1. INTRODUCTION

Compared with continuous time sliding mode control strategies, the design problem in discrete time is much less mature. Other than the early work in Sira-Ramírez [1991], much of the literature assumes all states are available [Furuta and Pan, 2000, Hui and Zak, 1999, Tang and Misawa, 2002]. Discrete sliding mode control schemes which have restricted themselves to output measurements alone have often been observer based schemes, with or without disturbance estimation [Lee and Lee, 1999, Tang and Misawa, 2000]. Recent exceptions have been the work in Monsees [2002] which considers both static and dynamic output feedback and the discrete time versions of certain higher-order sliding-mode control schemes [Bartolini et al., 2001, 2000].

In sliding mode control for continuous time systems, it has been shown that the relative degree condition associated with the solution of the existence problem can be weakened (see for example Hsu et al. [2006]). Another approach is to combine a classical sliding mode observer with sliding mode exact differentiators to generate additional independent output signals from the available measurements [Floquet et al., 2007]. In the discrete case, it has been shown that by using the output signal at the current time instant together with a limited amount of information from previous sample instants, the class of discrete time systems for which an output feedback based sliding mode controller can be developed is significantly broadened [Govindaswamy et al., 2008, 2009]. It has been shown that with this approach, discrete time output feedback based sliding mode control for systems with unstable invariant zeros is possible. It has also been shown that, by extending the outputs, the relative degree condition associated with the solution of the existence problem can be relaxed for output feedback based sliding mode control.

The above notion is further developed in this paper, by designing discrete time sliding mode controllers for uncertain systems using the extended outputs. By using the extended outputs, it is first shown that the sliding dynamics is a function of the disturbance and that the ideal sliding mode as defined for a nominal system is not possible. In this case, the reduced order dynamics is shown to be bounded about a region around the sliding surface. A procedure for synthesizing the control law will be shown. It is shown that a control law can be chosen such that the upper bound on the $\|H_2\|$ norm of a particular transfer function relating the disturbance input to the output is minimized. The paper is structured as follows. Section 2 gives the problem motivation. The existence problem is given in Section 3 and the closed loop stability analysis and the reachability problem is dealt with in Section 4. A case study is presented to illustrate the methodology.

2. MOTIVATION

Consider the uncertain discrete, linear, time invariant state space system representation as given below:

$$x_{k+1} = Ax_k + Bu_k + B_0d_k$$

$$y_k = [(y_1)_k ... (y_p)_k]^T = C_{x_k}, (y_i)_k = C_{i}x_k$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output vector, $u_k \in \mathbb{R}^m$ is the control input and $d_k \in \mathbb{R}^p$ is the disturbance input. It is assumed that $m \leq p$, the pair $(A, B)$ is controllable and without loss of generality, that $\text{rank}(C) = p$ and that column $\text{rank}(B) = m$. Assume

$$H_2 = \max_{\omega \in \mathbb{R}} \frac{\|Y(\omega)\|_2}{\|N(\omega)\|_2}$$

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that the uncertainties are lumped in $d_k$ and are such that for some real $\rho_0 > 0$, $\|d_k\| < \rho_0$. It is assumed that the matrix $A$ is invertible. It is also assumed that the usual matching condition $rank(B) = rank(B_d)$ is not satisfied. The nominal part of the plant (1)-(2) is defined as the representation when the disturbance $d_k = 0$. If the sliding surface $s$ is defined as:

$$ s = \{x_k \in \mathbb{R}^n : F \bar{C} x_k = 0\} \quad (3) $$

for some selected matrix $F \in \mathbb{R}^{n \times p}$ then it is well known for a unique equivalent control to exist, the matrix $F CB \in \mathbb{R}^{m \times m}$ must have full rank. As

$$ rank(FCB) \leq min\{rank(F), rank(CB)\} \quad (4) $$

it follows that both $F$ and $CB$ must have full rank. As $F$ is a design parameter, it can be chosen to be full rank. A necessary condition for $F CB$ to be full rank, and thus for solvability of the output feedback sliding mode design problem, becomes that $CB$ must have rank $m$. If this rank condition holds and any invariant zeros of the triple $(A, B, C)$ lie in the unit disk, then the existence of a matrix $F$ defining the surface (3), which provides a stable sliding motion with a unique equivalent control is determined from the stabilisability by output feedback of a specific, well-defined subsystem of the plant [Edwards and Spurgeon, 1998]. The aim here is to extend the existing results such that a sliding mode control based on output measurements can be designed for the system (1)-(2) when $rank(CB) < m$ and/or unstable invariant zeros are present in the triple $(A, B, C)$.

