Continuous-Time Identification of Linear Parameter Varying Model using an Output-Error Technique

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Abstract: In this paper, a practical solution for the identification of Continuous-Time (CT) Input-Output (IO) Linear Parameter Varying (LPV) systems is proposed. For this particular class, we formulate an output error identification problem and present a parameter estimation scheme in which a prediction error based cost function is minimized using nonlinear programming. Because the cost function possesses local minima, the success of any iterative parameter estimation algorithm depends on appropriate initial seeds. One approach would be to generate an initial estimate using CT Reinitialized Partial Moments (RPM) models extended to CT IO LPV systems. We assume that the inputs, outputs and scheduling parameters are directly measurable, and that the functional dependence of the system coefficients on the parameters is of a polynomial form. The performance of the proposed method is evaluated by Monte Carlo simulation analysis.

Keywords: Continuous-Time model; LPV models; Reinitialized Partial Moments; Output-Error identification.

1. INTRODUCTION

It is a fact that almost all real systems present a nonlinear behavior. However, the use of nonlinear models is often bulky and unsuitable for classical methods in the automatic control field. Typically, linear models are designed for a given operating area. In order to propose a model describing the system behavior in a global manner without the use of nonlinear models, it can be interesting to use an LPV approach which is a low cost method for the modeling of nonlinear processes.

The LPV systems are a particular class of nonlinear systems which have attracted considerable attention in recent years. This is because the LPV models can be used to approximate nonlinear systems with much lower system order but higher precision than linear model approximations. Moreover, the LPV models allow the extension of linear design techniques to nonlinear systems.

Due to the importance of the LPV representation, numerous algorithms have recently been developed to identify LPV models. Two main LPV model identification approaches exist:

- the local approach, which is based on the interpolation of local LTI models for fixed operating points of the system, that is, for constant values of the scheduling parameters. Several examples of the local LPV identification approach are found in the literature (Steinbuch et al. (2003); Lovera and Mercère (2007); De Caigny et al. (2009); Chouaba et al. (2010)).
- the global approach, which is based on the assumption that it is possible to perform one single experiment by exciting all the non-linearities of the system. A parameter-dependent model is then directly obtained (Lee and Poolla (1999); Bamieh and Giarré (2002); Verdult and Verhaegen (2005); Xiukun Wei and Re (2006); Giarré et al. (2006); Van Wingerden et al. (2007); Felici et al. (2007); Tóth (2008); Laurain et al. (2010)).

The present article follows the second approach. Many methods have been proposed to solve the identification problem of LPV systems for discrete-time models, and they are mainly characterized by the LPV model structure used: Input-Output representations, State space or Orthogonal Basis Functions (OBFs) based model, see (Tóth (2008)) for an overview. Nonetheless, in many areas of science and engineering, the identified dynamic models should be physically meaningful. As a result, there is a need for modeling approaches that are able to yield directly from the sampled data efficiently parameterized CT models that have clear physical interpretations. Although dynamical systems in the physical world are native of the CT domain, the attention in the system identification community was almost completely focused on the discrete-time model identification techniques until recently. The last few years have indeed witnessed considerable development in CT approaches to system identification from sampled data (Rao and Unbehauen (2006); Garnier et al. (2007); Tóth (2008)) in the LTI system context.

CT model identification in an LPV framework is a relatively unexplored area. That is why this article deals with LPV identification techniques for CT systems. Due
to the linear regressors based estimation, the usual model structure in existing methods is assumed to be equation error (EE). Despite their simplicity, this assumption is unrealistic in most practical applications since they suffer from defects in the assumption of linearity and asymptotic bias. They are then used to initialize the Output-Error technique (OE) for parameter estimation. They are based on the simulation model output and its sensitivity functions, with criterion minimization by a quadratic nonlinear programming. These techniques are cumbersome to implement but allow much more general applications with an asymptotically unbiased parameter estimation.

It can be noted that the output-error identification problem for LPV state-space models is examined in (Lee and Poolla (1999)) for discrete-time models.

