_i(\tilde{u}(k)), \text{ for } i = 1, \dots, M \\ & \tilde{u}_i(k) \in \Omega_i, \text{ for } i = 1, \dots, M \end{aligned} \quad (11)$$

The maximization problem (11) can be rewritten equivalently as

$$\begin{aligned} & \max_{\tilde{u}(k)} \sum_{i=1}^M w_i \log [d_i(k) - \phi_i(\tilde{u}(k))] \\ \text{subject to: } & d_i(k) > \phi_i(\tilde{u}(k)), \text{ for } i = 1, \dots, M \\ & \tilde{u}_i(k) \in \Omega_i, \text{ for } i = 1, \dots, M \end{aligned} \quad (12)$$

Thus, (11) can be solved in a distributed fashion by solving (12) following the feasible-cooperation approach presented

in Section 2. Consequently, the local optimization problem for subsystem  $i$  is given by the maximization problem

$$\begin{aligned} & \max_{\tilde{u}_i(k)} \sum_{r=1}^M w_r \log [d_r(k) - \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))] \\ \text{subject to: } & d_r(k) > \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)), \text{ for } r = 1, \dots, M \\ & \tilde{u}_i(k) \in \Omega_i \end{aligned} \quad (13)$$

In the next section, we propose a negotiation model to solve the FC-MPC game. Properties like convexity and feasibility are also discussed.

### 4. NEGOTIATION MODEL

A negotiation model consists of a sequence of steps whose outcome is the solution of the game in a cooperative or non-cooperative fashion. The negotiation model proposed in this work is based on the algorithm proposed by Nash (1953) for two-person cooperative games.

At each time step  $k$ , at each iteration  $q, q = 1, \dots, q_{\max}$ , the proposed steps to solve the FC-MPC game are:

1. Let  $d_i(k)$  denote the disagreement point of subsystem  $i$  at time step  $k$ . Then, given the initial conditions,  $x(k)$ , all subsystems compute their disagreement points  $d_i(k)$  according to (10) in a separated way.
2. After computing the disagreement points, each subsystem sends its disagreement point to the other subsystems.
3. Each subsystem solves the optimization problem (13). If (13) is feasible, let  $\tilde{u}_{i,q}^*(k)$  be an optimal solution (so it satisfies the constraints, i.e.,  $d_r(k) > \sigma_r(\tilde{u}_{i,q}^*(k), \tilde{u}_{-i,q-1}(k))$ , for  $r = 1, \dots, M$ ). If (13) is not feasible, subsystem  $i$  decides not to cooperate. In this step, if  $q = 1$ , then  $\tilde{u}_i^d(k)$  is considered as initial condition for subsystem  $i$ , for solving (13). Otherwise,  $\tilde{u}_{i,q-1}(k)$  is considered as initial condition for subsystem  $i$ , for solving (13).
4. The subsystems that decide to cooperate update their control actions by a convex combination  $\tilde{u}_{i,q}(k) = w_i \tilde{u}_{i,q}^*(k) + (1 - w_i) \tilde{u}_{i,q-1}(k)$ . The subsystems that decide not to cooperate select their control actions by  $\tilde{u}_{i,q}(k) = w_i \tilde{u}_i^d(k) + (1 - w_i) \tilde{u}_{i,q-1}(k)$ .
5. Each subsystem sends its control actions to the other subsystems. If  $\|\tilde{u}_{i,q}(k) - \tilde{u}_{i,q-1}(k)\| \leq \xi$  ( $\xi > 0$ ) for all subsystems, or if  $q = q_{\max}$ , or if the maximum allowable time for the computation of the optimal control input  $\tilde{u}^*(k) = [\tilde{u}_1^*(k), \dots, \tilde{u}_M^*(k)]^T$  is reached, the first element of the control sequence  $\tilde{u}_{i,q}(k)$  is applied and each subsystem returns to step 1. Else, each subsystem returns to step 3.

