Comparative Analysis of Stability and Robustness between Integer and Fractional-Order PI Controllers for First Order plus Time Delay Plants

F.J. Castillo-Garcia*, V. Feliu-Batlle*, R. Rivas-Perez** and L. Sanchez

*School of Industrial Engineering, Univ. of Castilla-La Mancha, Ciudad Real 13071, Spain (Tel. 0034 926295300; e-mail: fernando.castillo@uclm.es, vicente.feliu@uclm.es)
** Department of Autom. and Computer Science, Havana Polytechnic Univ., CUJAE, Marianao, C. Habana 19390, Cuba (e-mail: rivas@electrica.cujae.edu.cu)
***School of Industrial Engineering, Univ. of Castilla-La Mancha, Toledo 45071, Spain (Tel. 0034 925 268800; e-mail: luis.sanchez@uclm.es)

Abstract: This work carries out a comparative analysis of the stability robustness of PI and $P^\alpha$ controllers when applied to first order plus time delay plants. An analytical result shows that both controllers have exactly the same region of feasible frequency specifications. Nevertheless, the robustness of both controllers is quite different. Depending on the set of frequency specifications and the non integer order of the integral action of the fractional-order controller, the $P^\alpha$ controller can or not provide higher robustness to plant parameters variations than the PI controller. The regions where each controller is most robust are calculated in this paper. Two simulated examples illustrate this opposite robustness behaviour of the $P^\alpha$ controller with respect to the PI controller.

Keywords: Fractional order controller, frequency specifications, time delay, stability, robustness.

1. INTRODUCTION

First-order plus time delay plants are very often used to model industrial processes like chemical, thermal, diffusion, mechanics, viscoelasticity or transport processes (Artstein Z., 1982; Richard, 2003; Stephanopoulos, 1984).

PID (proportional integral derivative) conventional controller is the most popular control strategy used in industry because of its simplicity, performance robustness, and the availability of many effective and simple tuning methods based on minimum plant model knowledge (Ziegler et al., 1942). Some surveys have shown that the 90% of the industrial control loops belong to PI or PID structures (Koivo et al., 1991; Yamamoto et al., 1991).

On the other hand, fractional calculus is a mathematical tool that has found application in the subject of automatic control in recent years (Oustaloup, 1991). The generalization of the conventional PI, PD and PID industrial controllers allows us to get more sophisticated controllers that use fractional-order derivatives and integration operators, as $P^\alpha$, $P D^\beta$ and $P^\alpha ID^\beta$.

It’s well known that the use of fractional-order regulators to control integer order plants may improve the performance and robustness of the controlled systems (Monje et al., 2004; Oustaloup et al., 2006; Podlubny et al., 1999).

In fact, a lot of research about improving the robustness of control systems using fractional-order controllers has been carried out in the last years, e.g.: Monje et al., 2004; Chen et al., 2004, 2006; Pommier-Budinger et al., 2008; Lu et al., 2010. But a systematic analysis of the improvements that can be achieved in controlling first order plus time delay plants is still missing.

This paper presents a comparative study of the stability and robustness of PI and $P^\alpha$ controllers. Analytic results show how both controllers have the same region of feasible frequency specifications but their robustness is clearly different. Depending on the frequency specifications and the value of $\alpha$, the $P^\alpha$ controller can provide higher or lower robustness than the PI controller to the closed loop system.

The paper is organized as follows. Section 2 carries out a comparative stability analysis between PI and $P^\alpha$ controllers when they are used to control a first order plus time delay plant. Section 3 presents the robustness analysis of both controllers. Section 4 develops two simulated examples which illustrate both cases: when the $P^\alpha$ provides more robustness than the PI controller, and when the PI is more robust than the $P^\alpha$ controller. Finally, Section 5 resumes the main conclusions obtained during the development of this work.