In this paper, CT LPV IO structure is used to identify LPV system models and for the estimation of the parameters. The method using nonlinear programming, proposed in (Richalet et al. (1971); Walter and Pronzato (1997)) for LTI systems, will be extended to be used with LPV systems. This can be done by using a Kronecker product framework. Since OE methods suffer from the essential problem of initialization, a possible solution is to use a pre-estimation step to provide the initialization near the global optimum. This approach makes use of an EE algorithm to initialize the OE algorithm. Particularly, the properties of the reinitialized partial moments (Trigeassou (1987); Tohme (2008)) extended to CT IO LPV systems are shown.

This paper is organized in the following way. The identification problem is formulated in Section 2. In Section 3, the LPV identification method is outlined and some analysis are given. In Section 4, a Monte Carlo simulation study is used to demonstrate the robustness of the method. Finally, Section 5 gives some concluding remarks and indicates future works.

2. PROBLEM STATEMENT

In this section, we present the model structure considered in this paper. Furthermore, assumptions are listed and notations are introduced.

The output of a SISO CT Linear Parameter Varying system \( H(s, \rho) \) is given by (see Fig. 1)

\[
y(t) = H(s, \rho)u(t) + v(t) \tag{1}
\]

where

\[
H(s, \rho) = \sum_{j=0}^{n_a} b_j(\rho) s^j + \sum_{i=0}^{n_b} a_i(\rho) s^i + \rho \cdot \cdot \cdot
\tag{2}
\]

\( \rho \) represents the measurable scheduling variable and \( s \) is the differential operator. The structure \( H(s, \rho) \) of the CT LPV IO system is assumed to be asymptotically stable for all \( \rho \) in the operating area. Assume that the input and output data \{\( u(t), y(t) \)\} are generated by the LPV system. In the proposed model structure, the noise \( v(t) \) is a white noise uncorrelated with the input \( u(t) \). We assume that the disturbance \( v(t) \) is a stationary stochastic process with zero mean and bounded variance. The coefficients \( b_j(\rho), a_i(\rho) \) are defined as follows

\[
b_j(\rho) = b_j^0 + f_1(\rho)b_j^1 + \ldots + f_{r-1}(\rho)b_j^{r-1}
\]

\[
a_i(\rho) = a_i^0 + f_1(\rho)a_i^1 + \ldots + f_{r-1}(\rho)a_i^{r-1}
\tag{3}
\]

3. LPV MODEL IDENTIFICATION

The model \( H(\hat{\theta}, \rho) \) is in CT representation, thus it is preferable to use an Output-Error technique to estimate its parameters. The parameters are adapted with a nonlinear optimization algorithm which minimizes a quadratic criterion according to the iterative Levenberg-Marquardt algorithm (Marquardt (1963)). This algorithm achieves a compromise between the stability of the Gradient method and the fast convergence rate of the Gauss-Newton method. The advantage of such an algorithm is that no particular hypothesis is needed on the output noise specifications.

3.1 OE algorithm

The OE method deals with minimizing an objective function, usually a quadratic criterion, that is based on the output error defined by

\[
\varepsilon_{OE} = y - \hat{y}(\hat{\theta}) = y - H(\hat{\theta}, \rho)u \tag{5}
\]

This is the error between the measured system output \( y \) and the model output \( \hat{y}(\hat{\theta}) \).

Let us define

\[
\hat{\theta}_{ii} = [\hat{a}_0^0, \ldots, \hat{a}_{r-1}^0, \ldots, \hat{a}_0^{r-1}, \ldots, \hat{a}_{r-1}^{r-1}, \hat{\theta}_0, \ldots, \hat{\theta}_0^{r-1}, \ldots, \hat{\theta}_{n_a-1}, \hat{\theta}_{n_a-1}^{r-1}]^T
\tag{6}
\]

\[
\Phi_M(\hat{\theta}_{ii}(\rho)) = \phi^T(I_n \otimes F) \tag{7}
\]

Fig. 1. CT LPV IO system with noisy output data.

where \( \{b_j, a_i, l = 0, \ldots, r - 1\} \) are constant values,

\( \{f_l(\rho), l = 0, \ldots, r - 1\} \) are functions of the on-line measurable variable \( \rho \). The values of these functions can be calculated directly from \( \rho \), e.g. \( \cos(\rho) \), or \( \rho \) itself. We assume that the functions \( \{b_j, a_i\} \) in (3) are linear combinations of a set of known fixed basis functions \( \{f_1, \ldots, f_{r-1}\} \).