At time step  $k+1$  the initial conditions for subsystem  $i$  for solving (10) are determined by the shifted control sequence  $\tilde{u}_{i,0}(k+1) = [u_{i,q_{\text{end}}}^*(k+1, k), \dots, u_{i,q_{\text{end}}}^*(k+N_u, k), 0]^T$ , where  $u_{i,q_{\text{end}}}^*(k+1, k)$  denotes the optimal value of the control inputs for subsystem  $i$  at iteration  $q_{\text{end}}$  at the time step  $k+1$  given the conditions at time step  $k$ . From the negotiation model we have that only the disagreement point is communicated by the subsystems at each time step

$k$  and that the subsystems transmit only their sequence of control actions at each iteration  $q$ .

Now we have:

*Proposition 1.* The maximization problem (13) is a concave optimization problem.

**Proof.** The optimization problem (13) can be written as

$$\min_{\tilde{u}_i(k)} \left( - \sum_{r=1}^M w_r \log [d_r(k) - \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))] \right)$$

subject to:  $d_r(k) > \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)), \forall r \in N$   
 $\tilde{u}_i(k) \in \Omega_i$  (14)

The function  $-\log[g(x)]$  is a convex function if  $g(x) > 0$  and  $-g(x)$  is a convex function (Rockafellar, 1970, Theorem 5.1). In the case of the optimization problem (14), the function  $\log [d_r - \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))]$  satisfies  $[d_r(k) - \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))] > 0$  because of the constraint  $d_r(k) > \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))$ , for  $r = 1, \dots, M$ , and it is a convex function because  $d_r(k)$  is fixed and  $\sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))$  is a convex function with respect to  $\tilde{u}_i(k)$ , for  $r = 1, \dots, M$ .

Since the function

$$C_i = - \sum_{r=1}^M w_r \log [d_r(k) - \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))]$$

is a convex combination of  $M$  convex functions, it is also a convex function with respect to  $\tilde{u}_i(k)$ . Hence, the optimization problem (14) is convex and therefore the optimization problem (13) is concave.  $\square$

From Proposition 1, it follows that the optimization problem (13) can be solved efficiently using interior point methods like active-set algorithms, linear search algorithms, or gradient-based algorithms (Boyd and Vandenberghe, 2004).

*Proposition 2.* The input sequences for subsystem  $i$  generated by the negotiation model at time step  $k$  from a feasible starting point are feasible for time steps  $k+1, k+2, \dots$  for the original MPC problem, i.e.,  $\tilde{u}_{i,q}(k) \in \Omega_i$  for  $i = 1, \dots, M$ .

**Proof.**

Since  $\sigma_i(\tilde{u}_i(k), \tilde{u}_{-i}(k))$  is a convex function, and because  $\Omega_i$  and  $\Omega_{-i}$  are both convex sets,  $\tilde{u}_{-i}^*(k) \in \Omega_{-i}$  and  $\tilde{u}_i^d(k) \in \Omega_i$  exist for all iterations  $q = 1, \dots, q_{\text{end}}$ , for all  $i = 1, \dots, M$ .

Let  $\tilde{u}_{i,0}(k) \in \Omega_i$  denote the initial solution at time step  $k$ . By the negotiation model, if at iteration  $q = 1$  at time step  $k$  subsystem  $i$  decides not to cooperate, its control actions are given by the convex combination  $\tilde{u}_{i,1}(k) = w_i \tilde{u}_i^d(k) + (1 - w_i) \tilde{u}_{i,0}(k)$ . Since  $\tilde{u}_{i,0}(k) \in \Omega_i$ ,  $\tilde{u}_i^d(k) \in \Omega_i$  and  $\Omega_i$  is convex, any convex combination of  $\tilde{u}_{i,0}(k)$  and  $\tilde{u}_i^d(k)$  also belongs to  $\Omega_i$ . Then,  $\tilde{u}_{i,1}(k) \in \Omega_i$  for  $i = 1, \dots, M$ .

