On the Global CLF Stabilization of Systems with Polytopic Control Value Sets
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Abstract: Our main purpose in this paper is to address the problem of the global asymptotic stabilization of affine systems with control value sets given by polytopes $U$ with $0 \in \text{int}U$. An important polytope is the $m$-dimensional hyperbox $U = [-r_1, r_1] \times \ldots \times [-r_m, r_m]$, with $r_j > 0$. Without loss of generality, we have made a slight modification on (3)-(4) by changing the sign.

1. INTRODUCTION

Consider the multiple input continuous-time affine system

$$
\dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x),
$$

where $x \in \mathbb{R}^n$, $f, g_j : \mathbb{R}^n \to \mathbb{R}^n$, for $j = 1, \ldots, m$, are $C^a(\mathbb{R}^n)$ vector fields ($a \geq 0$), the control value set (CVS) $U \subset \mathbb{R}^m$ is a convex polytope with $0 \in \text{int}U$, and $u = (u_1, \ldots, u_m)^T \in U$ with $^T$ denoting transposition. We will assume that $f(0) = 0$. By an admissible feedback control $u : \mathbb{R}^n \to U$ we will understand any continuous function $u : \mathbb{R}^n \to U$. The main objective of this paper is to address the global asymptotic stabilization (GAS) of the affine system (1) by means of an admissible feedback control $u(x)$.

In control theory, a Lyapunov control function $V(x,u)$ is a generalization of the concept of Lyapunov function $V(x)$ used on the stability analysis of a dynamical system. A Lyapunov function is used to prove that a system is asymptotically stable, whereas a Lyapunov control function is used to prove that a control system is feedback stabilizable. The concept of control Lyapunov function (CLF) was introduced by Artstein (1983), opening the possibility of using it as a tool for solving GAS problems (see also Sontag (1983)).

Formally, a function $V : \mathbb{R}^n \to \mathbb{R}$ is called a control Lyapunov function (CLF) for system (1) with controls taking values in $U$ if and only if (iff) it is a $C^k(\mathbb{R}^n)$ ($k \geq 1$) function which is proper positive definite ($V(0) = 0$ and $V(x) > 0$ iff $x \neq 0$) and proper (for any $c \geq 0$, $V^{-1}(c)$ is a compact set), such that

$$
\forall x \neq 0 \exists u \in U \ V(x,u) < 0. \tag{2}
$$

Let us restate (2) into the equivalent representation

$$
\forall x \neq 0 \inf_{u \in U} V(x,u) = \inf_{u \in U} \{a(x) - b(x) \cdot u\} < 0, \tag{3}
$$

where $\xi^1, \xi^2$ denotes the inner product of $\xi^1$ and $\xi^2$, and $a(x) := L_f V(x)$ & $b(x) := (b_1(x), \ldots, b_m(x))$, with $b_j(x) := - L_{g_j} V(x)$, for $j = 1, 2, \ldots, m$ (4) denote the Lie derivatives of $V(x,u)$ with respect to (w.r.t.) the vector fields that define the system (1).

In order to design feedback control functions continuous at the origin, the concept of small control property (SCP) was introduced in Artstein (1983): For each $\epsilon > 0$ there is a $\delta > 0$ such that the inequality $a(x) - b(x) \cdot u < \epsilon$ holds for a certain $u$ with $\|u\|_{\mathbb{R}^n} < \delta$, whenever $0 < \|x\|_{\mathbb{R}^n} < \delta$.

Theorem 1. (Artstein’s Theorem (1983)). Assume that the CVS $U \subset \mathbb{R}^m$ is convex and the system (1) is such that $f(x)$ and $(x,u) \mapsto \sum_{j=1}^{m} u_j g_j(x)$ are continuous. There exists a continuous feedback control $u : \mathbb{R}^n \to U$ that renders system (1) GAS iff there exists a CLF satisfying the SCP.

It is known that an asymptotically null controllable with bounded controls (ANCBC) linear system is GAS using arbitrarily small controls; whereas an unstable linear system cannot be GAS if the feedback is bounded, see Sussmann et al. (1994) and Suárez et al. (1997) where explicit control formulae were also proposed. Now, based on Artstein’s Theorem, in the former case there is a CLF, but in the latter the CLF is only local. The nonlinear passive systems are also GAS using arbitrarily small controls (see Solis-Daun et al. (2000) for an explicit formula). In general, the control design problem under prescribed conditions (e.g. regular control inputs restricted to $U$) in the framework of general stabilization strategies for nonlinear systems is
a very difficult task. Although Artstein’s result made a considerable impact on stabilization theory, it cannot be used as a control design tool since its proof was based on a nonconstructive procedure (partitions of unity). However, there has been a great activity in designing feedback controls by means of CLF’s due to an explicit formula when $U = \mathbb{R}^m$, obtained in Sontag (1989): the universal formula. Motivated by Artstein and Sontag’s results, increasing attention has been devoted to the construction of CLF’s for special classes of systems (see e.g. Sepulchre et al. (1997) and Malisoff & Mazenc (2009)), and to the problem of obtaining new explicit feedback control formulas. The latter problem has two objectives: (1) to design stabilizing controls handling performance specifications (see Freeman & Kokotovic (1996)), or (2) to construct formulas w.r.t. more general CV’s, e.g. considering input hard constraints (see Lin & Sontag (1991), Malisoff & Sontag (2000), Malisoff (2000), Solís-Daun, Suárez, & Alvarez-Ramírez (2000), Suárez, Solís-Daun, & Aguirre (2001), Suárez, Solís-Daun, & Alvarez-Ramírez (2002), Curtis (2003) and Leyva & Solís-Daun (2009) and (2010)). The following open problem was stated in Sontag (1998): “Find universal formulas for CLF stabilization, for general (convex) control-value sets $U$,” i.e. solve the synthesis problem for almost smooth (of class $C^\infty_c(\mathbb{R}^m \times \{0\})$) and continuous on $\mathbb{R}^n$) or almost real analytic feedback stabilizers with values in convex sets.

