Control of disturbed LPV systems in a LMI setting

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Abstract: This paper is concerned with the control of a discrete-time LPV system affected by bounded disturbances. First, an input-to-state stable observer-based controller is synthesized. In the sequel, we compute the invariant ellipsoid that assures constraints satisfaction and has the maximal projection on the initial state subspace. In order to increase the system robustness (quantified in terms of ellipsoidal set volume), a Youla parameter is designed. This parameter enlarges the invariant set projection but slows the closed-loop performances. A compromise between robustness and performance is obtained by imposing a new bound on the Lyapunov function decreasing rate.

Keywords: LMI, S-procedure, ISS, Observer-based systems, Youla-Kučera parameter, Invariant sets, LPV systems.

1. INTRODUCTION

Linear parameter-varying (LPV) systems have gained a lot of attention in the control community due to their implication in treating nonlinear controller synthesis problems. The design of gain-scheduled controllers for non-linear systems is proposed in (Apkarian et al., 1994), (Becker et al., 1993), (Scorletti and Ghaoui, 1995). An observer-based controller is synthesized in (Heemels et al., 2009), the parameters are not exactly known and the parameter uncertainty is maximized while still guaranteeing closed-loop stability. (Xie and Etsaka, 2004) designs a stable observer-based controller for an LPV system that does not suffer from disturbances.

Invariant sets framework has significantly developed in control engineering over the last decades. The existence of an invariant set (ellipsoidal or polyhedral set) is equivalent with the existence of a Lyapunov function and hence with a stability test. In our case, the bounded disturbances presence and the LPV mode imposes the use of the “input-to-state stability” (ISS) notion in the case of norm bounded disturbances. ISS for this class of disturbances implies that the origin is an asymptotically stable point for the undisturbed LTI system and also that all state trajectories are bounded for all bounded disturbance sequences. More exactly, the state converges to the 0-reachable set (Blanchini and Miani, 2008). Furthermore, if the disturbance fades then the system asymptotically converges to the origin (Limou et al., 2008). In (Liberzon, 1999) the problem of achieving disturbance attenuation in the ISS and integral-ISS sense for nonlinear systems with bounded controls is considered. ISS for discrete-time nonlinear systems has been studied in (Jiang and Wang, 2001).

The control law has the form:

\[ u(k) = -F(\theta(k)) \hat{x}(k), \quad (1) \]

where \( F(\theta(k)) \in \mathbb{R}^{m \times r} \) is the feedback gain.

Consider the following discrete-time LPV system:

\[ x(k+1) = A(\theta(k))x(k) + Bu(k) + B_\omega \omega(k), \]
\[ y(k) = Cx(k) + Bu(k), \]

with an observer used to estimate the state:

\[ \hat{x}(k+1) = A(\theta(k))\hat{x}(k) + Bu(k) + L(\theta(k))(y(k) - \hat{y}(k)), \]
\[ \hat{y}(k) = C\hat{x}(k), \]

where \( x(k) \in \mathbb{R}^{n_x} \) is the state, \( u(k) \in \mathbb{R}^{m} \) the input, \( y(k) \in \mathbb{R}^{p} \) the output, \( \omega(k) \in \mathbb{R}^{r} \) the state noise, \( v(k) \in \mathbb{R}^{q} \) the output noise, \( \theta(k) \in \Theta \) the time varying parameter, \( \hat{x} \in \mathbb{R}^{n_x} \) the estimated state, \( L \in \mathbb{R}^{n_x \times p} \) the observer gain.

To the best of the authors’ knowledge the results presented in this paper (the synthesis of an observer-based controller and a Youla parameter for a LPV discrete-time system affected by disturbances) are novel. The observer-based controller assuring ISS for norm bounded disturbances is designed in Section 2. The inequalities giving this controller are obtained by considering an extension of the disturbances actuating in the system and by applying the S-procedure. In the sequel we search the invariant ellipsoidal projection satisfying input constraints and having the maximal projection on the initial state subspace. In Section 3, a Youla parameter that enlarges the maximal ellipsoidal projection (assuring therefore a better robustness) is synthesized. The Youla-based system performances are improved by imposing a new bound on the Lyapunov function decreasing rate. These results are expressed in a LMI form (Boyd et al., 1994) and are validated in Section 4 with a numerical example. In Section 5 some concluding remarks are drawn.