If at iteration  $q = 1$  at time step  $k$  subsystem  $i$  decides to cooperate, the control actions of subsystem  $i$  are given by the convex combination  $\tilde{u}_{i,q}(k) = w_i \tilde{u}_{i,q}^*(k) + (1 - w_i) \tilde{u}_{i,0}(k)$ . Since  $\tilde{u}_{i,0}(k) \in \Omega_i$ ,  $\tilde{u}_{i,q}^*(k) \in \Omega_i$ , and  $\Omega_i$  is convex, any convex combination of  $\tilde{u}_{i,0}(k)$  and  $\tilde{u}_{i,q}^*(k)$  also belongs to  $\Omega_i$ . Then,  $\tilde{u}_{i,1}(k) \in \Omega_i$  for  $i = 1, \dots, M$ .

At iteration  $q = 2$ ,  $\tilde{u}_{i,1}(k) \in \Omega_i$ , then  $\tilde{u}_{i,2}(k) \in \Omega_i$  independently of the decision of subsystem  $i$  about to cooperate or to not. This remains for the subsequent iterations  $q = 2, 3, \dots, q_{\text{end}}$ . Therefore, the sequence of control actions for subsystem  $i$  generated by the negotiation model from a feasible starting point remains feasible for all iteration  $q = 1, \dots, q_{\text{end}}$ , at time step  $k$ .

For time step  $k+1$ , the initial conditions for subsystem  $i$  for solving (10) are given by the shifted control sequence  $\tilde{u}_{i,0}(k+1) = [u_{i,q_{\text{end}}}^{*T}(k+1, k), \dots, u_{i,q_{\text{end}}}^{*T}(k+N_u, k), 0]^T$ . Recall that  $\Omega_i = \prod_{j=0}^{N_u} \Lambda_i$  and that  $0 \in \Lambda_i$ . Since  $\tilde{u}_{i,q_{\text{end}}}^*(k)$  is feasible, we have  $u_{i,q_{\text{end}}}(k, j) \in \Lambda_i$  for  $j = 1, \dots, N_u$ . Hence,  $\tilde{u}_{i,0}(k+1) \in \Omega_i$ . Therefore, the input sequences generated by the negotiation model for subsystem  $i$ , ( $i = 1, \dots, M$ ) belong to  $\Omega_i$  for the time step  $k+1$ . The same conclusions can be derived for the subsequent time steps  $k+2, k+3, \dots$ . Thus, the input sequences generated by the negotiation model for subsystem  $i$  are feasible for the time steps  $k, k+1, \dots$   $\square$

Recall that the origin belongs to  $\Omega = \Omega_1 \times \dots \times \Omega_M$ . Then, a possible choice for the initial condition for subsystem  $i$  at time step  $k=0$  is  $\tilde{u}_{i,0}(0) = 0$ , for  $i = 1, \dots, M$ .

Regarding the optimality of the proposed FC-MPC approach we should demonstrate that the solution of (13) is a nonsymmetric Nash solution of the DMPC problem. Then, it has to be shown that such a solution is unique and belongs to the Pareto front of the feasible set  $\{\tilde{u}_i(k) \in \Omega_i : d_r(k) > \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k))\}$ . The proof of these statements will be included in future work.

In the next section, simulation results of the application of the proposed control scheme to a plant with two continuous stirred tank reactors followed by a flash separator are presented.

## 5. APPLICATION: TWO REACTOR CHAIN WITH FLASH SEPARATOR

The example of a chain of two continuous stirred tank reactors (CSTRs) followed by a nonadiabatic flash separator was taken from Venkat et al. (2006b). All the simulations were performed using Matlab. For solving the optimization problem (13) the active-set algorithm provided by the *fmincon* function of the optimization toolbox of Matlab was used. The description of the system is presented below.

Consider a plant with two CSTRs followed by a nonadiabatic flash separator, as shown in Fig 1. In each of the CSTRs, the desired product  $B$  is produced through the irreversible first-order reaction  $A \xrightarrow{k_1} B$ ,  $k_1$  being the Arrhenius constant of the reaction. An undesirable side reaction  $B \xrightarrow{k_2} C$  results in the consumption of  $B$  and in the production of the unwanted side product  $C$  (here,  $k_2$  is the Arrhenius constant of this reaction). The product stream from CSTR-2 is sent to a nonadiabatic flash separator to separate the excess of  $A$  from the product  $B$  and the side product  $C$ . The model of the plant is given in (Venkat et al., 2006b). To apply the proposed distributed control scheme, each reactor and the flash separator are considered as subsystems, whose model is given by (3).