The latter problem has been addressed by Sontag and coworkers for specific compact convex CV’s: In Lin & Sontag (1991) it was proposed a universal formula for feedback control functions taking values in the Euclidean open unit ball; whereas in Malisoff & Sontag (2000) that result was extended to the “Minkowski” open unit balls,

$$\int B_{\psi}(1) := \left\{ u \in \mathbb{R}^m : \|u\|_p < 1 \right\},$$  
where $\|u\|_p := \left( \sum_{i=1}^m |u_i|^p \right)^{\frac{1}{p}}$,

with $p = 2r/(2r - 1)$ for some $r = 1, 2, \ldots, m$. The formula proposed in Malisoff & Sontag (2000) is

$$u(x) := \lambda_p(x) \cdot \left( x \right)^r \cdot \left( \frac{1}{b_{m}^{r-1}(x)} \right)^{\frac{1}{p}},$$

where $\lambda_p(x)$ is parameterized by $r$ and $\lambda_p : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\lambda_p(a, \theta) := \begin{cases} \frac{a + \sqrt{\theta + \frac{a^2}{2}}} {1 + \frac{1}{b_{m}^{r-1}(a)}} \theta, & \text{if } \theta > 0, \\ 0, & \text{if } \theta = 0. \end{cases}$$

Furthermore, it was proved that the feedback defined by (6)-(7) is almost smooth, whenever $\lambda_p(x)$ and $b_m(x)$ are smooth. Finally, that formula was also used to approximately solve the feedback stabilization problem for CVS given by the open square $\int B_{\psi}(1) = (-1, 1)^2$ and an open cross-polytope (or $m$-octahedron) $\int B_{\psi}(1)$ at expenses of small overflows in the control values. In Malisoff (2000), the “overflow problem” for $(-1, 1)^2$ was addressed.

Considering CVS defined by Cartesian products of $p, r$-weighted symmetric closed balls

$$B_{\psi}(1) := \left\{ u \in \mathbb{R}^m : \|u\|_{p,r} \leq 1 \right\},$$

where $\|u\|_{p,r} := \left( \sum_{i=1}^m |u_i/r_i|^p + \ldots + |u_m/r_m|^p \right)^{\frac{1}{p}}$,

with $r_j > 0$, for $j = 1, \ldots, m$, Solís-Daun, Suárez, & Alvarez-Ramírez (2000) applied a regularization method to obtain a family of differentiable controls for the GAS of nonlinear passive systems (1). The designed feedbacks satisfy both magnitude and rate constraints, while yielding large gains. In order to obtain large gains, while satisfying the constraints, the control design is modified. The appropriate rescaling of the controls is obtained through solving parametric nonlinear programs. In Suárez, Solís-Daun, & Alvarez-Ramírez (2002) it was defined a family of global stabilizers $u(x)$ taking values in the CVS given by

$$B_{\psi}(1) := \left\{ u \in \mathbb{R}^m : \|u\|_{p,r} \leq 1 \right\},$$

where $\|u\|_{p,r} := \left( \sum_{i=1}^m |u_i/r_i|^p + \ldots + |u_m/r_m|^p \right)^{\frac{1}{p}}$,

for $1 < p < \infty$, each $r_j$ is a function defined by

$$r_j(\zeta) := \begin{cases} r_j^\frac{1}{p}, & \text{if } \zeta \geq 0, \\ 0, & \text{if } \zeta < 0. \end{cases}$$

with $r_j^\frac{1}{p} > 0$, for $j = 1, \ldots, m$. The design functions $u(x)$ are everywhere continuous for any $p > 1$. Furthermore, continuous feedback controllers were also derived for the $m$-dimensional $\mathbb{R}$-hyperbox $B_{\psi}(1) = \left[ -r_1, r_1 \right] \times \cdots \times \left[ -r_m, r_m \right]$. The design of controls for $B_{\psi}(1) = \left[ -r_1, r_1 \right] \times \cdots \times \left[ -r_m, r_m \right]$ was considered that comprehends many of the control formulas proposed in the literature.