An extended output matrix $\bar{C}$ is then constructed without any a priori assumptions on the system (1)-(2) relating to the stability of the invariant zeros :

$$ \bar{C} = \begin{bmatrix} C_1^T & \cdots & C_p^T \end{bmatrix} (C_1 A^{-\mu_1 - 1} \cdots C_p A^{-\mu_p - 1})^T \quad (5) $$

such that $\bar{C}$ is full rank, $rank(\bar{C} B) = rank(B)$, and any invariant zeros of the triple $(A, B, \bar{C})$ lie inside the unit disk. Also, the $\mu_i$ are chosen such that $\tilde{p} = \sum_{i=1}^p \mu_i$ is minimal. Note that $\bar{C} = C \Rightarrow \mu = 0$ means that the original system is output feedback stabilisable using existing methods (see the work of Edwards and Spurgeon 1998). It is now important to show the relation between the invariant zeros of the original system triple $(A, B, C)$ and the invariant zeros of the augmented system triple $(A, B, \bar{C})$. It has been shown by Govindaswamy et al. [2008], that any invariant zeros of the triple $(A, B, \bar{C})$ are amongst the invariant zeros of the triple $(A, B, C)$. It has also been shown that if an appropriate choice of extended outputs is available (see for example Govindaswamy et al. [2009]), the invariant zeros from the original triple $(A, B, C)$ may disappear from the augmented system $(A, B, \bar{C})$. This is particularly useful if any of the invariant zeros of $(A, B, C)$ are unstable. Note that by augmenting the number of outputs to the dimension of the state, the transmission zeros may all be removed if the system is observable. However, the objective of this paper is to augment the outputs only sufficiently so that $\bar{C} B$ is full rank and any transmission zeros are stable. In general it is not required, and indeed may not be desirable, to augment the outputs to the dimension of the state; the greater the number of states the greater the complexity of the resulting controller. The next section will develop and analyze the sliding surface and the resulting sliding mode dynamics.

3. THE EXISTENCE PROBLEM

Assume that it is possible to choose $\bar{C}$ as stated in (5). The problem now becomes one of finding a suitable sliding variable $s_k$ that is a function of the augmented outputs only. Choose a set of past outputs along with the present outputs, and form a new extended output vector $\tilde{y}_k$ as shown below:

$$ \tilde{y}_k = \begin{bmatrix} (y_1)_k \cdots (y_p)_k \cdots (y_1)_{k-\mu_1 - 1} \cdots (y_p)_{k-\mu_p - 1} \end{bmatrix}^T \quad (6) $$

From the system (1)-(2), the above vector $\tilde{y}_k$, can be computed as (see for example Govindaswamy et al. [2008, 2009]):

$$ \tilde{y}_k = \bar{C} x_k - \begin{bmatrix} 0_{p \times l_i} & M_{d(\tilde{p}-p) \times q_1} \end{bmatrix} U_k - \begin{bmatrix} 0_{p \times q_1} \\ M_d(\tilde{p}-p) \times q_1 \end{bmatrix} D_k $$

with $l_i = m(\tilde{p} - p)$ and $q_1 = q(\tilde{p} - p)$.

$$ U_k = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_p^T \end{bmatrix}^T, \quad D_k = \begin{bmatrix} d_1^T & d_2^T & \cdots & d_p^T \end{bmatrix}^T \quad (7) $$

with $U_k \in \mathbb{R}^{l_i}$ and $D_k \in \mathbb{R}^{q_1}$, and for $i = 1, ..., p$ :

$$ \bar{u}_i = \begin{bmatrix} u_{i-1}^T & u_{i-2}^T & \cdots & u_{k-\mu_1 - 1} \end{bmatrix}^T \quad (8) $$

and

$$ \bar{d}_i = \begin{bmatrix} d_{i-1}^T & d_{i-2}^T & \cdots & d_{k-\mu_1 - 1} \end{bmatrix}^T \quad (9) $$

$$ M_i = diag\{M_{d1}, ..., M_{dp}\} \quad (10) $$

such that $\bar{C} = C \Rightarrow M_i = \begin{bmatrix} 0_{p \times l_i} & \cdots & 0_{p \times l_i} \end{bmatrix}$ and the matrix $M_{d4}$ is

$$ M_{d4} = \begin{bmatrix} C_i A^{-1} B & 0 & \cdots & 0 \\ C_i A^{-2} B & C_i A^{-1} B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_i A^{l_i - 1} B & \cdots & C_i A^{-2} B & C_i A^{-1} B \end{bmatrix} \quad (11) $$

with $M_{d4} \in \mathbb{R}^{(l_i - 1) \times (q(\mu_i - 1))}$ and the matrix $M_d$ is

$$ M_d = diag\{M_{d1}, ..., M_{dp}\} \quad (12) $$

and $M_{d4} \in \mathbb{R}^{(l_i - 1) \times (q(\mu_i - 1))}$. Let

$$ \Gamma_u = \begin{bmatrix} 0_{p \times l_i} \\ M_{d(\tilde{p}-p) \times l_i} \end{bmatrix} \quad \text{and} \quad \Gamma_d = \begin{bmatrix} 0_{p \times q_1} \\ M_d(\tilde{p}-p) \times q_1 \end{bmatrix} \quad (13) $$