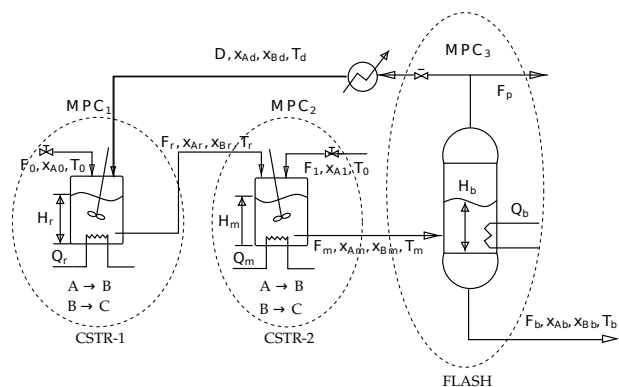


Fig. 1. Two reactor chain followed by nonadiabatic flash (Venkat et al., 2006b)

The performance of the proposed control scheme will be evaluated for a scenario in which a 15.79 percent increase in the level of  $H_m$  is made at time  $t = 200s$ , and a set-point change corresponding to a 5.56 percent decrease in the level of  $H_r$  is made at time  $t = 400s$ . Figs. 2 to 4 show the performance of the proposed control scheme for these set-point changes. In this simulation we have taken the prediction horizon  $N_p = 25$ , the control horizon  $N_u = 10$ , the weight of each subsystem  $w = 0.33$ , the sample time  $T_s = 10s$ , and the maximum iterations per time step  $q_{max} = 5$ .

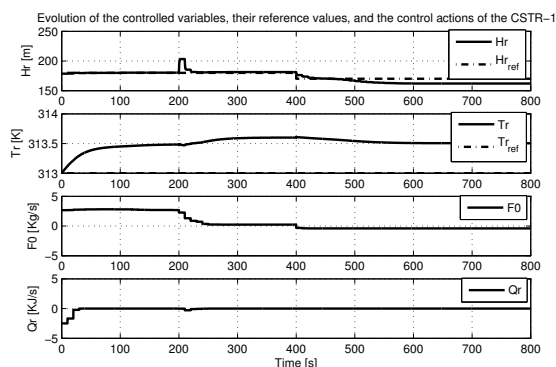


Fig. 2. Evolution of the level and the temperature of CSTR-1, their reference values, and the control inputs.

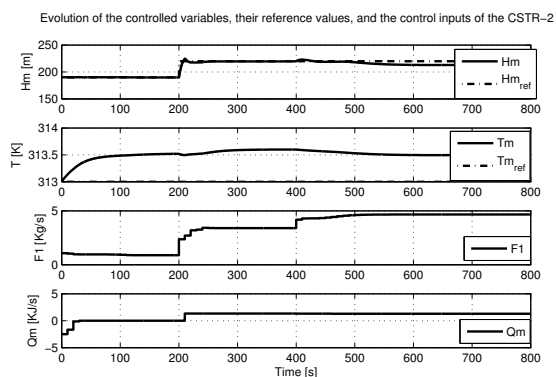


Fig. 3. Evolution of the level and the temperature of CSTR-2, their reference values, and the control inputs.

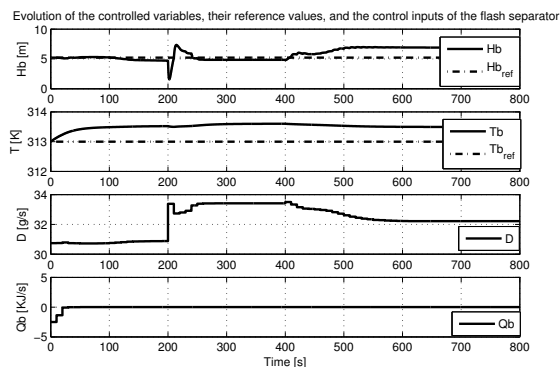


Fig. 4. Evolution of the level and the temperature of the nonadiabatic flash, their reference values, and the control inputs.