Let us define the cost function

\[
J(\theta) = \frac{1}{N} \sum_{k=1}^{N} \left\| y_k - \hat{y}_k(\hat{\theta}) \right\|^2 \tag{4}
\]

where \( \hat{y}(\hat{\theta}) = H(\hat{\theta}, \rho)u(t) \) is the predicted output, \( H(\hat{\theta}, \rho) \) has the same structure of the system (2), \( \hat{\theta} \) is the parameter vector to be identified and \( N \) is the number of samples.

The goal is to estimate the constant coefficients \( \{b_j^l\}, \{a_i^l\} \) of polynomials from sampled measurements \( \{y(kT_s), u(kT_s), p(kT_s)\}_L \) generated by the system \( H(s, \rho) \), where \( T_s \) is the sampling period. For this, we formulate an OE identification problem and present a parameter estimation scheme in which a prediction error based cost function is minimized using nonlinear programming.
with 
\[ \phi = \left[ -\hat{y}(\hat{\theta}_{ii}), \ldots, -s^{n_{a-1}} \hat{y}(\hat{\theta}_{ii}), u, \ldots, s^{n_{b}} u \right]^T \]
\[ F = (1 f_1(\rho) \ldots f_{r-1}(\rho)) \]
where \( I_n \) is the identity matrix of size \( n = n_a + n_b + 1 \), 
\( \hat{\theta}_{ii} \in \mathbb{R}^{n(r-1) \times 1} \) is a vector containing all coefficients to be identified, \( \phi^T \in \mathbb{R}^{1 \times n} \), \( F \in \mathbb{R}^{1 \times r-1} \) and \( \Phi_M(\hat{\theta}_{ii}(\rho)) \in \mathbb{R}^{1 \times n(r-1)} \). Here \( \otimes \) denotes a Kronecker product and \( r \) is the polynomials order.

The model based on the estimation of parameters at the \( i \)th iteration is given by
\[ \hat{y}(\hat{\theta}_{ii}(\rho)) = \Phi_M(\hat{\theta}_{ii}(\rho)) \hat{\theta}^T_{ii} \]  
(8)

The Levenberg-Marquardt algorithm is extended to identify IO LPV systems in CT domain and it is defined by the following iterative equation
\[ \hat{\theta}_{ii+1} = \hat{\theta}_{ii} - \left[ J'_{\theta\theta} + \lambda I \right]^{-1} J'_{\theta y} \]
(9)
where \( J'_{\theta y} \) and \( J'_{\theta\theta} \) are the gradient and the approximated Hessian defined respectively by
\[ J'_{\theta y} = \frac{2}{N} \sum_{k=1}^{N} \varepsilon_{OE}^k(\hat{\theta}) \sigma(\hat{\theta}) \]
(10)
\[ J'_{\theta\theta} = \frac{2}{N} \sum_{k=1}^{N} \sigma(\hat{\theta}) \sigma(\hat{\theta})^T \]
(11)
\[ \sigma(\hat{\theta}) \] is the vector of the sensitivity function \( \sigma(\hat{\theta}) = \frac{\partial y}{\partial \theta} \), and \( \lambda \) is a scalar to control the convergence. The nonlinear regression equation (8) incorporates the successive derivatives of the model output and input. Thus, this regression form is not used when the OE algorithm is implemented. It is replaced by the estimated IO model which is defined by
\[ \hat{y}(\hat{\theta}_{ii}(\rho)) = \frac{\hat{B}(s, \hat{\theta}_{ii}(\rho))}{\hat{A}(s, \hat{\theta}_{ii}(\rho))} u(t) \]  
(12)
where
\[ \hat{B}(s, \hat{\theta}_{ii}(\rho)) = \sum_{j=0}^{n_b} \hat{b}_j(s) s^j \]
\[ \hat{A}(s, \hat{\theta}_{ii}(\rho)) = \sum_{i=0}^{n_a} \hat{a}_i(s) s^i + s^{n_a} \]
The sensitivity functions are computed as follows
\[ \sigma_{\hat{a}_i}(\hat{\theta}_{ii}(\rho)) = -s^{i} \frac{\hat{y}(\hat{\theta}_{ii}(\rho))}{\hat{A}(\hat{\theta}_{ii}(\rho))} \]
(13)
\[ \sigma_{\hat{b}_j}(\hat{\theta}_{ii}(\rho)) = s^{j} \frac{u(\hat{\theta}_{ii}(\rho))}{\hat{A}(\hat{\theta}_{ii}(\rho))} \]
(14)
and the sensitivity function vector can be written as
\[ \sigma = [\sigma_{\hat{a}_{0}}, \ldots, \sigma_{\hat{a}_{n_a-1}}, \sigma_{\hat{b}_0}, \ldots, \sigma_{\hat{b}_{n_b}}] \otimes F \]
(15)
Unfortunately, the property of asymptotic convergence is achieved at the cost of the minimization of a nonlinear quadratic criteria, which can lead to a local optimum (Landau et al. (2001)). This fundamental problem can be partially solved by running the optimization algorithm with different initial points. Consequently, the optimum values of the parameters are selected. However, this step is costly in terms of the computation time. Another solution is to use an EE algorithm to provide the initialization near the global optimum.