Defining the sliding manifold in terms of known variables gives:

$$ s_k = F \tilde{y}_k + \Gamma_u U_k = F \bar{C} x_k - \Gamma_d D_k \quad (14) $$

Now to analyze the stability of the sliding motion, it is convenient to introduce a coordinate transformation $x \rightarrow Tx = \bar{x}$ with the following structure [Edwards and Spurgeon, 1998]:

$$ \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \bar{B}_d = \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix} \quad (15) $$

$$ \bar{C} = \begin{bmatrix} 0 & T \end{bmatrix} \quad (16) $$
where $T \in \mathbb{R}^{\tilde{p} \times \tilde{p}}$ is an orthogonal matrix. The matrix \( \hat{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)} \) and has the following structure

\[
\hat{A}_{11} = \begin{bmatrix}
\hat{A}_{11}^0 & \hat{A}_{11}^T_2 & \hat{A}_{11}^M_2 \\
0 & \hat{A}_{22}^0 & \hat{A}_{22}^T \\
0 & 0 & \hat{A}_{22}^T
\end{bmatrix}
\]

with \( \hat{A}_{11}^0 \in \mathbb{R}^{r \times r} \), \( \hat{A}_{22}^0 \in \mathbb{R}^{(n-p-r)\times(n-p-r)} \) and \( \hat{A}_{22}^T \in \mathbb{R}^{(r-m)\times(n-p-r)} \) for some \( r \geq 0 \) and the pair \((\hat{A}_{22}^0, \hat{A}_{22}^T)\) is completely observable. The remaining sub-blocks in the system matrix are partitioned accordingly. The corresponding switching surface matrix is given by

\[
[\hat{F}_1, \hat{F}_2] = FT
\]

where \( T \) is the matrix from equation (13). As a result

\[
F\hat{C} = [\hat{F}_1 C_{\hat{F}_1} F_2]
\]

where

\[
C_{\hat{F}} = \begin{bmatrix} 0_{(\tilde{p}-m)\times(\tilde{p}-p)} \end{bmatrix}
\]

Therefore \( F\hat{C} \hat{B} = F_2 B_2 \) and the square matrix \( F_2 \) is nonsingular. The disturbance distribution matrix \( \hat{\Gamma}_d \) in the new coordinates \( \hat{\Gamma}_d = \Gamma_d \) is of the form

\[
\begin{bmatrix}
\hat{\Gamma}_{d_1} & \hat{\Gamma}_{d_2}
\end{bmatrix} = \Gamma_d
\]

Consider the nominal plant for the system (1)-(2), when \( d_k = 0 \), in the new co-ordinate system. The canonical form in (13) can then be viewed as a special case of the regular form normally used in sliding mode controller design, and it can be shown that the reduced order sliding motion in this co-ordinate system is governed by a free motion with system matrix

\[
\hat{A}_{11}^* = \hat{A}_{11} - \hat{A}_{12} K C_{\hat{F}}
\]

which must therefore be stable and the gain \( K \in \mathbb{R}^{n\times\tilde{p}} \) is defined as \( K = F_2^{-1} F_1 \). However, for the system (1)-(2) with uncertainty, the sliding surface (12) has a disturbance term acting on it explicitly and hence it is clear that the sliding motion as given in (18) is not possible. Hence, it is now pertinent to show that the reduced order motion in the presence of disturbance will be confined within a region around the sliding surface \( \hat{s} \). To show this, let:

\[
\hat{s}_k = 0
\]

so that

\[
(\hat{x}_2)_{k+1} = F_2^{-1} F_1 C_{\hat{F}} (\hat{x}_1)_{k} + \hat{B}_{d_1} \hat{d}_k
\]

Substituting for \( (\hat{x}_2)_{k} \) in \( (\hat{x}_1)_{k+1} \), one can obtain

\[
(\hat{x}_1)_{k+1} = \hat{A}_{11}^* (\hat{x}_1)_{k} + \hat{A}_{12} F_2^{-1} \hat{\Gamma}_d F_2 \hat{d}_k + \hat{B}_{d_1} \hat{d}_k
\]

The solution for \( (\hat{x}_1)_{k+1} \) is:

\[
(\hat{x}_1)_{k} = \hat{A}_{11}^* (\hat{x}_1)_{0} + \sum_{j=0}^{k-1} \hat{A}_{11}^* (\hat{A}_{12} F_2^{-1} \hat{\Gamma}_d F_2 \hat{d}_j + \hat{B}_{d_1} \hat{d}_j)
\]

and thus

\[
\|\hat{x}_1\|_k \leq \|\hat{A}_{11}^*\| \|\hat{x}_1\|_0 + \sum_{j=0}^{k-1} \|\hat{A}_{11}^*\| \|\hat{A}_{12}\| \|\hat{\Gamma}_d\| \|\hat{d}_j\| + \|\hat{B}_{d_1}\| \|\hat{d}_j\|
\]