From Figs. 2 to 4 it is possible to conclude that the proposed control scheme stabilizes the closed-loop system. As a response to the set-point change of  $H_m$ , CSTR-1 and CSTR-2 jointly decide to decrease and increase their feed flow rates, respectively, in order to softly drive the system to the new desired operating point, while the flash separator decides to increase the recycle flow in order to regulate its level and the level of the CSTR-1. This indicates a cooperative behavior among the MPC controllers. In Fig. 5 the computation time incurred by the computation of the solution of the FC-MPC problem as a game is presented.

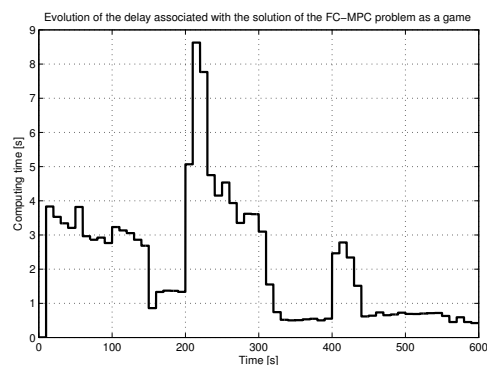


Fig. 5. Evolution of the delay associated with the solution of the FC-MPC problem as a game.

From Fig. 5 it is possible to conclude that the computation of the FC-MPC problem as a game in the case of the chain of reactors presented in this section is always less than the sample time, i.e. less than 10s.

## 6. CONCLUSIONS

In this work, the distributed model predictive control problem was considered. A new formulation for the feasible cooperation model predictive control approach was proposed based on concepts of game theory. In this new formulation, a disagreement point was introduced, allowing the subsystems to decide whether to cooperate or not, depending on the benefit perceived by each subsystem by the cooperative behavior. The benefit of each subsystem was computed as the difference between the disagreement point and the value of the local cost function when the

subsystems decide to cooperate. The convexity and the feasibility of the proposed method were demonstrated. However, the stability of the proposed approach remains as an open problem.

The proposed control scheme was tested using a chain of two reactors followed by a nonadiabatic flash. The reference values of the reactors were changed in different directions at different times, keeping the values of the references of the other subsystems constant. In this case, the three subsystems cooperate in order to jointly select the best control actions in the sense of the local performance without decreasing the entire system performance.