In Curtis (2003), it was introduced a method for algorithmically parameterizing stabilizing control laws that obey polytopic input constraints, given a known CLF. In Leyva & Solís-Daun (2009) it was proposed an explicit formula for continuous feedback laws taking values in the interval $[-r_1, r_1]$, that renders an affine system GAS. The case of positive feedback controls in an interval $[0, r_1]$ was also addressed. Finally, in Leyva & Solís-Daun (2010), those results were further extended deriving admissible feedback controllers for the $m$-dimensional $\mathbb{R}$-hyperbox $B_{\psi}(1) = \left[ -r_1, r_1 \right] \times \cdots \times \left[ -r_m, r_m \right]$ where $r_j = 0$ was addressed.

In this paper, we generalize the results obtained in Suárez et al. (2002), and Leyva & Solís-Daun (2009) and (2010). The goal is to give a solution to the problem: “Address the global CLF stabilization of system (1) with polytopic CVS $U$ by means of admissible controls”. In general, the designed feedback controllers are continuous, in accordance with Artstein’s theorem, and take values in the $U$-function.

The paper is organized as follows. In section 2, we introduce some convexity concepts in order to keep the paper self-contained. In section 3, we study first some properties of feedback control $\Xi(x)$ that gives the best rate w.r.t. a
clf in terms of the properties of the cvx $U$; then, control $\varphi(x)$ is analyzed for the important class of polytopic cvx. In view that the control $\varphi(x)$ is generally discontinuous with values on $\partial U$ -the boundary of the control value set $U$, in section 4 we consider the conditions that feedback controls of a general form $u(x) = (u_1, \ldots, u_m)^T$, with $u_j(x) = \varphi_j(x) \varphi(x)$, should satisfy in order to be admissible and achieve GAS of system (1).

2. BASIC DEFINITIONS OF CONVEX THEORY AND PRELIMINARIES.

We will need the following convexity results (for an exposition on convex theory, see e.g. Rockafellar (1972), sections 13-15): Assume that $U \subset \mathbb{R}^m$ is a convex set. A function $\mu : \mathbb{R}^m \rightarrow \mathbb{R}$ is called a Minkowski functional (or gauge) iff it is a positive semi-definite (i.e., $\mu(0) = 0$ and $\mu(u) \geq 0$), positively homogeneous (i.e., $\mu(\lambda u) = \lambda \mu(u)$, for any real $\lambda \geq 0$) and convex function. Hence, for some convex set $\emptyset \neq U \subset \mathbb{R}^m$, a Minkowski functional is defined by

$$\mu(u) := \inf \{ r \geq 0 : u \in rU \} .$$

and vice versa if $\mu(u)$ is closed ($\mu_{|\text{dom}=\emptyset}$ is finite and it is lower semi-continuous), then there is a unique convex set $U \neq \emptyset$ such that $U = \{ u \in \mathbb{R}^m : \mu(u) = 1 \}$ -a level set.

We have the following equivalences (Rockafellar (1972), p. 128 et seq.): (1) $\mu$ is finite everywhere iff $0 \in \mathbb{R}^m$ and (2) it is positive definite iff $U$ is compact.

If $\mu$ is finite everywhere and positive definite, then its polar and the polar set of $U$ are defined, respectively, by

$$\mu^{*}(u^{*}) := \sup_{u \neq 0} \frac{u^{*} \cdot u}{\mu(u)}$$

and

$$U^{*} := \{ u^{*} \in (\mathbb{R}^m)^{*} : \mu^{*}(u^{*}) \leq 1 \} ,$$

where $(\mathbb{R}^m)^{*}$ denotes the dual space of $\mathbb{R}^m$.

Gauges polar to each other satisfy the following property:

$$u^{*} \cdot u \leq \mu^{*}(u^{*}) \mu(u), \forall u \in \text{dom} \mu \& \forall u^{*} \in \text{dom} \mu^{*},$$

where, $u \in \mathbb{R}^m$ and $u^{*} \in (\mathbb{R}^m)^{*}$. The previous expression is the “best” inequality in the sense that it cannot be tightened by replacing $\mu$ or $\mu^{*}$ by lesser functions on larger domains. In particular, if $\mu$ is a $p$-norm, then $\mu^{*}$ is a $q$-norm ($\frac{1}{p} + \frac{1}{q} = 1$) and the above relation reduces to the well known Hölder’s inequality: $u^{*} \cdot u \leq ||u^{*}||_q ||u||_p$.

The support function of $U$, denoted $\varphi_U$, is defined as

$$\varphi_U(u^{*}) := \sup_{u \in U} u^{*} \cdot u ,$$

which is a positively homogeneous and convex function, and dom $\varphi_U$ is a convex cone in $(\mathbb{R}^m)^{*}$ with the apex at 0.

If $0 \in \text{int} U$, then $U$ and its polar $U^{*}$ share the same properties, expressed in the following theorem.

Theorem 2. $U$ is a compact convex set with $0 \in \text{int} U$ iff $U^{*}$ is a compact convex set with $0 \in \text{int} U^{*}$. Furthermore, $U^{**} = U$, and if $\mu$ and $\mu^{*}$ are the gauges of $U$ and $U^{*}$, respectively, then $\mu^{*} = \varphi_U$ and vice versa.

Hereafter, we will identify the dual space $(\mathbb{R}^m)^{*}$ with $\mathbb{R}^m$ using the usual inner product, and denote the dual $u^{*}$ by $b$.