If \( \hat{A}_{11}^* \) is designed by the choice of \( F_1 \) such that it has eigenvalues within the unit circle, then it can be shown that

\[
\|\hat{A}_{11}^*\| \leq \gamma \lambda^k
\]

where \( \gamma > 0 \) and \( 0 < \lambda < 1 \). It can further be shown that:

\[
\sum_{j=0}^{k-1} \|\hat{A}_{11}^*\| \leq \gamma \sum_{j=0}^{k-1} \lambda^j \leq \gamma \frac{1}{1-\lambda}
\]

Hence bounds on \( \|\hat{x}_1\|_k \) can be written as:

\[
\|\hat{x}_1\|_k \leq \gamma \lambda^k \|\hat{x}_1\|_0 + \gamma \frac{1}{1-\lambda} \|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \|\hat{d}_j\| + \|\hat{B}_{d_1}\| \|\hat{d}_j\|
\]

If \( \|\hat{D}_j\| < \rho_2 \), for some scalar \( \rho_2 > 0 \) then the reduced order sliding motion will be uniformly ultimately bounded by

\[
\sum_{j=0}^{k-1} \|\hat{A}_{11}^*\| \|\hat{A}_{12}\| \|F_2^{-1}\| \|\hat{\Gamma}_d\| \|\hat{d}_j\| + \|\hat{B}_{d_1}\| \|\hat{d}_j\|
\]

Note that, for the existence of a stable bounded sliding motion, \( \hat{A}_{11}^* = \hat{A}_{11} - \hat{A}_{12} K C_{\hat{F}} \) must be stable, which implies that the pair \((\hat{A}_{11}, \hat{A}_{12})\) is controllable and \((\hat{A}_{11}, C_{\hat{F}})\) is observable. The former is ensured as \((\hat{A}, \hat{B})\) is controllable.

The observability of \((\hat{A}_{11}, C_{\hat{F}})\), is not so straightforward. To show that, define the matrix \( \hat{A}_{122} \) as the bottom \((n-m-r)\) rows of the matrix \( \hat{A}_{12} \) and the matrix \( \hat{A}_{11} \) as the corresponding \((n-m)\) rows and the last \((n-m-r)\) columns of the matrix \( \hat{A}_{11} \). If \( \hat{C}_{\hat{F}} \) is the last \((n-m-r)\) columns of \( C_{\hat{F}} \) then the triple \((\hat{A}_{11}, \hat{A}_{122}, \hat{C}_{\hat{F}})\) is both observable and controllable and the spectrum of \( \hat{A}_{11}^* \) represents the invariant zeros of the triple \((\hat{A}, \hat{B}, \hat{C})\) [Edwards and Spurgeon, 1998]. A number of approaches to design the sliding surface by establishing a static output feedback gain are available, for example Mehdi et al. [2004], Arzelier et al. [2003]. The technique in Mehdi et al. [2004] introduces slack variables to decouple the Lyapunov matrix and the static output feedback gain. With the additional slack variables and a chosen state space variable, an LMI problem is solved to obtain the static output feedback controller. The following theorem is required to formulate the design of the sliding surface using this method.

**Theorem 1.** Let the matrix \( A_0 \) be defined as \( A_0 = \hat{A}_{11} + \hat{A}_{122} K_0 \). A static output feedback gain \( K \) is stabilising for the triple \((\hat{A}_{11}, \hat{A}_{122}, \hat{C}_{\hat{F}})\) if and only if there exist a positive definite matrix \( \Xi = \Xi^T > 0 \) in \( \mathbb{R}^{n \times n} \), non-singular matrices \( G_1 \in \mathbb{R}^{m \times n} \) and \( E_4 \in \mathbb{R}^{n \times n} \), non-null matrices \( E_1 \in \mathbb{R}^{n \times n} \) and \( L \in \mathbb{R}^{m \times p} \) and arbitrary matrices \( E_2 \in \mathbb{R}^{n \times n} \), \( E_3 \in \mathbb{R}^{m \times n} \) such that the following LMI:

\[
\begin{bmatrix}
E_1 A_0 + A_0^T E_1^T & -\Xi & \ast & \ast \\
E_2 A_0 & -\Xi & \ast & \ast \\
E_3 A_0 + A_0^T E_3^T + (L\hat{C}_{\hat{F}} - G_1 K_0) A_0^T E_3^T & E_4 A_0 - E_1^T & \ast & \ast \\
E_4 A_0 - E_1^T & \ast & \ast & \ast \\
E_3 A_122 + A_0^T E_3^T - (G_1 + G_1^T) & E_4 A_122 - E_1^T & \ast & \ast \\
E_4 A_122 - E_1^T & \ast & \ast & \ast \\
\end{bmatrix} < 0
\]
is feasible for a given state feedback gain $K_o$ that stabilises the pair $(\hat{A}_{11}, \hat{A}_{122})$ with the static output feedback gain given by

$$K = G^{-1} L \tag{26}$$

For a conclusive proof of the above theorem, refer to [Mehdi et al., 2004]. The section below will now consider selection of a suitable control law that would stabilise the closed loop dynamics of the plant.