## REFERENCES

- Akira, O. (2005). A noncooperative approach to general n-person cooperative games. Discussion Papers 2005-01, Graduate School of Economics, Hitotsubashi University.
- Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press.
- Camponogara, E., Jia, D., Krogh, B.H., and Talukdar, S. (2002). Distributed Model Predictive Control. *IEEE Control Systems Magazine*, 22(1), 44–52.
- Camponogara, E. and Talukdar, S.N. (2007). Distributed model predictive control: Synchronous and asynchronous computation. *IEEE Transactions on Systems, Man, and Cybernetics, Part A*, 37(5), 732–745.
- Di Palma, F. and Magni, L. (2004). A multi-model structure for model predictive control. *Annual Reviews in Control*, 28(1), 47–52.
- Doan, D., Keviczky, T., Necoara, I., and Diehl, M. (2008). A Jacobi algorithm for distributed model predictive control of dynamically coupled systems. Technical Report arXiv:0809.3647. Comments: Shorter version submitted to 2009 American Control Conference.
- Dokucu, M., Park, M., and Doyle III, F. (2008). Multi-rate model predictive control of particle size distribution in a semibatch emulsion copolymerization reactor. *Journal of Process Control*, 18(1), 105–120.
- Du, X., Xi, Y., and Li, S. (2001). Distributed model predictive control for large-scale systems. *Proceedings of the 2001 American Control Conference*, 3142–3143.
- Dunbar, W. and Desa, S. (2007). Distributed MPC for dynamic supply chain management. *Assessment and Future Directions of Nonlinear Model Predictive Control*, 358, 607–615.
- Dunbar, W. and Murray, R. (2006). Distributed receding horizon control for multi-vehicle formation stabilization. *Automatica*, 42(4), 549–558.
- Hennet, J. (2003). A bimodal scheme for multi-stage production and inventory control. *Automatica*, 39(5), 793–805.
- Jia, D. and Krogh, B.H. (2002). Min-max feedback model predictive control for distributed control with communication. *Proceedings of the 2002 American Control Conference*, 4507–4512.
- Laabidi, K., Bouani, F., and Ksouri, M. (2008). Multi-criteria optimization in nonlinear predictive control. *Mathematics and Computers in Simulation*, 76(5-6), 363–374.
- Li, S., Zhang, Y., and Zhu, Q. (2005). Nash-optimization enhanced distributed model predictive control applied to the Shell benchmark problem. *Information Sciences*, 170(2-4), 329–349.
- Nash, J.F. (1950a). Equilibrium points in N-persons games. *Proceedings of the National Academy of Sciences of United States of America*, 36(1), 48–40.
- Nash, J. (1950b). The bargaining problem. *Econometrica*, 18(2), 155–162.
- Nash, J. (1953). Two-person cooperative games. *Econometrica*, 21(1), 128–140.
- Necoara, I., Doan, D., and Suykens, J.A.K. (2008). Application of the proximal center decomposition method to distributed model predictive control. *Proceedings of the 2008 IEEE Conference on Decision and Control*, 2900–2905.
- Negenborn, R.R., van Overloop, P.J., Keviczky, T., and De Schutter, B. (2009). Distributed model predictive control for irrigation canals. *Networks and Heterogeneous Media*, 4(2), 359–380.
- Peters, H.J.M. (1992). *Axiomatic Bargaining Game Theory*. Kluwer Academic Publishers, Dordrecht.
- Rockafellar, R. (1970). *Convex Analysis*. Princeton University Press, Princeton, New Jersey.
- Stephanopoulos, G., Karsligil, O., and Dyer, M. (2000). Multi-scale aspects in model-predictive control. *Journal of Process Control*, 10(2-3), 275–282.
- Talukdar, S., Jia, D., Hines, P., and Krogh, B.H. (2005). Distributed model predictive control for the mitigation of cascading failures. *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, 4440–4445.
- Tatjewski, P. (2008). Advanced control and on-line process optimization in multilayer structures. *Annual Reviews in Control*, 32(1), 71–85.
- Vargas-Villamil, F. and Rivera, D. (2000). Multilayer optimization and scheduling using model predictive control: application to reentrant semiconductor manufacturing lines. *Computers & Chemical Engineering*, 24(8), 2009–2021.
- Venkat, A.N., Rawlings, J.B., and Wright, S.J. (2006a). Implementable distributed model predictive control with guaranteed performance properties. *Proceedings of the 2006 American Control Conference*, 613–618.
- Venkat, A., Hiskens, I., Rawlings, J., and Wright, S. (2008). Distributed MPC strategies for automatic generation control. In *Proceedings of the IFAC Symposium on Power Plants and Power Systems Control*, 25–28.
- Venkat, A., Rawlings, J., and Wright, S. (2006b). Stability and optimality of distributed, linear model predictive control. Part I: State feedback. *Texas-Wisconsin Modeling and Control Consortium Technical Report*, 3.
- Wang, F. and Cameron, I. (2007). A multi-form modelling approach to the dynamics and control of drum granulation processes. *Powder Technology*, 179(1-2), 2–11.
- Wang, Y., Zhou, D., and Gao, F. (2008). Iterative learning model predictive control for multi-phase batch processes. *Journal of Process Control*, 18(6), 543–557.
- Wisniewski, P. and Doyle, F. (1998). Control structure selection and model predictive control of the Weyerhaeuser digester problem. *Journal of Process Control*, 8(5-6), 487–496.
- Wu, W., Xu, J., and Hwang, J. (2009). Multi-loop nonlinear predictive control scheme for a simplistic hybrid energy system. *International Journal of Hydrogen Energy*, 34(9), 3953–3964.