2. OBSERVER-BASED DESIGN FOR A DISTURBED LPV SYSTEM

2.1 System description

Consider the following discrete-time LPV system:

\[ x(k+1) = A(\theta(k))x(k) + Bu(k) + B_\omega \omega(k), \]
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with an observer used to estimate the state:

\[ \hat{x}(k+1) = A(\theta(k))\hat{x}(k) + Bu(k) + L(\theta(k))(y(k) - \hat{y}(k)), \]
\[ \hat{y}(k) = C\hat{x}(k), \]

where \( x(k) \in \mathbb{R}^{n_x} \) is the state, \( u(k) \in \mathbb{R}^{m} \) the input, \( y(k) \in \mathbb{R}^{p} \) the output, \( \omega(k) \in \mathbb{R}^{r} \) the state noise, \( v(k) \in \mathbb{R}^{q} \) the output noise, \( \theta(k) \in \Theta \) the time varying parameter, \( \hat{x} \in \mathbb{R}^{n_x} \) the estimated state, \( L \in \mathbb{R}^{n_x \times p} \) the observer gain.

The control law has the form:

\[ u(k) = -F(\theta(k)) \hat{x}(k), \quad (1) \]

where \( F(\theta(k)) \in \mathbb{R}^{m \times r} \) is the feedback gain.
The varying parameter $\theta$ is fully known and available. This parameter lies in a polytope $\Theta$ of vertices $\theta_i, i=1,N: \theta(k) = \sum_{i=1}^{N} \mu_i(k) \theta_i; \mu_i(k) \geq 0, \sum_{i=1}^{N} \mu_i(k) = 1$. (2)

For the system with observer the following augmented state representation will be used:

$$x_o(k+1) = A_o(\theta(k))x_o(k) + B_o(\theta(k))u(k),$$ (3)

where: $x_o(k) = \begin{bmatrix} x(k) \end{bmatrix}, (k) = x(k) - \hat{x}(k)$ the estimation error, $A_o(\theta(k)) = \begin{bmatrix} A \beta I & 0 & \alpha P \end{bmatrix}$, $B_o(\theta(k)) = \begin{bmatrix} B \alpha P \end{bmatrix}$ and $n(k) = \begin{bmatrix} \omega(k) \end{bmatrix}$ the disturbance vector.

The command can be written in the following form:

$$u(k) = -F_o(\theta(k)) \cdot x_o(k)$$ (4)

with $F_o(\theta(k)) = [F(\theta(k)) - F(\theta(k))]$.

It is clear that the parameter-varying matrices range in a polytope of matrices whose vertices are the images of the vertices $\theta_1, \ldots, \theta_N$:

$$[A(\theta(k)), F(\theta(k)), L(\theta(k)), A_o(\theta(k)), B_o(\theta(k)), F_o(\theta(k))] = \bigg\{ \sum_{i=1}^{N} \mu_i(k) [A_i, F_i, L_i, A_o, B_o, F_o] = \sum_{i=1}^{N} \mu_i(k) [A(\theta_i), F(\theta_i), L(\theta_i), A_o(\theta_i), B_o(\theta_i), F_o(\theta_i)], \mu_i(k) \geq 0, \sum_{i=1}^{N} \mu_i(k) = 1 \bigg\}$.

(5) The disturbance vector is assumed to be bounded:

$$n^T n \leq 1.$$

2.2 Design of a stabilizing observer-based LPV control

In this section we state the theorem giving an ISS observer-based LPV controller for the LPV system affected by bounded disturbances (5).