4. CLOSED LOOP ANALYSIS AND REACHABILITY PROBLEM

To analyze the stability of the closed loop, introduce a nonsingular state transformation of the system $\dot{x} \rightarrow \hat{T}\hat{x}$ where $\hat{T}$ is given as :

$$\hat{T} = \begin{bmatrix} I_{n-m} & 0 \\ -KC_f & I_m \end{bmatrix} \tag{27}$$

where the matrices $I_{n-m}$ and $I_m$ are identity matrices of dimension $n - m$ and $m$ respectively and the matrix $KC_f \in \mathbb{R}^{m \times (n-m)}$. With the above transformation, the system (1)-(2) is of the form:

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \hat{B}_d = \begin{bmatrix} \hat{B}_{d1} \\ \hat{B}_{d2} \end{bmatrix}, \hat{F} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \tag{28}$$

Here, $\hat{A}_{11} = \hat{A}_{111}$ with $\hat{A}_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ and the rest of the sub-matrices in the matrix $\hat{A}$ are conformably partitioned. The control law $u_k$ is chosen here as a function of the known signals $\bar{y}_k$ and $U_k$ and has the form :

$$u_k = -(F\hat{C}\hat{A}^{-1}B)^{-1}s_k \tag{29}$$

In the new coordinate system given in (28) the corresponding control $\bar{u}_k$ is:

$$\bar{u}_k = -(F\hat{C}\hat{A}^{-1}B)^{-1}s_k \tag{30}$$

Now, define the disturbance input $w_k \in \mathbb{R}^{(q+n)}$ such that $\bar{w}_k = \begin{bmatrix} d_k^T \\ \hat{D}_k^T \end{bmatrix}$. Define a new output variable

$$\bar{Y}_k = \bar{y}_k - \begin{bmatrix} 0_{p \times t_1} \\ M_{(p-\hat{p}) \times t_1} \end{bmatrix} \bar{U}_k = \hat{C}\bar{x}_k - \hat{\Gamma}_1 w_k \tag{31}$$

with $\bar{Y}_k \in \mathbb{R}^p$ and where the matrix $\hat{\Gamma}_1$ is such that

$$\hat{\Gamma}_1 = \begin{bmatrix} 0_{p \times q} \\ 0_{(p-\hat{p}) \times \hat{q}} \end{bmatrix} \hat{M}_{\hat{d}(\hat{p}-\hat{p}) \times \hat{q}} \tag{32}$$

Define the matrix $\hat{B}_{w} \in \mathbb{R}^{n \times (q+n)}$ such that $\hat{B}_w = [\hat{B}_d 0]$ and let $\hat{B}_D = B(F\hat{C}\hat{A}^{-1}B)^{-1}[0 \ \hat{\Gamma}_1] + \hat{B}_w$. Now consider the closed loop system formed from the system (28) with the control (30)

$$\begin{align*}
\dot{\bar{x}}_{k+1} &= \hat{A}_{cl}\bar{x}_k + \hat{B}_D\bar{w}_k \\
\bar{Y}_k &= \hat{C}\bar{x}_k - \hat{\Gamma}_1 w_k \\
\bar{z}_k &= \begin{bmatrix} \bar{Y}_k \\ \bar{s}_k \end{bmatrix} \tag{33}
\end{align*}$$

with $\hat{A}_{cl}$ stable by design and where $\bar{z}_k$ is a performance output. The transfer function of the closed loop system with the disturbance input $\bar{w}_k$ and the performance output $\bar{z}_k$ is given as

$$T_{zw} = \begin{bmatrix} \hat{C} \\ F\hat{C} \end{bmatrix} ((zI - \hat{A}_{cl})^{-1}\hat{B}_D - \hat{\Gamma}_1)$$

The objective here is to minimize the $H_2$ norm $\|T_{zw}\|_2$ of the closed loop system with the disturbance input $\bar{w}_k$ and performance output $\bar{z}_k$ such that the $\|T_{zw}\|_2$ is less than some prescribed positive value. Let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix that satisfies

$$0 = P - A_{cl}^T P A_{cl} - C^T \hat{C} - C^T F^T \hat{C} \tag{34}$$

Then the norm $\|T_{zw}\|_2$ (see for example Chen and Francis [1994]) can be computed as

$$\|T_{zw}\|_2^2 = trace(B_D^T P B_D + \hat{\Gamma}_1^T \hat{\Gamma}_1 + \hat{\Gamma}_1^T F^T \hat{C}) \tag{35}$$