2. STABILITY ANALYSIS

1 This research was supported by the Spanish Government research Programme Project DPI2009-09956 (MCyT), by the Junta de Comunidades de Castilla-La Mancha project PCI-08-0135; and the European Social Fund.
2.1 Plant and controllers.

All the work presented in this paper is focused on controlling first order plus delay (FOPD) plants. Their transfer functions can be written as:

\[ G_p(s) = \frac{K_p}{T_p s + 1} e^{-L_s}, \]  

(1)

where \( K_p, T_p \) and \( L_s \) are the nominal parameters of the plant: gain, constant time and time delay, respectively.

In order to obtain generic results, expression (1) is normalized by scaling the time \( t \) by \( T_0 = t/T_0 \) and the process output \( y \) by \( K_0 = y/K_0 \). Then a normalized transfer function yields:

\[ G_{\text{on}}(s) = \frac{1}{s + 1} e^{-L_s'}, \]  

(2)

where \( L_{\text{on}} \) is the nominal normalized time delay (\( L_{\text{on}} = L_0/T_0 \)).

Assuming parameters variations in (1) and (2) the transfer functions can be written as:

\[ G(s) = \frac{K}{T s + 1} e^{-L_s} \quad \text{and} \quad G_u(s) = \frac{K}{T_n s + 1} e^{-L_s'}. \]  

(3)

A conventional unity feedback control scheme is used to control (2), as shown in Fig. 1.

![Fig. 1. Closed loop control scheme.](image)

In this figure \( R(s) \) may be a conventional PI controller or a fractional-order \( \text{PI}^\alpha \) controller. Both controllers can be expressed as:

\[ R(s, \alpha) = \frac{K_p}{s^\alpha} + \frac{K_i}{s} \]  

(4)

where \( K_p \) and \( K_i \) are the proportional and integral gains of the controllers and \( \alpha \) is the non-integer order of the integral action (\( 0 < \alpha < 2 \)). Note that a particular case of (4) is the PI controller, \( R(s, 1) \).

Note that \( \text{PI}^\alpha \) controller has an extra degree of freedom that PI has, so an \( \alpha \) value could be selected in order to provide an additional specification, e.g., to maximize robustness against plant parameters changes.

2.2 Frequency tuning of the controllers

Due to the integral action of both controllers a zero steady state error is ensured if the controllers provide a stable behaviour.

In order to tune the velocity and overshoot of the time response of the normalized plant (2) to a step command, two frequency specifications are designed: the gain crossover frequency, \( \omega_c \), in relation to the time response velocity, and the phase margin, \( \phi_m \), in relation to the time response overshoot.

The design equations, that allow obtaining the controller parameters fulfilling the two before frequency specifications, can be expressed in a compact form as the complex equation:

\[ R(j \omega) G_{\text{on}}(j \omega_c) = e^{-j(\phi_m-\pi)}. \]  

(5)

If equation (5) is particularized for the \( \text{PI}^\alpha \) controller structure (4), the following expressions are obtained:

\[ K_p(\alpha) = \frac{1}{\sin (\frac{\pi}{2} \alpha)} \left[ \omega_c \sin (\phi_m + L_{on} \omega_c - \frac{\pi}{2}(1 - \alpha)) \right. \]
\[ \left. - \cos (\phi_m + L_{on} \omega_c - \frac{\pi}{2}(1 - \alpha)) \right] \]  

(6)

and

\[ K_i(\alpha) = \frac{\omega_c}{\sin (\frac{\pi}{2} \alpha)} \left[ \omega_c \cos (\phi_m + L_{on} \omega_c) + \sin (\phi_m + L_{on} \omega_c) \right] \]  

(7)

Particularising (6) and (7) for \( \alpha = 1 \) the PI controller parameters are:

\[ K_p(1) = \omega_c \sin (\phi_m + L_{on} \omega_c) - \cos (\phi_m + L_{on} \omega_c) \]  

(8)

and

\[ K_i(1) = \omega_c \left[ \omega_c \cos (\phi_m + L_{on} \omega_c) + \sin (\phi_m + L_{on} \omega_c) \right] \]  

(9)

2.3 Stability conditions

The fulfilment of equations (6)-(9) does not guarantee the stability of the controlled system neither in the PI or \( \text{PI}^\alpha \) controllers. Next additional conditions are obtained in order to guarantee stable control.