Moreover, based on these convexity results, we will assume that $U \subset \mathbb{R}^m$ is a compact convex set with $0 \in \text{int} U$.

An important class of compact non-strictly convex sets is given by the convex polytopes. Recall that a (convex) polytope $U$ can be defined as a bounded set obtained from the intersection of a finite number of closed halfspaces, called the $H$-representation; or described equivalently in terms of the convex hull of its vertices $V = \{ v_1, \ldots, v_k \}$, $v_i \in \mathbb{R}^m$, termed the $V$-representation. From the point of view of computational complexity, it makes a great difference which type of representation is given, e.g. the $m$-hyperbox can be described by $2m$ halfspaces or by $2^m$ vertices. The problem of transforming one representation into the other, known as the convex hull problem, is non-trivial. In order to transform a polytope from its $H$-representation into its $V$-representation -find all its vertices! - in Avis and Fukuda (1996) it was introduced reverse search techniques to solve the problem in polynomial time.

The class of convex polytopes has many desirable properties. For instance, any compact convex body $U \subset \mathbb{R}^m$ (int $U \neq \emptyset$) can be “sandwiched” between two polytopes arbitrarily close to $U$. This implies that this class is dense in the class of all compact convex bodies in $\mathbb{R}^m$. It is well known that the polar set of a polytope is also a polytope. Furthermore, $U$ is a polytope iff its support function $\varphi_U$ is piecewise linear. Observe that for a point $v \in \mathbb{R}^m$, its support function is a linear function $\varphi_U(b) = v \cdot b$. Then, if $U$ is a polytope, the domains of linearity of $\varphi_U$ correspond to its vertices (for the supremum is achieved at one of the vertices). These domains of linearity tile $\mathbb{R}^m$ into a union of polytopal cones $C_i$ with the apex at 0. This tiling is called the fan of the polytope $U$.

3. ON THE GLOBAL CLF STABILIZATION W.R.T. CONVEX POLYPOTES

Now, returning to our control problem, observe that problem (3) is solvable iff there exists a feedback control $u(x)$ taking values in $U$, such that $a(x) < b(x) \cdot u(x), \forall x \neq 0$. On the other hand, for any control $u(x)$ taking values in $U$, we have that $b(x) \cdot u(x) \leq \mu^{*}(b(x)) \mu(u(x))$. Thus, if we set $u(x) = \varphi(x)$, with $\mu(\varphi(x)) \equiv 1$ (i.e. it takes values in $\mathbb{R}^m$), then $\varphi(x)$ satisfies the equation

$$b(x) \cdot \varphi(x) = \mu^{*}(b(x)) .$$

Consequently, if we can find an appropriate clf, then any feedback control $\varphi(x)$ satisfying (15) accomplishes the equivalence between problem (3) and the inequality

$$a(x) < \mu^{*}(b(x)) , \quad \forall x \neq 0 .$$

We will call a feedback $\varphi(x)$ to be a best rate (optimal) control law w.r.t. a CLF $\varphi$ for system (1) (with controls taking values in $U$) iff it satisfies the equation

$$a(x) - b(x) \cdot \varphi(x) = \inf_{u \in U} \{ a(x) - b(x) \cdot u \} < 0 ,$$

for all $x \neq 0$. Hence, problem (3) is solvable if there exists a feedback control $\varphi(x)$. However, from (15), it follows that this control is a singular function at the set

$$N_b := \{ x \in \mathbb{R}^m : b(x) = 0 \} ,$$

because if $b(x) = 0$ then $\varphi(x)$ is arbitrary. Then, $\varphi(x)$ is not admissible since it is a singular function.

Now, we proceed on the existence and continuity of the control $\varphi(x)$. Taking into account the polar $\mu^{*}$ defined by (12) and temporarily leaving aside the dependence on $x$,
we have that equation (15) can be restated as the following m-parameterized optimization program
\[ P(b) := \sup \{ b \cdot u : u \in U(b) \}, \] (19)
where \( b \in \mathbb{R}^m \) and \( U(b) := \{ u \in \mathbb{R}^m : \mu(u) = 1 \} \) is the constraint set. Clearly, since \( U \) is compact, there exists at least one global solution of \( P(b) \), denoted by \( \omega(b) \). On the other hand, the optimization program \( P(b) \) is precisely the support function of set \( U \), \( \omega(b) \), which is convex and positively homogeneous. Moreover, from Theorem 2 we have that \( \omega_t(b) = \mu^*(b) \). Recall that \( \omega(b) \) is a homogeneous function of degree \( \alpha \), if \( \omega(rb) = r^\alpha \omega(b) \), for any real \( r \). In particular, if \( h(b) \) is a positively homogeneous function (i.e., \( \alpha = 1 \) and \( r \geq 0 \)), then a non smooth extension of the so-called Euler’s Theorem for homogeneous functions (see Blanchini & Miani (2008), p. 132) states that
\[ h(b) = w \cdot b, \quad \forall w \in \partial h(b), \] (20)
where the set \( \partial h(b) \) is the subdifferential of \( h \) at \( b \) and each element \( w \in \partial h(b) \) is called a subgradient of \( h \) at \( b \). Clearly, if \( h(b) \) is differentiable, then \( \partial h(b) \) is a singleton and \( w = \nabla h(b) \) is the gradient of \( h \) at \( b \). Now, let \( h(b) = \omega(b) \) -the support function \( U \). Thus, if we assume that \( \omega_t(b) \) is of class \( C^1(D) \), where \( D \subset \mathbb{R}^m \) is an open set, then \( \omega(b) \) is uniquely defined in \( D \) and it is given by
\[ \omega(b) = (\nabla \omega_t(b))^\top, \] (21)
thus, it is continuous on \( D \) and homogeneous of degree \( 0 \).