**Lemma 1.** (ISS-Lyapunov function) Goulart et al. (2005) The system (3) is ISS for the norm bounded disturbances (5) if there exist $K_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot), \delta(\cdot)$ and a continuous function $V: \mathbb{R}^n \to \mathbb{R}$, $V(0) = 0$ such that:

$$\alpha_1(|| x(k) ||) \leq V(k) \leq \alpha_2(|| x(k) ||),$$ (6)

$$V(k+1) - V(k) \leq -\alpha_3(|| x(k) ||) + \delta(|| n(k) ||).$$ (7)

**Theorem 1.** Let the discrete-time LPV system (3) affected by bounded disturbances. If there exist $G_i = G_i^T \succ 0, P_i = P_i^T \succ 0, Q_i, Q_{ro}, Y_i, Z_i, i=1,N, \alpha > 0, \beta > 0$ and $\gamma > 0$ such that the following inequalities are satisfied:

$$\begin{bmatrix} G_0 + Q_{ro} - G_i & 0 & \alpha P_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha Q_0 & 0 & \alpha G_i & 0 & 0 & 0 \\ A_i Q_{ro} - B_i & [B_{ro}^T] & 0 & 0 & 0 & 0 \\ P_i & 0 & 0 & 0 & 0 & 0 \\ Q_P A_i - Z_i & [Q_P B_w - Z_i B_w]^T & 0 & 0 & 0 & 0 \end{bmatrix} \succ 0, \quad (8)$$

Further we prove that if:

$$\Delta V(k) < 0, \quad (13)$$

for any $x_o$ and $n$ satisfying:

$$x_o(k) T P x_o(k) \geq 1 \quad \text{and} \quad n(k) T n(k) \leq 1 \quad (14)$$

the condition (7) is obtained for the case of norm bounded disturbances (5).

For the sequel of the demonstration the following extensions will be made:

$$B \to B = \begin{bmatrix} B_w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad n \to n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad n_1 = \begin{bmatrix} n_1 \omega_1 \\ v_1 \end{bmatrix}, \quad n_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}, \quad (15)$$

With these extensions we assume the system and the observer can be affected by different disturbances. This augmentation does not change our problem and it is a necessary tool for manipulating the inequalities in order to obtain (8) and (9).

From the S-procedure (Boyd 2009, Boyd et al. (1994) pp.82-84) we have that (14) implies (13) if there exist $\alpha > 0, \beta \geq 0$ and $\gamma \geq 0$ such that:

$$\Delta V(k) - \alpha (1 - x_o(k) T P x_o(k)) - \beta (n_1^T n_1 - 1) - \gamma (n_2^T n_2 - 1) < 0.$$ (16)

For $\alpha \geq \beta + \gamma$ (Boyd et al. (1994) pp.82-84), (16) can be written as in (7) with $\alpha_3 = \alpha \delta || x(k) ||$ (a sufficiently small positive scalar) and $\delta = \max(\beta, \gamma)$. By applying Schur complement (16) becomes:

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} P \end{bmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha P \end{bmatrix} \end{bmatrix} \end{bmatrix} \succ 0, \quad (17)$$

and

$$\alpha - \beta - \gamma \geq 0. \quad (18)$$

4150
Following the steps in (Boyd et al. (1994) pp. 117-118) we prove first that the inequality (17) has a solution if and only if the inequalities (8), (9) and (10) do. Suppose that the inequalities (17) and (18) have a solution $\mathcal{P}$, $\alpha > 0$ and $\beta \geq 0$, where:

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21}^{T} & \mathcal{P}_{22} \end{bmatrix} > 0.$$  \hfill (19)

For $\mathcal{P}$ in the form above, we pre- and post-multiply (17) with, respectively, $\Gamma$ and $\Gamma^{T}$,

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$  \hfill (20)

Now, based on Schur complement it can be easily seen that the inequality (17) implies:

$$\begin{bmatrix} \mathcal{P}_{11} & 0 & \alpha \mathcal{P}_{11} \mathcal{A} \mathcal{F}^{T} \mathcal{P}_{11}^{+} \\ 0 & \beta I & 0 \\ \alpha \mathcal{P}_{11} & 0 & \mathcal{P}_{11}^{+} \end{bmatrix} > 0,$$  \hfill (21)

$$\alpha - \beta \geq 0.$$  \hfill (22)

We pre- and post-multiply (21) with diag($Q(k), I, P_{11}^{-1}, (P_{11}^{+})^{-1}$) and diag($Q(k), I, (P_{11}^{-1})^{+}, (P_{11}^{+})^{-1}$). Since $(Q_{G} - P_{11}^{-1})P_{11} = (Q_{G} - P_{11}^{-1})^{+} \geq 0$ the following relaxation can be made: $Q_{G}^{+} P_{11} Q_{G} \geq Q_{G}^{+} + Q_{G} - P_{11}^{-1}$. By making the notations $G(\theta(k)) = P_{11}^{+}$, $G(\theta(k) + 1) = (P_{11}^{+})^{-1}$ and $Y(\theta(k)) = F(\theta(k))Q(k)$ and because the new inequality is affine in all the variables, condition (8) results.