Suppose the open loop transfer function without the time delay term:

\[ G_{\text{on}}(s) = \frac{1}{s+1}, \]  

(10)

if we tune the controller \( R(s) \) in order to fulfil the frequency specification \( \phi_m \) and \( \omega_c \), then:

\[ R(j \omega_c) G_{\text{on}}(j \omega_c) = -e^{j\phi_m}. \]  

(11)

On the other hand, the open loop transfer function with the time delay term evaluated in the critically stable case is:

\[ R(j \omega_c) G_{\text{on}}(j \omega_c) e^{-jL_{on} \omega_c} = -1, \]  

(12)
so it can be easily deduced combining (11) and (12) that if the frequency tuning of \( R(s) \) for plant (10) provides a stable controlled system, the maximum allowed time delay before instability is produced yields:

\[
L_{\text{max}} = \frac{\phi_n}{\omega_c}. \tag{13}
\]

Under this point of view the stability of the controlled system can be analyzed taking into account the open loop transfer function without the time delay term, \( R(s)G'(s) \), and then obtaining from (13) the maximum delay allowed before the system turns unstable. Its Nyquist diagram is drawn in Figure 2. From the three cases shown in this figure, some conclusions can be stated about stability conditions.

\[
\omega_c \alpha + 1 + \omega_c \frac{K'(\alpha)}{K_p(\alpha)} \cos \left( \frac{\pi}{\alpha} \right) + \frac{K(\alpha)}{K_p(\alpha)} \sin \left( \frac{\pi}{\alpha} \right) = 0, \tag{18}
\]

which allows us to calculate \( \omega_c \). Condition (17) can be written as:

\[
K_p(\alpha) > -1 - \frac{\omega_c}{\tan \left( \frac{\pi}{\alpha} \right)}. \tag{19}
\]

It can be proven that (15) is more restrictive than (19) so the stability condition can be represented only by means of (15).

Although usual values of \( \omega_c \) and \( \phi_n \) lead to positive values of \( K_p(\alpha) \), low phase margin and low gain crossover frequency values may lead to negative values of \( K_p(\alpha) \), in both cases, integer and fractional-order PI controllers, so \( K_p(\alpha)<0 \) condition must be taken into account.

Note that (15) is independent of the value of \( \alpha \), so the stability condition for the \( P^\alpha \) controller is the same as the \( PI \) controller, and therefore both controllers exhibit the same region of feasible frequency specifications.

Figure 3 represents the feasible region for both controllers (15), where \( \xi = \phi_n + L_{0n}\omega_c \).

3. ROBUSTNESS ANALYSIS

3.1 Robustness definition

The robustness of a controller is defined in this paper by means of the ‘robustness region’. This is a 3D region in a 3D space whose coordinates are gain \( K \) (x axis), time constant \( T \) (y axis), and time delay \( L \) (z axis). The volume of this region will be used to compare the two studied controllers. Note that this is a normalized volume as we consider parameter variations in the normalized model (2), which are represented by \( G_n(\alpha) \) in (3).

Assume the normalized plant (2), then the \( PI \) and \( P^\alpha \) controllers can be tuned inside the region defined by expression (15). Inside this region, stability is guaranteed.
Once the controller parameters are calculated from (6), (7), or (8), (9), in the PI and PIα cases respectively, we will determine the corresponding robustness regions.

We have found that a practical way of calculating such region is obtaining the surface \( L_{\alpha}^{\text{max}} = f(K_{\alpha}, T_n) \), i.e. to obtain the maximum allowable delay (which does not destabilize the closed-loop system) for a given couple of values \( K_{\alpha} \) and \( T_n \). This representation of the controller robustness will be used to compare both controllers PI, and PIα in function of the value of \( \alpha \).