### 3.2 Initialization by EE algorithms

The classical approach to initialize the OE algorithms is to use an EE method. In this context, the LPV CT RPM model is presented. The principle is based on the partial moments, i.e. integrations on a truncated horizon. Then, this approach can be classified into the integration-based methods (Garnier et al. (2003)). However, by considering a time-shifting window mechanism called reinitialization, this model is formulated like a data filtering. It means that this approach, called CT RPM model, can be also classified into the filtering-based methods. Actually, it is this implicit filter which gives good properties to the model. The CT RPM model properties have been used for two decades in different application fields such as electrical engineering (Cuirault et al. (1995, 1996); Da Cunha et al. (1996)) or electronics (Djamai et al. (2006)). A complete description of this model is given in (T ohme (2008); Ouvrard and Trigeassou (2010)) in the LTI system context. The RPM output model is linear in parameters. Then, the parameter estimation is given by Least-Squares (LS). Unfortunately, the estimate is biased but the instrumental variable (IV) method with an auxiliary model can be used to eliminate the bias.

Consider the \( n_{a} \)th order CT Input-Output defined in (1). The actual response of this system can be modeled by the CT LPV RPM model defined by
\[ \hat{y}(t) = \sum_{j=0}^{n_b} \hat{b}_j(t) \beta^j_{a}(t) + \sum_{i=0}^{n_a-1} \hat{a}_i(t) \alpha^i_{a}(t) + \gamma^y(t) \]  
(16)
where
\[ \beta^j_{a}(t) = m(t) * u(t) \]
\[ \alpha^i_{a}(t) = -m(t) * y(t) \]
\[ \beta^j_{b}(t) = \frac{d^j m(t)}{dt^j} * u(t) \] for \( 1 \leq j \leq n_b \)
\[ \alpha^i_{b}(t) = -\frac{d^i m(t)}{dt^i} * y(t) \] for \( 1 \leq i \leq n_a \)
\[ \gamma^y(t) = \left( \delta(t) - \frac{d^n m(t)}{dt^n} \right) * y(t) \]
\[ m(t) = \left( T - t \right) t^{n_a-1} \]
\[ \text{with} \quad t \in \left[ 0, T \right] \]
m(t) is a FIR filter called CT RPM filter, \( \hat{T} \) is a design parameter called reinitialization time.