The following theorem can now be stated to compute the $H_2$ norm for $T_{zw}$

**Theorem 2.** The system (31)-(33) is stable with $\|T_{zw}\|_2 < \delta$, with $\delta > 0$, if and only if there exists positive definite matrices $\hat{P} \in \mathbb{R}^{n \times n}, \hat{W} \in \mathbb{R}^{(q+n) \times (q+n)}$ and a matrix $\hat{F}_2 \in \mathbb{R}^{n \times m}$ such that the following set of LMI are satisfied

$$trace(\hat{W}) < \delta, \begin{bmatrix} \hat{P} & \hat{P} \hat{B}_D & \hat{P} \hat{B}_2 & 0 \\ \hat{B}_D^T \hat{P} & 0 & 0 & 0 \\ \hat{B}_2^T \hat{P} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} > 0 \tag{36}$$

$$\begin{bmatrix} \hat{P} & 0 & 0 & 0 \\ \hat{P} \hat{A}_{cl} & 0 & 0 & 0 \\ \hat{A}_{cl}^T \hat{P} & \hat{P} & \hat{C}^T \hat{C} & \hat{C}^T F^T \hat{C} \tag{37}$$

The first inequality can be obtained by letting (35) less than some positive definite matrix $\hat{W}$ and then applying the Schur complement. The second inequality can be obtained by replacing the equality sign with an inequality in (34) and then applying the Schur complement. Note that the matrix $\hat{F}_2$ here is unknown. To compute the matrix $\hat{F}_2$, solve the set of linear matrix inequalities (36)-(37) with $\hat{P}, \hat{W}$ and $\hat{F}_2$ as the decision variables.

**Remark 1** The matrix $\hat{F}_2$ does not affect the control $\bar{u}_k$. By simple algebraic manipulation it can be shown that:

$$\bar{u}_k = -(F\hat{C}\hat{A}^{-1}B)^{-1}F\bar{Y}_k \tag{38}$$

$$= -([K_{lm}]\hat{C}\hat{A}^{-1}B)^{-1}([K_{lm}]\bar{Y}_k) \tag{39}$$

Hence the claim is proved. Now using the control law (30), the closed loop state space matrix for the nominal system (28) can be computed as:

$$\hat{A}_{cl} = \hat{A} - \begin{bmatrix} 0 \\ B_2 \end{bmatrix} ([0 \ F_2] \hat{A}_1^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}) [0 \ F_2] \tag{40}$$

$$\hat{A}_{cl} = \hat{A} - \begin{bmatrix} 0 \\ I_m \end{bmatrix} ([A_{22} - A_{21}\hat{A}_1^{-1}A_{12}] \begin{bmatrix} 0 \\ I_m \end{bmatrix}) \tag{41}$$

$$\hat{A}_{cl} = \begin{bmatrix} \hat{A}_{11} \\ \hat{A}_{21} \hat{A}_1^{-1}A_{12} \end{bmatrix} \tag{42}$$

**Remark 2** The closed loop dynamics for the nominal plant (31)-(32) when $\bar{w}_k = 0$ is a reduced order dynamics. Introduce a new change of coordinates of the form $\bar{x} \rightarrow T_e \bar{x} = x^e$, where $T_e$ has the form

$$T_e = \begin{bmatrix} I_{n-m} & -\hat{A}_{11}^{-1}A_{12} \\ 0 & I_m \end{bmatrix} \tag{43}$$

The closed loop matrix $\hat{A}_{cl}$ in the new co-ordinates can be written as:

$$\hat{A}_{cl}^e = \begin{bmatrix} \hat{A}_{11} \\ \hat{A}_{21} \hat{A}_1^{-1}A_{12} \end{bmatrix} \tag{44}$$
\[ A_{cl} = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{12}^{-1}\bar{A}_{21} & 0 \\ \bar{A}_{21} \end{bmatrix} \]  
(44)

The closed loop eigenvalues in this case are clearly seen to be the eigenvalues of the matrix \( A_{11} + A_{12}^{-1}A_{21} \). Hence stability of the closed loop system implies the stability of the matrix \( A_{11} + A_{12}^{-1}A_{21} \). It is now necessary to show that the control (30) forces the state trajectories towards the sliding surface \( s_k \) for any initial condition and that the sliding dynamics are bounded within a region around the sliding surface. The section below will now show the existence of a sliding mode.