The inequality (17) also implies:

$$\begin{bmatrix} \mathcal{P}_{22} & 0 & \alpha \mathcal{P}_{22} \mathcal{A} \mathcal{F}^{T} \mathcal{P}_{22}^{+} \\ 0 & \gamma I & 0 \\ \alpha \mathcal{P}_{22} & 0 & \mathcal{P}_{22}^{+} \end{bmatrix} > 0,$$  \hfill (23)

$$\alpha - \gamma \geq 0.$$  \hfill (23)

We pre- and post-multiply (23) with diag($I, I, I, Q_{F}(P_{22}^{+})^{-1}$). Since $(Q_{P} - P_{22}^{+})(P_{22}^{+})^{-1} (Q_{P} - P_{22}^{+}) \geq 0$ the following relaxation can be made: $Q_{P}(P_{22}^{+})^{-1} Q_{P} \geq Q_{P}^{+} - P_{22}$. By making the notations $G(\theta(k)) = P_{22}^{+}$, $G(\theta(k) + 1) = P_{22}$ and $Z(\theta(k)) = Q_{P}(L(\theta(k))$ and because the new inequality is affine in all the variables, inequality (9) results.

"⇐" Suppose that $P(\theta(k)), G(\theta(k)), Q_{F}, Q_{G}, Y(\theta(k)), Z(\theta(k)), \alpha, \beta$ and $\gamma$ satisfy the inequalities (8), (9) (written in $\theta(k)$) and (10) and define $F(\theta(k)) = Y(\theta(k))Q_{G}^{+}$ and $L(\theta(k)) = Q_{P}^{+} Z(\theta(k))$. The objective is to prove that there exists a $\lambda > 0$ such that $\mathcal{P} = \begin{bmatrix} \lambda G(\theta(k))^{-1} & 0 \\ 0 & P(\theta(k)) \end{bmatrix}$ satisfies (17) and (18), assuring therefore ISS for (5).

Because $\beta$ is a variable, we can suppose $\beta \rightarrow \lambda \beta$. By congruence of (17) with $\Gamma$, and after with diag($Q_{G}^{+}, I, G(\theta(k)), G(\theta(k) + 1), I, I, I, Q_{F} G(\theta(k) + 1))$ we obtain via Schur complement that the observer-based system is ISS for the bounded disturbances (5) if $\lambda > 0$ satisfies:

$$(\theta(k)) - \lambda \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S(\theta(k)) \end{bmatrix} \begin{bmatrix} - \lambda \beta - \gamma \geq 0. \end{bmatrix} \hfill (24)$$

These inequalities are satisfied for any positive $\lambda$ such that: $\nu > \lambda \eta$ and $\alpha - \nu \geq \lambda \beta$ where: $\nu = \varsigma \cdot \epsilon$ (a sufficiently small positive scalar)

$$\eta = \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{\max}(S_{i}(9)^{-1} S_{j}^{T}).$$

Since $(8)(\theta(k)), (9)(\theta(k))$ and (10) are feasible, such $\lambda$ will always exist.

S-procedure introduces new variables. Because one of this variable $\alpha$ multiplies $P, G_{1}$ and $Q_{G}$, the inequalities (8) and (9) are now BMIs (bilinear matrix inequalities). Since $\alpha$ is a scalar, an $\alpha_{opti}$ can be found by executing a simple loop. In order to avoid the loop the PENBMI solver (Koevara and Stingl, 2006) can be used.

The results can be extended to parameter dependent input and output matrices, $B(\theta(k))$ and $C(\theta(k))$, by using the relaxations stated in (Sala and Ariño, 2007). 2.3 Invariant ellipsoid calculus

An invariant set is a subset of the state space with the property that, if it contains the system state at some time, then it will contain it also in the future (Bianchini and Miani, 2008). For the observer-based controller (11), we search the maximal invariant set (the biggest state region) satisfying input constraints despite the disturbance presence and the LPV mode. Input constraints, disturbances presence and the LPV mode influence the invariant set volume (larger the possible disturbance signal is, the smaller the maximal ellipsoid gets).