The polynomials \( \hat{b}_j(t), \hat{a}_i(t) \) are functions of the on-line measurable variables \( \rho \), and whose structure is defined in (3). The model (16) can be rewritten in the following linear regression form
\[ \hat{y}(t) = \psi(t) \hat{\theta}^{RPM} + \gamma^y(t) \]  
(18)
where
\[ \psi(t) = \varphi^T(t) (I_n \otimes F) \]
\[ \varphi(t) = \left[ \alpha^0_{a}(t), \ldots, \alpha^{n_a-1}_{a}(t), \beta^{n_b}_{a}(t), \ldots, \beta^{n_b}_{n_b}(t) \right]^T \]
\[ \hat{\theta}^{RPM} = \left[ \hat{a}^{0}_{0}(t), \ldots, \hat{a}^{n_a-1}_{0}(t), \ldots, \hat{a}^{n_a-1}_{n_a-1}(t), \ldots, \hat{b}^{0}_{0}(t), \ldots, \hat{b}^{n_b-1}_{n_b-1}(t) \right]^T \]  
(20)
\( \hat{\theta}_{RPM} \in \mathbb{R}^{n(r-1) \times 1}, \varphi^T(t) \in \mathbb{R}^{1 \times n} \) and \( \psi(t) \in \mathbb{R}^{1 \times n(r-1)} \). The output model given by (18) has two properties:

- linearity with respect to the parameters,
- filtering of the noisy output.

In this work, the case of noise-free scheduling parameters is considered. As in (Bamieh and Giarré (2002)), we propose a minimization of a quadratic criterion for the estimation of the system parameters. This criterion can be minimized using a number of well known algorithms including the standard least squares algorithm. The LS-estimate of \( \hat{\theta}_{RPM} \) is given by:

\[
\hat{\theta}_{RPM} = \left[ \sum_{k=K}^{N} \psi(kT_e)\varphi^T(kT_e) \right]^{-1} \sum_{k=K}^{N} \psi(kT_e) (y(kT_e) - \gamma^y(kT_e))
\]

The consistency of input-output LPV identification is investigated in (Butcher et al. (2008); Xiukun Wei and Re (2006); Bamieh and Giarré (2002)). To get a consistent identification, the scheduling parameters have to be noise-free for the case of an output dependent LPV model. Since the least squares technique does not give consistent estimates, the instrumental variable iterative scheme can be used to remove the bias. Consider the instrument given by the so-called auxiliary model as follows

\[
\xi(t) = \sum_{j=0}^{n_s} \hat{b}_j(\rho)s^j u(t) + \sum_{i=0}^{n_a-1} \hat{a}_i(\rho)s^i y(t)
\]

with the parameters estimated in the previous iteration.

Hence, the IV regressor is built

\[
Z(t) = \xi^T(t) (I_n \otimes F)
\]

\[
\zeta(t) = \left[ \alpha_0^\xi(t), \ldots, \alpha_{n_s-1}^\xi(t), \beta_0^\xi(t), \ldots, \beta_{n_a}^\xi(t) \right]^T
\]

The IV-estimate is given by

\[
\hat{\theta}_{IV} = \left[ \sum_{k=K}^{N} Z(kT_e)\psi^T(kT_e) \right]^{-1} \sum_{k=K}^{N} Z(kT_e) (y(kT_e) - \gamma^y(kT_e))
\]

A few iterations of the IV-estimate must be performed to remove the bias. 

**Remark:** The CT RPM model requires the selection of a design parameter which is the reinitialization parameter \( \hat{T} \). A wide experience in RPM handling has shown that the quality of the RPM model is not very sensitive to this choice (See Töhme (2008), for instance). The choice of \( \hat{T} \) is not more difficult than any design parameters of CT system identification methods (Garnier et al. (2003)).

In practice, the value of \( \hat{T} \) is chosen empirically by increasing \( \hat{T} \) and by comparing the results with a standard test, such as the quadratic criterion or the residuals.

### 4. SIMULATION EXAMPLE

In this section, we illustrate the proposed approach by a numerical example.