The previous stability condition outputs two different cases:

\( K_{\alpha}(\alpha)>0 \) and \( K_{\alpha}(\alpha)>0 \): the system remains stable for \( 0 \leq K_{\alpha}<\infty \) and \( 0 \leq T_n<\infty \).

\( K_{\alpha}(\alpha)>0 \) and \(-1 - \omega_c/(\tan(\pi\alpha)/2) < K_{\alpha}(\alpha) < 0 \): the system remains stable for \( K < -1 / K_{\alpha}(\alpha) \).

Thus, the robustness volume will be defined over \( 0 \leq K_{\alpha}<\infty \) and \( 0 \leq T_n<\infty \) in case a), and over \( K_{\alpha} < -1 / K_{\alpha}(\alpha) \) in case b).

Case a) is much more interesting in order to obtain robust controllers than case b) because for any pair of values \( K_{\alpha}, T_n \), there exists an interval of time delays \( 0 \leq L_n \leq L_{\alpha}^{\text{max}} \) for which the closed loop system remains stable, while in case b) there are values of \( K_{\alpha} \) and \( T_n \) for which the system without delay is unstable and the aforementioned stable delay interval does not exist. Then from now on we will consider only case a).

As we demonstrated in the previous section, the maximum value of the time delay that keeps stable the closed-loop system is given by (13).

Therefore we need to determine \( \phi_{\alpha n} \) and \( \omega_c^* \) in order to obtain \( L_{\alpha}^{\text{max}} \) from (13), expression that also holds if \( G_{\alpha}(s) = K_{\alpha}(T_n s + 1) \) is used instead of \( G_{\alpha n}(s) \).

Considering the plant \( G_{\alpha}(s) \), the gain crossover frequency must fulfill:

\[
\left| \frac{K_{\alpha} \left( K_{\alpha}(\alpha) + (j \omega_c^*)^2 K_{\alpha}(\alpha) \right) }{(j \omega_c^*)^2 \left( 1 + j \omega_c^* T_n \right)} \right| = 1 .
\]  

The gain crossover frequency, \( \omega_c^* \), can not be obtained analytically from expression (20) because its rearrangement gives the equation:

\[
\left( \frac{\omega_c^*}{T_n} \right)^2 - \frac{2 K_{\alpha} \left( K_{\alpha}(\alpha) \cos \left( \frac{\pi}{2} \alpha \right) - \left( K_{\alpha}(\alpha)^2 / T_n \right) - K_{\alpha} \left( K_{\alpha}(\alpha) \right)^2 / T_n^2 \right)}{1 - \left( K_{\alpha}(\alpha)^2 / T_n \right)} = 0
\]

that must be solved by a numerical method.

Once \( \omega_c^* \) has been obtained, the phase margin can be easily calculated from:

\[
\phi_{\alpha n} = \frac{1}{K_{\alpha}(\alpha) \left( \frac{\omega_c^*}{T_n} \right)^2 \sin \left( \frac{\pi}{2} \alpha \right) - \left( K_{\alpha}(\alpha)^2 / T_n \right) - K_{\alpha} \left( K_{\alpha}(\alpha) \right)^2 / T_n^2} - \frac{\left( K_{\alpha}(\alpha)^2 / T_n \right) + K_{\alpha} \left( K_{\alpha}(\alpha) \right)^2 / T_n^2}{\left( \frac{\omega_c^*}{T_n} \right)^2 \cos \left( \frac{\pi}{2} \alpha \right) + \sin \left( \frac{\pi}{2} \alpha \right)} .
\]  

Thus, substituting the solution of (21), and (22), in (13), the maximum allowed value of the time delay, \( L_{\alpha}^{\text{max}} \), that keeps stable the closed-loop system is univocally defined. It can be observed that (21) and (22) depend on the value of \( \alpha \), so the ‘robustness volume’ can be noticeably increased by choosing an appropriate value of \( \alpha \).