Because of their association with powerful tools such as the Lyapunov equation or LMIs, we approximate the real invariant set by an ellipsoidal set:

$$E = \{ x_{0} | x_{0}^{T} Q^{-1} x_{0} \leq 1 \}.$$  \hfill (25)

where $Q \in R^{(2n_{e})x(2n_{e})}$ is a symmetric positive-definite matrix (not necessarily block diagonal). Since we are interested more in the initial system state, $x(k)$, we maximize the invariant set projection onto $x$-subspace instead of maximizing the entire set volume. The projection of $E$ onto $x$ is given by:

$$E_{x} = \{ x | x^{T} (TQ^{T}T)^{-1} x \leq 1 \} = T x_{x}.$$  \hfill (26)

The projection of an ellipsoid on a subspace is an ellipsoid (Bianchini and Miani, 2008).

The constraints on the input are:

$$\| u \| \leq u_{\max}.$$  \hfill (27)

Theorem 2. Consider an ISS observer-based LPV system (3) face of bounded disturbances (5). The maximization of $E_{x}$ subject to input constraint (27) is achieved by solving:

$$\min_{G_{\alpha, Q, Q_{e}, \alpha, \beta}} - \log \det (TQ^{T}T)$$  \hfill (28)
subject to:
$$ \begin{pmatrix} Q_o + Q_o^T - G_{oi} & 0 & \alpha Q_o^T \ A_{oi}^T \\ \beta I & 0 & B_{oi}^T \\ \alpha Q_o & 0 & G_{oj} \end{pmatrix} \succ 0, \quad (29) $$

$$ \begin{pmatrix} Q_o + Q_o^T - G_{oi} & Q_o^{T^2} F_{oi} \\ F_{oi}^T Q_o & u_{max} I \end{pmatrix} \succ 0, \quad i = 1, \ldots, N. \quad (31) $$

**Proof.** The conditions (17) and (18) guarantee the state invariance for bounded disturbances. By congruence of (17) with $$\text{diag}(Q_o^T, I, P^{-1}, (P^{-1})^{-1})$$, considering $$\mathcal{G} = P^{-1}$$, $$\mathcal{G}^+ = (P^+)^{-1}$$ and $$Q_o G^{-1} Q_o^T \geq Q_o + Q_o^T - \mathcal{G}$$ an affine inequality is obtained leading to (29). In order to reduce the calculation load one can consider non extended noise signal and $$\alpha = \beta > 0$$ (Boyd et al. (1994) pp.82-84).

For $$\mathcal{F} = F_{oi}(\theta(k))$$ we have:
$$ ||u||^2 = || \mathcal{F} x_o ||^2 \leq || \mathcal{F}^{P^{-1/2}} ||^2 || P^{1/2} x_o ||^2 = \lambda_{max}(\mathcal{F}^{P^{-1/2}}) (x_o^T P x_o) \leq \lambda_{max}(\mathcal{F}^{P^{-1/2}}), $$
where $$\lambda_{max}$$ is the maximal eigenvalue. Now, by applying Schur complement and by congruence with diag($$Q_o^T, I$$) we obtain that $$||u||^2 \leq u_{max}$$ if LMI (31) is feasible.

The ellipsoid volume is inversely proportional with the eigenvalues product (that is the determinant). For rendering the problem concave, the “logarithm” operator is used. Because the most MatLab tools can only determine the minimum of a convex problem our optimization criterion becomes $$\min \quad - \log \det(\mathcal{T} Q^T T).$$

### 3. A YOULA PARAMETER DESIGN FOR A DISTURBED LPV SYSTEM

#### 3.1 System description

In order to improve the robustness of the observer-based system a Youla parameter is introduced in the closed-loop:
$$ x_y(k+1) = A_y(\theta(k)) x_y(k) + B_y(\theta(k)) \tilde{y}(k), $$
$$ \tilde{u}(k) = C_y(\theta(k)) x_y(k) + D_y(\theta(k)) \tilde{y}(k), $$
$$ \tilde{y}(k) = y(k) - \tilde{y}(k), $$
where $$x_y \in \mathbb{R}^n$$ is the state of Youla parameter, $$\tilde{u} \in \mathbb{R}^m$$ is its output of $$Q$$ and $$\tilde{y} \in \mathbb{R}^p$$ the input. Fig. 1 gives a block-diagram overview of the structure.