### 4.1 The Reachability Problem

To show that the control law (30) forces the states of the uncertain system to the sliding surface \( s_k \) and maintain an ideal sliding motion, consider \( s_{k+1} \).

\[ \dot{s}_{k+1} = F\bar{C}\dot{A}_d\bar{x}_k + F\bar{C}\bar{B}(F\bar{C}^{-1}\bar{B})^{-1}\bar{C}_d\dot{D}_k - \bar{C}_d\bar{D}_{k+1} + F\bar{C}\bar{B}_d\bar{d}_k \]  
(45)

Define the matrix \( \phi \) such that \( \phi = F_2\bar{A}_2\bar{A}_1^{-1}\bar{A}_{12}\bar{F}_2^{-1} \), and let the matrix \( \phi_1 = F_2(\bar{A}_{22} - \bar{A}_{11}\bar{A}_{12}\bar{F}_2^{-1}) \). The term \( F\bar{C}\dot{A}_d\bar{x}_k \) can be written as

\[ F\bar{C}\bar{A}_d\dot{x}_k = F_2\bar{A}_2(\bar{x}_1)_k + \phi \dot{s}_k \]

Hence \( s_{k+1} \) can be written as

\[ s_{k+1} = F_2\bar{A}_2(\bar{x}_1)_k + \phi \dot{s}_k + \phi_1 \bar{C}_d\dot{D}_k - \bar{C}_d\bar{D}_{k+1} + F\bar{C}\bar{B}_d\bar{d}_k \]  
(46)

The solution for the above difference equation is given as:

\[ \dot{s}_k = \phi^k \dot{s}_0 + \sum_{j=0}^{k-1} \phi^{k-j-1}(F_2\bar{A}_2(\bar{x}_1)_j + \phi_1 \bar{C}_d\dot{D}_j - \bar{C}_d\bar{D}_{j+1} + F\bar{C}\bar{B}_d\bar{d}_j) \]  
(47)

For the nominal case, it can be seen that if \( \phi \) is a stable matrix, then \( \|s_k\| \rightarrow 0 \) as \( k \rightarrow \infty \) and the first term in (47) tends towards zero for a large enough \( k \), as \( \bar{x}_k \) to 0 by design for a large enough \( k \). In the presence of disturbance, it would be required to show that \( \|s_k\| \) would be bounded. For that consider the term \( \sum_{j=0}^{k-1} \phi^{k-j-1} \) in (47). If the matrix \( \phi \) is non-singular and has eigenvalues \( \lambda_0 \), of magnitude 0 < \( \lambda_0 \) < 1 and if \( \|\phi\| < 1 \), then \( \sum_{j=0}^{k-1} \|\phi^{k-j-1}\| \leq \sum_{j=0}^{\infty} \|\phi\|^j = (1 - \|\phi\|)^{-1} \). Hence, taking norm on both sides of (47) gives,

\[ \|s_k\| \leq \bar{\Lambda}(\phi)^k \|\dot{s}_0\| + \sum_{j=0}^{k-1} \|\phi^{k-j-1}\| \|\bar{C}_d\bar{D}_j - \bar{C}_d\bar{D}_{j+1} + F\bar{C}\bar{B}_d\bar{d}_j\| \]

where \( \bar{\Lambda}(\phi) \) is the maximum eigenvalue of \( \phi \) and denotes the spectral radius of \( \phi \). Hence

\[ \|s_k\| \leq \bar{\Lambda}(\phi)^k \|\dot{s}_0\| + \sum_{j=0}^{\infty} \|\phi\|^j \|\bar{C}_d\bar{D}_j - \bar{C}_d\bar{D}_{j+1} + F\bar{C}\bar{B}_d\bar{d}_j\| \]

If Theorem 2 holds, then the term in the above equation that depends on \( \|\dot{x}\| \) will be uniformly bounded. The term \( \|\dot{d}_j\| < \rho_0 \) by assumption and if \( \|\bar{C}_d\bar{D}_j\| < \rho_2 \) then from (50), it can be seen that \( \|s_k\| \) will be uniformly ultimately bounded by the right hand side of (50). In the case when \( \|\phi\| > 1 \), assume that the eigenvalues of \( \phi \) are distinct. Then the matrix \( \phi \) can be decomposed as \( \phi = U_\phi A_\phi U_\phi^{-1} \), where the matrix \( U_\phi \) is an orthogonal matrix. The above analysis can then be performed by considering the fact that \( \sum_{j=0}^{k-1} \|\phi^{k-j-1}\| = \|\sum_{j=0}^{k-1} U_\phi A_\phi^{k-j-1}U_\phi^{-1}\| \leq \|U_\phi\| \sum_{j=0}^{\infty} \|A_\phi\|^j \|U_\phi^{-1}\| = \|U_\phi\| \sum_{j=0}^{\infty} (1 - \|A_\phi\|)^{-1} \|U_\phi^{-1}\| \), since \( \|A_\phi\| < 1 \) by design. It is clear from (50) that the properties of \( \phi \) and \( F_2 \) determine how close the system dynamics are to the ideal sliding mode dynamics. From the definition of \( \phi \), it can be seen that the value \( \phi \) is dependent on \( F_2 \) for a given stabilising gain \( K \). Hence to minimize the effect of the disturbance on \( s_k \), \( F_2 \) can be designed by constraining its norm to be less than some positive value when solving the linear matrix inequalities (36)-(37). The above control design paradigm will now be applied to an example problem in the below section to show its effectiveness.