3.2 Example

As example, Fig. 4 shows the ‘robustness volume’ of a PIα controller tuned for \( \omega_c = 2 \) rad/s, and \( \xi = 1.509 \), from \( \phi_{\alpha n} = 75^\circ \) and \( L_{\alpha} = 0.1 \), (in this case \( K_{\alpha}(\alpha), K_{\alpha}(\alpha) = 0 \) with \( \alpha = 0.9 \).
Note that (24) is a particular case of (22) when $\alpha = 1$.

Figure 5 shows the ‘robustness volume’ of the $PI$ controller.

Fig. 5. $PI$ controller: robustness volume.

In order to compare the robustness of both controllers we can subtract both ‘robustness volumes’ defined by (13). Figure 6 shows the obtained results.

Fig. 6. $PI$ vs. $PI^\alpha$ controller: comparison of both robustness volumes.

Therefore, in order to maximize the robustness, an efficient criteria to select the parameter $\alpha$ is to decrease its value as low as possible but avoiding to stay inside the design region where $PI$ controller is more robust than $PI^\alpha$ controller.

4. CASE OF STUDY

Let us suppose the same frequency specifications of Section 3: $\omega_c = 2$ rad/s and $\phi_m=75^\circ$. If we tune the $PI$ controller, (8) and (9), and the $PI^\alpha$ controller with $\alpha=0.8$, (6) and (7), the resulting controllers are:

$$R(s,1) = 1.9344 + \frac{2.2432}{s}$$

and

$$R(s,0.8) = 1.5700 + \frac{2.0533}{s^{0.8}}.$$  

Evaluating (13) for a plant with $K_n=0.1$ and $T_n=9.1$ (inside the region where $PI^\alpha$ is more robust) the limits for $L_n^{\max}$, obtained from (21) and (22), are:

$$L_n^{\max}(PI) = 5.6296, \quad L_n^{\max}(PI^\alpha) = 11.4793,$$  

so that the $PI^\alpha$ controller is more than twice robust than the $PI$ controller. Figure 8 represents the time responses of both controllers with $0.1 < L_n < L_n^{\max}(PI)$.

Nevertheless, if we select $K_n=9.7$ and $T_n=8.7$, the limits now for $L_n^{\max}$ using (21) and (22) are:

$$L_n^{\max}(PI) = 0.4878, \quad L_n^{\max}(PI^\alpha) = 0.4448$$

and the $PP^\alpha$ controller can not improve the robustness of the $PI$ controller. Figure 9 represents the time responses of both controllers with $0.1 < L_n < L_n^{\max}(PI)$.

5. CONCLUSIONS

In this paper a comparative stability and robustness analysis between $PI$ and $PI^\alpha$ controllers have been developed.
The analytical results conclude that both controllers have exactly the same feasible frequency design region (15), shown in Fig. 3, when they are applied to control a first order plus delay plant.

A new way to represent the robustness to parameter variations in $K$, $T$ and $L$, has been introduced: the ‘robustness volume’.

The ‘robustness volume’ of both controllers has been analytically calculated. Comparative results conclude that there are two different regions of robustness: one in which the $PI^\alpha$ is more robust than $PI$ and another where happens the opposite.

The limits between both regions depend on the frequency specifications, $\omega_c$ and $\phi_m$, and on the value of the non integer integral action, $\alpha$, of the $PI^\alpha$ controller. The region where the $PI^\alpha$ is more robust than the $PI$ controller becomes greater when $\omega_c$, $\phi_m$, and $\alpha$ increase.

To conclude, two examples have been developed where the time response of both controllers have been simulated. The first example shows a case where it is easily obtained a robustness improvement by using a $PI^\alpha$ with a small decrease of the value of $\alpha$ (with respect to $\alpha = 1$). The second example represents a case where both controllers have exactly the same robustness.

REFERENCES