#### 4.1 The system

Consider the following LPV system given by

\[
H(s, \rho) = \frac{b_0(\rho)}{a_0(\rho) + a_1(\rho)s + s^2}
\]

where the coefficients dependence on the scheduling parameter is chosen as polynomial in a single parameter \( \rho \).

\[
b_0(\rho) = 1 + 0.4\rho + 0.01\rho^2
\]

\[
a_0(\rho) = 2 + \rho + \rho^2
\]

\[
a_1(\rho) = 1 + 0.1\rho + 2\rho^2
\]

The LPV CT noise-free output \( y(t) \) is simulated with a sampling period \( T_s = 0.1\) s and a length \( N = 1000 \). A zero-mean white noise \( v(kT_e) \) is added to the output samples with a signal-to-noise ratio of 10. The input \( u(t) \) is PRBS signal. The corresponding output \( y(t) \) and the noise-free scheduling signal \( \rho(t) \) are plotted in Figure 2. The location of the poles of the LPV system in the complex plane are represented in Figure 3. In this figure, there are only poles with a positive imaginary part.

![Fig. 2. A data set.](image-url)
(2) the initial model is used as a seed for estimating parameters of a more accurate LPV model via the Levenberg-Marquardt algorithm.

In order to give statistical properties of the estimator based on the LPV model \( \hat{H}(\theta, \rho) \) in the case of the noisy output measurements, a Monte-Carlo simulation (MCS) with \( N_{\text{exp}} = 100 \) random realizations was carried out. The variance of the additive noise \( \sigma^2 \) is adjusted to obtain a signal-to-noise ratio of \( SNR = 10 \). Table 1 shows the statistical properties of the parameters, the mean value of the estimated parameters, their standard deviation and their normalized root mean square error (RMSE) defined as

\[
RMSE = \sqrt{\frac{1}{N_{\text{exp}}} \sum_{q=1}^{N_{\text{exp}}} \left( \frac{\theta_j^0 - \hat{\theta}_j(q)}{\theta_j^0} \right)^2} \quad (27)
\]

with \( \hat{\theta}_j(q) \) the \( j \)th element of the estimate of the parameter vector \( \theta \), while the superscript ‘0’ denotes the true value of the parameter.

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>RPM</th>
<th>( \hat{a}_i )</th>
<th>OE</th>
<th>( \hat{\theta}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>2.156</td>
<td>2.136</td>
<td>2.017</td>
<td>0.0024</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0.945</td>
<td>0.939</td>
<td>0.82</td>
<td>0.0148</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>1</td>
<td>-0.0706</td>
<td>1.27</td>
<td>0.0102</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>1</td>
<td>1.171</td>
<td>1.085</td>
<td>0.004</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>0.1</td>
<td>0.37</td>
<td>0.035</td>
<td>0.020</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.4</td>
<td>0.69</td>
<td>2.11</td>
<td>0.0152</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.04</td>
<td>0.169</td>
<td>0.04</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

Table 1. Monte Carlo simulation results \( (SNR = 10) \)

![Fig. 3. Poles with positive imaginary part of the LPV system.](image)

It can be seen from the table that the OE method gives unbiased parameter estimates of the system parameters with better standard deviations than the RPM method. The simulated and estimated outputs of the final model and the initial RPM model as well as the corresponding residuals obtained during one of these simulations are drawn in Figures 4 and 5. From those simulations, it can be concluded that firstly the final LPV model is an accurate representation of the LPV system during the variations of the parameter and secondly that the nonlinear programming step is necessary, since the errors in the initial LPV RPM model test are larger.

![Fig. 4. The simulated output \( (SNR = 10) \) and the estimated output of the initial RPM model.](image)

![Fig. 5. The simulated output \( (SNR = 10) \) and the estimated output of the final model.](image)

5. CONCLUSION

CT IO model identification in an LPV framework is a relatively unexplored area. This paper presents an attempt to solve this problem. A new method to identify CT IO models in an LPV framework has been presented. It is based on the use of output-error technique. Despite defects due to the volume of computation time and non-uniqueness of the optimum, the technical output error have the fundamental advantage of near-universal applicability, whether with respect to the types of systems or application areas. Numerical simulations have illustrated the performance of
the proposed method in the case where zero-mean white noise is added to the output samples. Even in the case of heavy noisy output measurements, we showed the ability of the proposed method to obtain unbiased estimates. The advantage of the developed method is that the LPV models can represent a very large class of industrial processes. Finally, as a continuation of the presented work, extensions of the method to MIMO continuous-time LPV system identification will be investigated.

REFERENCES