![State-space controller with Youla parametrization](image)

Fig. 1. State-space controller with Youla parametrization.

Now the control law has the form:
$$ u(k) = -F(\theta(k)) \tilde{x}(k) - \tilde{u}(k). $$

For the Youla-based LPV system the following augmented state representation will be used:
$$ x_y(k+1) = A_y(\theta(k)) x_y(k) + B_y(\theta(k)) n(k) $$

where:
$$ A_y(\theta(k)) = \begin{bmatrix} A_o(\theta(k)) - B_o D_y(\theta(k)) C_c & -B_c C_q(\theta(k)) \\ B_o(\theta(k)) C_c & \end{bmatrix}, $$
$$ B_y(\theta(k)) = \begin{bmatrix} B_{wx} - L_y(\theta(k)) B_o - D_y(\theta(k)) B_o \end{bmatrix}, $$
$$ C_c = [0 \quad C_{wx}]. $$

The command is written in the following form:
$$ u(k) = -F_y(\theta(k)) x_y(k) - F_{wx}(\theta(k)) n(k), \quad (35) $$

with:
$$ F_y(\theta(k)) = [F_o(\theta(k)) + D_o(\theta(k)) C_c \quad C_y(\theta(k))], $$
$$ F_{wx}(\theta(k)) = [0 \quad D_y(\theta(k)) B_o]. $$

The augmented system has a polytopic description:
$$ \{ A_y(\theta(k)), B_y(\theta(k)), F_y(\theta(k)) \} = \{ \sum_{i=1}^{N} \mu_i(\theta) [A_{yi}, B_{yi}, F_{xi}, F_{ni}], \mu_i(\theta) \geq 0, \sum_{i=1}^{N} \mu_i(\theta) = 1 \}. $$

The disturbance has the same bounds as in (5).

#### 3.2 Youla synthesis for robustness requirements

For the Youla LPV system, consider the invariant ellipsoid:
$$ E_y = \{ x_y \in \mathbb{R}^n | x_y^T Q_y x_y \leq 1 \}. $$

The goal here is to synthesize a Youla parameter that maximizes the projection of $$E_y$$ on the initial state subspace:
$$ E_{xy} = \{ x | x^T (Z Q_y Z^T)^{-1} x \leq 1, Z = x_x \}. $$

The gain in robustness will be given by the difference of volume between the observer-based system ellipsoidal projection and the Youla-based system ellipsoidal projection. The same input constraints as in (27) are considered.

**Theorem 3.** Consider the discrete-time LPV system (34) subject to input constraints (27) and affected by bounded disturbances (5). The design of the Youla parameter that maximizes the projection $$E_{xy}$$ is achieved by solving:
$$ \min \quad - \log \det(\mathcal{T} Q^T T) $$

subject to:
$$ LMI_{i,j} \succ 0, \quad (37) $$
$$ G_{i,11} \succ Q, $$
$$ \begin{bmatrix} X + X^T & & \ast & \ast \\ - & & & \\ \ast & & - & \ast \\ 0 & \ast & & 0 \end{bmatrix} \succ 0, $$
$$ i, j = 1, N, \quad (39) $$

and
$$ \alpha - \beta \geq 0, \quad \alpha > 0, \quad \beta \geq 0, $$

where:
$$ \begin{bmatrix} R & Y^T X & & \\ X & -Y^T U & & \\ M_i & D_q C_c X + C_q U & & \\ N_i & -Y^T B_c D_q + V^T B_{q1} & & \end{bmatrix}, $$
$$ \begin{bmatrix} H_i & -Y^T A_o X & -Y^T B_c D_q C_c X + V^T B_{q1} C_c X - \end{bmatrix}, $$

and $$\ast$$ is referring to the symmetrical value. Assuming that the maximization problem has a solution one gets:
The Youla parameter that maximizes the projection $E_{xy}$ guaranteeing (44) is achieved by solving:

$$\min_{x,y,M_i,N_i,\theta_i,d_{qi},\alpha,\beta} \log \det(TQT^T)$$

subject to:

$$\begin{bmatrix} LMI_{i,j} \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \begin{bmatrix} \star \star \star \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \begin{bmatrix} G_{j,11} \end{bmatrix}$$

(38),(39) and (40), with the notations made in (41) and $C_j = [C \ 0]$. Assuming that the maximization problem has a solution, the Youla parameter is obtained as in (42).