In particular, on the basis of our control problem, we have that since the support function of a polytope \( U \) is piecewise linear (hence, piecewise real analytic), from (21) we obtain that \( \omega(b) \) is constant on each open polytopal cone \( \text{int} C_t \), and it is equal to the vertices of the polytope \( U \). Specifically, if \( v_i \) is a vertex of \( U \) and \( C_t \) is its corresponding polytopal cone, then for all \( b \in \text{int} C_t \), \( \omega(b) = v_i \). Hence, the importance to find the vertices of a polytope.

In order to throw some light into the developed ideas, we illustrate the introduced concepts with an example.

**Example 3.** Consider the equilateral triangle of length \( l = 2\sqrt{3} \) defined by \( U = \text{convhull}(v_1, v_2, v_3) = \text{convhull}(\mathbf{v}(\sqrt{3}, 1), (-\sqrt{3}, 1), (0, -2)) \) in \( \mathbb{R}^2 \). Then, its support function is given by
\[ \omega(b_1, b_2) = \begin{cases} 
 v_1 \cdot b = \sqrt{3} b_1 + b_2, & \text{if } b \in C_1, \\
 v_2 \cdot b = -\sqrt{3} b_1 + b_2, & \text{if } b \in C_2, \\
 v_3 \cdot b = -2b_2, & \text{if } b \in C_3,
\end{cases} \] (22)
where, \( C_1 = \{ (b_1, b_2) \in \mathbb{R}^2 : b_1 \geq 0 \& b_2 \geq -b_1/\sqrt{3} \} \), \( C_2 = \{ (b_1, b_2) \in \mathbb{R}^2 : b_1 \leq 0 \& b_2 \geq b_1/\sqrt{3} \} \) and \( C_3 = \{ (b_1, b_2) \in \mathbb{R}^2 : b_2 \leq 0 \& \sqrt{3} b_2 \leq b_1 \leq -\sqrt{3} b_2 \} \).

Then, \( U^* \) can be defined in the \( H \)-representation by \( U^* = \{ (b_1, b_2) \in \mathbb{R}^2 : (\sqrt{3} b_1 + b_2 \leq 1) \& (-\sqrt{3} b_1 + b_2 \leq 1) \} \) or in the \( V \)-representation by
\[ U^* = \text{convhull}((0, 1), (-\sqrt{3}/2, -1/2), (\sqrt{3}/2, -1/2)), \] (23)
which is an equilateral triangle of length \( l = \sqrt{3} \).

Hence, from the formula (21), we obtain
\[ \omega(b_1, b_2) = \begin{cases} 
 v_1 = (\sqrt{3}, 1), & \text{if } b \in C_1, \\
 v_2 = (-\sqrt{3}, 1), & \text{if } b \in C_2, \\
 v_3 = (0, -2), & \text{if } b \in C_3,
\end{cases} \] (24)
Observe that \( \omega(b) \) is discontinuous at the set \( \cup \partial C_t = \{ (b_1, b_2) \in \mathbb{R}^2 : (b_1 = 0 \& b_2 \geq 0) \vee (b_1 \leq 0 \& b_2 = b_1/\sqrt{3}) \vee (b_1 \geq 0 \& b_2 = -b_1/\sqrt{3}) \} \)-set resembling an inverted \( Y \).

Now, let us obtain a formula for \( \omega(b) = (\omega_1(b), \omega_2(b)) \) in terms of step functions. In fact, consider the following table for the values of its components \( \omega_1(b) \) and \( \omega_2(b) \):
\begin{align*}
\omega_1 & \quad \sqrt{3} - \sqrt{3} 0 \\
\omega_2 & \quad 1 \quad 1 -2
\end{align*}

Observe that the set \( U \) can be represented as the graph of the function \( b_2 = -|b_1|/\sqrt{3} \). Then, a switch of \( \omega(b) \) at \( \partial U \) can be registered by means of the formula \( \text{sign}(b_2 + |b_1|/\sqrt{3}) \).

Analogously, a switch of \( \omega(b) \) at the common boundary between \( C_1 \) and \( C_2, \partial C_1 \cap \partial C_2 = \{ (b_1, b_2) \in \mathbb{R}^2 : b_1 = 0 \& b_2 \geq 0 \} \), is given by \( \text{sign}(b_1) \).