### 5. MOTIVATIONAL EXAMPLE

Consider the state space representation of the plant given in Govindaswamy et al. [2008] below:

\[ A = \begin{bmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0, 0, 1] \]  
(51)

The disturbance transmission matrix here is taken as

\[ B_d = [0.1, 0]^T \]

It is assumed that \( |a| < 1 \) and \( |b| > 1 \). Note that this system is not output feedback stabilisable using a static output feedback control law and the existence problem cannot be solved for the plant with the given output. Also, note that the invariant zeros for the system triple (51) are at \( a \) and \( b \). Here \( a \) and \( b \) were chosen as \( 0.1 \) and \( -1.2 \) respectively. The original system thus possesses an unstable transmission zero at \( -1.2 \). Now construct the augmented matrix \( \tilde{C} \) by extending the output once. The augmented matrix \( \tilde{C} \) in this example is

\[ \tilde{C} = \begin{bmatrix} 0 & 0 & 1.0000 \\ 0 & 1.0000 & 1.2000 \end{bmatrix} \]  
(52)

The invariant zero for the triple \((A, B, \tilde{C})\) is at \(-1.2\) which is again an invariant zero of the triple \((A, B, C)\). Here the orthogonal transformation matrix \( T \) is

\[ T = \begin{bmatrix} -0.7682 & -0.6402 \\ 0.6402 & -0.7682 \end{bmatrix} \]

and the subsystem \((\tilde{A}_{11}, \tilde{A}_{122}, \tilde{C}_f)\) obtained for the sliding surface design is \( \tilde{A}_{11} = -1.6918, \tilde{A}_{122} = -0.4098 \) and
The control input $u_k$, sliding surface $s_k$ and the plant output $y_k$

$\bar{C}_f = 1$. The initial state feedback gain $K_0$ is obtained by pole placement and the poles are placed at -0.985. Using the algorithm given by Mehdi et al. [2004], the gain $K$ for the reduced order subsystem ($\bar{A}_{11}, \bar{A}_{12}, \bar{C}_f$) is calculated as $K = -6.0289$ with $\lambda(\bar{A}_{11} - \bar{A}_{12}K\bar{C}_f)$ at 0.7790. The plant was then transformed to the canonical form given in (28). The matrices $P$, $W$ and $F_2$ were computed by applying the LMI solver to the feasibility problem (36)-(37) and were obtained as:

$$P = \begin{bmatrix} 0.3837 & 0.2029 & 0.7098 \\ 0.2029 & 51.3205 & -12.7841 \\ 0.7098 & -12.7841 & 6.7797 \end{bmatrix}$$

$$W = \begin{bmatrix} 6.7275 & -0.0943 \\ -0.0943 & 6.8625 \end{bmatrix}$$

and $F_2 = 0.1114$ respectively. The trace($W$) can be computed as 13.500. The output feedback gain $F$ is then obtained as $F_2 \begin{bmatrix} K \\ I \end{bmatrix} T^T = \begin{bmatrix} -0.5872 \\ 0.3444 \end{bmatrix}$. Here the matrices $\phi$, $\phi_1$ and $\Gamma_d$ that affect the sliding dynamics are found to be equal to $-0.6954$, $-1.2836$ and $[0 \ 0.3091]$ respectively. The closed loop eigenvalues are $[0.1000 \ 0.0836 \ 0.0000]$. The simulations were performed with the output feedback gain in (30) and with the initial conditions $[0.1 \ 0 \ 0]^T$. The disturbance chosen here is of the form $\dfrac{1}{2}(\sin(y(k)) + y(k-1))^2 \|y_k\|$ which in this case is bounded by 4.374. The sampling time here is taken to be $0.01$ s and the simulation results are shown in Figure 1.

Fig. 1. The control input $u_k$, sliding surface $s_k$ and the plant output $y_k$

6. CONCLUSION

An output feedback based sliding mode control design for discrete time systems has been developed in this paper. It has been shown that discrete time controllers can be realized via the extended outputs for non square systems with uncertainties. The conditions for the existence of a sliding mode have been given. A procedure for synthesizing a control law has been given. The control law has been chosen such that the norm of a particular closed loop transfer function is minimized. The control law has then been applied on a motivational problem in the presence of uncertainties to show its effectiveness.

REFERENCES