Proof. Making the notation $C_z = [C_f \ 0]$ and $B_z = [0 \ \beta]$, one has: $\|C_f(k)\|^2 + \|B_z v(t)\|^2 = [x_y \ \theta y]^T [C_f \ 0] [C_f^T \ 0] [x_y \ \theta y] = 0$. For $A_y = A_y(\theta(k)), B_y = B_y(\theta(k))$ we get that (44) is satisfied if:

After applying Schur theorem, we pre- and post-multiply this inequality with $\text{diag}(\Pi^T Q^T y, I, I)$ and then $(\Pi Q y, I, I)$, LMI (39) is derived.

The optimization criterion is obtained by analogy with (28) since $x = Z x_y = T x_o$. ■

3.3 Compromise between robustness and performance

The synthesis of a Youla parameter in terms of finding the maximal invariant ellipsoid projection may lead to slow closed-loop performances. An effective way to improve the response transients is to consider Lyapunov functions $V(k) = x^T P x_k$ with a decrease rate bigger than the output norm scaled by $\frac{1}{\gamma}$:

$$\frac{V(k) - V(k+1)}{\gamma} \geq \|C x(k)\|^2 + \|B_v v(k)\|^2 \geq \frac{1}{\gamma} \|y(k)\|^2.$$ (44)

**Theorem 4.** Consider the discrete-time LPV system (34) with input constraints (27) and affected by bounded disturbances (5). For a given scalar $\gamma \geq 0$, the design of

$$\begin{bmatrix} \frac{X + X^T - G_i,11}{I + R - G_i,21} \ Y + Y^T - G_i,22 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{A_{oi} X - B_{oi} M_i}{H_i} \ \frac{A_{oi} - B_{oi} d_{qi} C_v}{N_i} \end{bmatrix}$$

This projection is represented by the dark ellipsoid in Fig. 2 and was found for an optimal $\alpha = \beta = 0.0042$.
By introducing the Youla parameter a new maximal projection is computed. This projection corresponds to the blue ellipsoid in Fig. 2 and was obtained for an optimal $\alpha = \beta = 0.0084$. For lack of space we skip the values obtained for the Youla parameter.

![Fig. 2. Maximal projection.](image)

Considering the performance constraints (44) with $\gamma = 1$, the maximal projection is now given by the magenta ellipsoid in Fig. 2. This projection and the associated Youla parameter were obtained for an optimal $\alpha = \beta = 0.0213$.

In Fig. 3 we have plotted the time-domain representation of the output for a step disturbance (with the amplitude 0.1) in the output signal at time 0.05s and a switch between the two vertices at time 0.1s. By adding the large closed-loop transients may result, a compromise between performance and robustness is reached by imposing a new bound on the Lyapunov function decreasing rate.

![Fig. 3. Output time-domain simulation.](image)

Youla parameter (and so, some degrees of freedom) we obtain an improvement in terms of volume (Fig. 2). The gain is more significant when performance constraints are not considered. On the other hand, the Youla parameter leads to slow closed-loop performances (Fig. 3). These performances can be improved by imposing new bounds on the Lyapunov function decreasing rate.

These results were obtained using the software Yalmip (Lofberg, 2004) with the Lmilab solver in Matlab environment (the operator used for “log det” is “geomean”).

5. CONCLUSION

Considering a discrete-time LPV system affected by bounded disturbances, this paper proposes the design of an ISS observer-based controller. For this controller we search the maximal projection of the invariant ellipsoid that satisfies input constraints despite the disturbance presence. In the sequel we synthesize a Youla parameter that enlarges the invariant ellipsoidal projection. Because

REFERENCES