Consequently, a formula for \( \omega(b) \) is the following
\[ \omega(b_1, b_2) = (\omega_1(b_1, b_2), \omega_2(b_1, b_2)), \]
with \( \gamma(b_1, b_2) = r_1(\gamma(b_1, b_2)) \text{sign}(b_1) \) and \( \omega_2(b_1, b_2) = r_2(\gamma(b_1, b_2)) \text{sign}(b_1) \),
(25)

An important polytope is the \( m \)-dimensional hyperbox \( U = [-r_1, r_1] \times \cdots \times [-r_m, r_m] \), with \( r_j > 0 \). Hence, a compact and convex set with \( 0 \in \text{int} U \), so that it admits a representation in terms of a Minkowski functional.

In fact, define the function \( \psi_{\infty, r} : \mathbb{R}^m \to \mathbb{R} \) by
\[ \psi_{\infty, r}(u) := \sup \left\{ \frac{|u_j|}{r_j} \right\}, \] (26)
where \( r_j(\zeta) \) is defined in (10), for \( j = 1, \ldots, m \).

**Proposition 4.** The function \( \psi_{\infty, r}(u) \) is a positive definite Minkowski functional, and
\[ U = \{ u \in \mathbb{R}^m : \psi_{\infty, r}(u) \leq 1 \} \] (27)
is a representation of the hyperbox.

In particular, if \( r_j^* = r_j^+ \) for all \( j \), then \( \psi_{\infty, r}(\cdot) = ||\cdot||_{\infty, r} \) is a norm, and thus \( U \) is a symmetric hyperbox.

Corresponding to \( \psi_{\infty, r} \), we define the following gauge
\[ \psi_{1, 1/r}(b) := \sum_{j=1}^m r_j(b_j) |b_j|, \] (28)
where \( r_j(\zeta) \) is defined in (10), for \( j = 1, \ldots, m \).

**Proposition 5.** Gauges \( \psi_{\infty, r} \) and \( \psi_{1, 1/r} \) are polar to each other (\( \psi_{\infty, r} = \psi_{1, 1/r} \) and vice versa). Moreover,
\[ U^* := \{ u^* \in \mathbb{R}^m : \psi_{1, 1/r}(u^*) \leq 1 \}, \] (29)
is the polar set of the hyperbox \( U \) -a cross-polytope.

Since the support function of the hyperbox \( U \) is given by \( \omega_u = \omega_{1, 1/r} \), which is piecewise linear, then
\[ \omega(b) = (\nabla \psi_{1, 1/r}(b))^\top = (r_1 \text{sign} b_1, \ldots, r_m \text{sign} b_m)^\top, \] (30)
which is constant on each open orthant (polytopal cone).

Now, let us return to the dependence on the state variable \( x \in \mathbb{R}^n \). Hereafter, we define by \( r_j(x) := r_j(b(x)), j =\)
1,...,m, $\beta(x) := \mu^*(b(x))$, and $\varpi(x) := \omega(b(x))$ is the optimal control, with $\omega(b)$ given by (21) and $b(x)$ by (4).

Finally, observe that $\omega(b)$ is discontinuous at the union of the boundaries (faces) of the cones $C_i, \cup_i \partial C_i$. Then, assume that this union can be represented as the level sets of $m$ functions $\gamma_j(b)$ containing 0. Hence, we define $N_j = \{x \in \mathbb{R}^n : \gamma_j(b(x)) = 0\}$, (31)

for $j = 1,\ldots,m$, the representation of those level sets in the space $\mathbb{R}^n$, such that its intersection is the set $N_0$ defined in (18), i.e. $N_0 = \cap_j N_j$. Then, it should be clear that $\varpi(x)$ is a singular function on $\cup_j N_j$. Moreover, each $N_j$ is a switching surface for it. Therefore, $\varpi(x)$ is a bang-bang type feedback control, so that it is not admissible!

In the case of the $m$-dimensional hyperbox, we have that $\gamma_j(b(x)) = b_j(x)$, for $j = 1,\ldots,m$; whereas, for Example 3, $\gamma_1(b_1, b_2) = b_1(x)$ and $\gamma_2(b_1, b_2) = b_2(x) + |b_1(x)|/\sqrt{3}$.

4. A GENERAL DESIGN OF ADMISSIBLE CONTROLS FOR POLYTOPIC CVs

From the point of view of many applications, the use of the feedback control $\varpi(x)$ is not desirable because its computational implementation can induce “chattering” due to its discontinuity at each switching surface $N_j$. Indeed, the right hand side of the closed-loop system (1)-(21) becomes discontinuous at $N_j$, so that existence of its solutions w.r.t. initial conditions is not guaranteed.

In Solís-Daun, Aguirre, and Suárez (2010), considering compact and strictly convex CVs $U$ with $0 \in \text{int} U$, it was proposed general feedback controls of the form $u(x) := \rho(x)\varpi(x)$ that comprehend many of the designed control formulas proposed in the existing literature. However, that designing can only deal with the singularities of $\varpi(x)$ at the null set $N_0$. Henceforth, in the case of a polytopic CVs, we propose a feedback control of the general form

$u(x) = (u_1(x), \ldots, u_m(x))^T, \quad u_j(x) := \rho_j(x)\varpi_j(x), \quad (32)$

for $j = 1,\ldots,m$, where $\varpi(x)$ is obtained from (21) and $\rho(x) = (\rho_1(x), \ldots, \rho_m(x))$ is a rescaling vector function to be determined. The reason for this design obeys to the problem to deal with the discontinuities of $\varpi(x)$ at its switching surfaces $N_j$ by means of appropriate functions $\rho_j(x)$. Hence, we ask the conditions that function $\rho(x)$ should satisfy in order to guarantee the existence of an admissible feedback global stabilizer (32) for system (1).

**Hypothesis H.** Assume that $\rho : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function such that

(i) $\forall x \in \mathbb{R}^n, 0 \leq \rho_j(x) \leq 1$, for $j = 1,\ldots,m$,

(ii) $\rho_j(x) = 0$ if $x \in N_j$, for $j = 1,\ldots,m$, and

(iii) $\forall x \in \mathbb{R}^n \setminus N_0, \|\rho(x)\|_\infty > \frac{a(x)}{\beta(x)}$.

**Remark 6.** We need to impose some regularity conditions on $\rho(x)$ to obtain certain regularity of feedback control $u(x)$. The mild ones are continuity whenever $\gamma_j(b(x)) \neq 0$, for $j = 1,\ldots,m$, and $\rho(x)$ should be able to remove the singularities of $\varpi(x)$, i.e. $\forall x^* \in N_j$, $\lim_{x \to x^*} \rho_j(x)\varpi_j(x)$ exists. Now, since $0 \in N_0 = \cap_j N_j$ and the scp will be assumed, a necessary condition for the continuity of each $u_j(x) = \rho_j(x)\varpi_j(x)$ at $N_0$ is that $\lim_{x \to x^*} \rho_j(x)\varpi_j(x) = 0$, $\forall x^* \in N_j^0$, and $\rho_j(x) \big|_{N_j^0} = 0$, where $N_j^0$ is the connected component of 0 in $N_j$. Thus, to include the case that $N_j$ be a connected set, we have that both $\rho_j(x) \big|_{N_j^0} = 0$ and $\forall x^* \in N_j^0, \lim_{x \to x^*} \rho_j(x)\varpi_j(x) = 0$ must hold.

**Remark 7.** The continuity of the controllers at $x = 0$ is achieved via the scp 2: Given $\epsilon > 0$ there is $\delta > 0$ such that if $0 < |x| < \delta$, then $a(x) < b(x) \cdot u$ holds for a certain $u$ with $\mu(u) < \epsilon$. Hence, for such $u$, $a(x) < b(x) \cdot u(x) \leq \beta(x)\mu(u(x)) < \epsilon < \beta(x)$, so that $\lim_{x \to 0} a(x)\beta(x) = 0$.

**Remark 8.** From Condition (iii), we have that each $\rho_j(x)$ can be defined as a composition of continuous functions: $\rho_j(x) := g_j(a, \beta)(a(x), \beta(x))$, where $g_j : \mathbb{R}^2 \to [0,1]$. Furthermore, if we denote $\varphi = (g_1, \ldots, g_m)$, then for all $a, \beta \neq 0$, $\|\varphi(a, \beta)\|_\infty > a/\beta$ and $\varphi(a, 0) = (0,\ldots,0)$.

**Theorem 9.** Assume that $U \subset \mathbb{R}^m$ is a convex polytope with $0 \in \text{int} U$, $V(x)$ is a CLF w.r.t. system (1) with controls taking values in $U$ satisfying the scp, $\varpi(x)$ is the optimal control given by (21) and $\rho : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function satisfying Hypothesis H. Then, control (32) is an admissible feedback that renders system (1) GAS.

**Proof.** First of all, we have that $u(x)$ given by (32) is admissible: In fact, from Condition (i) we have that since $u_j(x) = \rho_j(x)\varpi_j(x) \leq \varpi_j(x)$, then $\mu(u(x)) \leq \mu(\varpi(x)) \equiv 1$. Thus $u(x)$ takes values in $U$.

Further, $\forall x \in \mathbb{R}^n \setminus \cup_j N_j$, we have that both $\rho(x)$ and $\varpi(x)$ are continuous, so is $u(x)$. In the case that $x \in N_j$, since $\rho_j(x)$ is continuous and $\varpi_j(x)$ is bounded, then from the scp and Condition (ii) it follows that $\forall x \in N_j$,

$$0 \leq \lim_{x \to x^*} |u_j(x)| = \lim_{x \to x^*} \rho_j(x) |\varpi_j(x)| = 0,$$

then each $u_j(x)$ is continuous at $N_j$ and $u_j(x) \big|_{N_j} = 0$.

Finally, w.r.t. the GAS of the system, we have:

(a) If $x \in N_0 \setminus \{0\}$, from Condition (iii) and (16) ($\beta(x) = 0 \Rightarrow a(x) < 0$), then $dV/dt = a(x) - b(x) \cdot u(x) = a(x) < 0$.

(b) If $x \in \mathbb{R}^n \setminus N_0$, then there is at least an active $\gamma_j(b(x))$, i.e. there is a $j$ ($1 \leq j \leq m$) such that $\gamma_j(b(x)) \neq 0$. From Condition (iii), we have that $dV/dt = a(x) - b(x) \cdot u(x) \leq a(x) - \mu^*(b(x)) \mu(\rho(x)\varpi_j(x), \ldots, \rho_m(x)\varpi_m(x) < 0$, if $a(x) < \mu^*(b(x)) \varphi(\varpi_j(x), \ldots, \varpi_m(x)) \neq 0$, (34)

where $\varphi(x) = \max_j \rho_j(x)$.

Hence, $u(x)$ is a global asymptotic stabilizer.

**Remark 10.** The regularity of control $u(x)$ given by (32) depends on the regularity of the rescaling function $\rho(x)$ because $\varpi(x)$ is piecewise constant on $\mathbb{R}^n \setminus \cup_j N_j$. Indeed, if we can construct a $\rho(x)$ which is smooth/real analytical on $\mathbb{R}^n$, then $u(x)$ will be smooth/real analytical for all $x \in \mathbb{R}^n \setminus \cup_j N_j$ and everywhere continuous. Moreover, from Conditions (ii)-(iii), it can be inferred that $\forall x \in N_0 \setminus \{0\}$, $a(x) < 0$, in accordance with inequality (16).

**Remark 11.** Among the conditions of Hypothesis H, item (iii) can be too restrictive, and it was assumed to guarantee the GAS of the system. Indeed, in Levy and Solís-Daun (2010) it was considered a relaxed condition instead. In that paper, we addressed the GAS of system (1) by means of feedback controls constrained to an m-dimensional r-hyperbox $B_m^r(\infty)$, with $r^m > 0$. In specific, it was proposed the next formula for an $\epsilon$-parameterized family of controls

Note, here, $\mu$ is a finite everywhere and positive definite gauge, hence it is equivalent to any norm.
u^ε(x) := (u^ε_1(x), \ldots, u^ε_m(x))^\top \quad \text{with} \quad u^ε_j(x) = g^ε_j(a(x), \beta(x)) \xi_j(x)

where \xi(x) is the optimal control given by (30), g^ε_j : \mathbb{R} \times [0, \infty) \to \mathbb{R} is defined as g^ε_j(a, \beta) = 0 \text{ if } |b_j| r_j = 0, \text{ or } g^ε_j(a, \beta) = 1 - \left(1 - \frac{|a| + |b_j| r_j}{2 \beta} \right) e^{(\tau^ε_j |b_j| r_j)},

if |b_j| r_j > 0, and τ^ε_j is a non-positive function given by

\tau^ε_j(x) = \left\{ \begin{array}{ll}
\frac{\ln \left(\frac{\lambda(x)}{\rho(m)} \right)}{\lambda(x)} - \varepsilon |b_j| r_j, & \text{if } \beta > 0, \\
0, & \text{if } \beta = 0,
\end{array} \right.

for j = 1, \ldots, m, where \lambda(x) = 1 - \frac{1}{2}((a(x) + a(x))/\beta(x), \beta(x) = \psi_{1/r}(b(x)) and \varepsilon > 0 is a tuning parameter.

It was proved that \rho^ε(x) given by (36) satisfies Hypothesis H, but a relaxed Condition (iii). Then, feedbacks u^ε(x) take values in int\mathcal{B}^n(\infty) and render the system (1) GAS, provided an appropriate CLF is known. Furthermore, if the CLF \mathcal{V}(x), and the vector fields f(x) and g_j(x), for j = 1, 2, \ldots, m, that define the system (1) smooth/real analytical, then it was shown that controls u^ε(x) are smooth/real analytical \forall x \in \mathbb{R}^n \setminus (\cup_j N_j), where N_j := \{x \in \mathbb{R}^n : a(x) = 0\}, and everywhere continuous.

5. CONCLUSION

In this paper, we address the problem of the global CLF stabilization of affine systems w.r.t. polytopic CVSs.

We show first that the CLF-optimization problem (3) is solvable if there exists a feedback control \xi(x) that takes values in \partial U: The best rate control law w.r.t. a CLF. Then, given a convex set \mathcal{U} \subset \mathbb{R}^m, we analyze the conditions for the existence and continuity of the feedback control \xi(x). It was observed that in the case of a polytopic CVS, \xi(x) is piecewise constant, with switching surfaces given by the level sets N_j defined in (31): It is a bang-bang type control.

Therefore, in view that feedback control \xi(x) associated to a polytopic CVS is not admissible (it is discontinuous at \cup_j N_j), we consider a general form of feedback controls \xi(x) = (u_1(x), \ldots, u_m(x))^\top, whose components are given by u_j(x) = \rho_j(x) \xi_j(x), where \rho_j(x) is a rescaling function used to regularize \xi_j(x). Then, we study the conditions that such controls should satisfy in order to be admissible (continuous and taking its values in U) and globally asymptotically stabilize system (1), provided an appropriate CLF is known. Furthermore, those feedback controls are smooth/real analytical on \mathbb{R}^n excepting at the switching surfaces N_j (j = 1, \ldots, m), and everywhere continuous, whenever all \rho_j(x) are smooth/real analytical.

Finally, the problem of designing an explicit formula for a continuous feedback control for the GAS of system (1) constrained to take values in a general polytope is still under development, and will appear in a future work.

REFERENCES