Inherent robustness of nonlinear discrete-time systems

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Abstract: The regional Input-to-State Stability and Input-to-State practical Stability of a system are studied under the assumption that the origin of the corresponding nominal model is an asymptotically stable equilibrium. The proposed results apply also to systems with a discontinuous model function and admitting only discontinuous Lyapunov functions. A numerical example is reported.

Keywords: Robustness, Input-to-State Stability, Nonlinear systems, Discontinuous systems.

1. INTRODUCTION

Stability properties of equilibria in discrete-time nonlinear systems may be lost in the presence of disturbances. As it is discussed in [6, 3, 8], discontinuity of the model function, and of the Lyapunov functions for the nominal system, can emphasize such a lack of robustness and unstable behaviors may arise even in the presence of arbitrarily small perturbations. Hence, the analysis of the disturbance entity tolerated by nominally stable equilibria (say, of inherent robustness) is of paramount importance. This is the subject of this paper where, following [22, 5, 9], robustness is analyzed in terms of regional Input-to-State Stability (ISS) and Input-to-State practical Stability (ISpS) notions.

Inherent robust stability for discrete-time nonlinear systems has been analyzed in [21, 1] with reference to asymptotically vanishing disturbances. In the former paper, the focus is on the robustness guaranteed by exponential stability, whilst in [1] it is proved the equivalence of global asymptotic stability and the so called Integral Input-to-State Stability. Nevertheless, in both studies, continuity properties of the model function are binding assumptions.

A field where the inherent robust stability analysis is of great interest is that of Model Predictive Control (MPC). In fact, while stabilizing MPC methods are well established [15, 20], it is also well-known that nominal MPC can be non robust to arbitrarily small disturbances, see [3]. The aforementioned discontinuity issue is crucial in MPC, where both the resulting feedback law and the available Lyapunov function can be discontinuous (see also [16]).

For this reason, attention has been recently focused on the development of MPC algorithms ensuring desired robustness properties [13, 19]. The main drawback of such techniques is their complexity which inhibits from widespread use in practice. For this reason, it is still of interest to draw the attention to the inherent robustness analysis of nominal MPC. Pioneering results in this vein are reported in [2, 14, 21] but they rely on regularity assumptions on the Lyapunov function or on the MPC law.

The main contribution of this paper consists of providing a characterization of stability properties in perturbed conditions which can be deduced by the behavior of a Lyapunov function \( V \) for the nominal system. More specifically, a function \( \Theta \) is constructed in terms of standard \( \mathcal{L}_\infty \)-functions used to bound \( V \) and its variation along trajectories [7]. The robustness analysis is then easily derived by the study of the behavior of such a function \( \Theta \), which thus represents a kind of robustness energy measure. The proposed approach allows one to successfully consider also the critical case of systems with a discontinuous model function and discontinuous Lyapunov functions.

The work of this paper extends preliminary results presented in [17]: while in such a referenced paper only additive disturbances are considered, in this work also the case of disturbances whose effect on the system is mediated by a state depending function is studied. Moreover, although the control synthesis issue is not treated in the present paper, the proposed results can be also used for such a purpose. This is done in [17] for systems under additive disturbance, where a formulation is presented of a nominal model predictive controller capable of ensuring inherent robustness for the resulting closed-loop dynamics.

Paper organization: in Section 2, the system is defined and recent results on the ISS and ISpS concepts are recalled; the inherent robustness analysis is presented in Section 3; Section 4 specializes the analysis to linear systems; a numerical example is reported in Section 5.

Notation and terminology: By \( \mathbb{N} \) and \( \mathbb{R}_+ \), we mean the sets of non negative integers and real numbers, respectively. A generic vector norm in \( \mathbb{R}^n \) is denoted by \( \| \cdot \| \) while, if \( \mathbb{R}^{n \times n} \ni P > 0 \), we let \( |x|_P = \sqrt{x^T F x} \); the Euclidean vector or induced matrix norm is denoted by \( \| \cdot \|_2 \). A signal taking values in \( \mathbb{R}^d \) is denoted by \( w = \{ w(0), w(1), \ldots \} \). For \( \mathcal{W} \subseteq \mathbb{R}^d \), let \( \mathcal{M}_\mathcal{W} \) be the set of signals taking values in \( \mathcal{W} \) and such that \( \| w \| = \sup_{k \in \mathbb{N}} |w(k)| < +\infty \). The interior part of a set \( S \subseteq \mathbb{R}^n \) is denoted by \( \text{int}(S) \). For \( r > 0 \), let \( \mathcal{B}_r = \{ z \in \mathbb{R}^n : |z| \leq r \} \). A function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is an \( \mathcal{A}_0 \)-function iff it is continuous, non-decreasing and...
such that $\varphi(0) = 0$. For the definition of $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{K}$-functions, we refer to [20]. A function $\delta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be non-decreasing iff, for any fixed $s \geq 0$, both $\delta(\cdot, s)$ and $\delta(s, \cdot)$ are non-decreasing functions. The identity function $s \mapsto s$ is denoted by $Id$.

2. PRELIMINARIES: SYSTEM DEFINITION AND ROBUST STABILITY PROPERTIES

Consider the perturbed nonlinear discrete-time system

$$x(k + 1) = \tilde{f}(x(k), k), \quad k \in \mathbb{N}, \quad x(0) = \tilde{x},$$

(1)

where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{W} \subseteq \mathbb{R}^d$ is a disturbance term and $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is a not necessarily continuous function such that, for any bounded set $S \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $\tilde{f}(S) \subseteq \mathbb{R}^n$ is bounded. The solution of system (1) at time $k$ for $x(0) = \tilde{x}$ and disturbance $w$ is denoted by $x(k, x, w)$. Let the corresponding nominal model be

$$x(k + 1) = f(x(k)), \quad k \in \mathbb{N}, \quad x(0) = \tilde{x},$$

(2)

where $f(x) = \tilde{f}(x, 0)$, and suppose that $f(0) = 0$. With $g(x, w) = \tilde{f}(x, w) - f(x, 0)$, $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, one has $f(x, w) = f(x) + g(x, w)$ and, $\forall x \in \mathbb{R}^n$, $g(x, 0) = 0$.

Let us introduce the stability properties considered in the following developments.

**Definition 1.** A set $\Gamma \subseteq \mathbb{R}^n$ is said to be *positively invariant* for system (2) iff $\forall x \in \Gamma$, $f(x) \in \Gamma$.

**Definition 2.** Consider system (2), let $\Gamma$ and $\Omega$ be two sets such that $\Omega \subseteq \Gamma$, $0 \in int(\Omega)$ and $\Gamma$ is positively invariant. A (not necessarily continuous) function $V : \mathbb{R}^n \to \mathbb{R}^+$ is a Lyapunov function in $(\Gamma, \Omega)$ iff there exist $\mathcal{K}_\infty$-functions $\alpha_1$, $\alpha_2$ and $\alpha_3$ such that:

$$V(x) \geq \alpha_1(|x|) \quad \forall x \in \Gamma$$

(3a)

$$V(x) \leq \alpha_2(|x|) \quad \forall x \in \Omega$$

(3b)

$$V(f(x)) - V(x) \leq -\alpha_3(|x|) \quad \forall x \in \Gamma.$$  

(3c)

**Proposition 1.** [7] Consider system (2) and let $V$ be a Lyapunov function in $(\Gamma, \Omega)$, then the origin is an asymptotically stable equilibrium in $\Gamma$.

**Definition 3.** A set $\Gamma \subseteq \mathbb{R}^n$ is said to be *robust positively invariant with respect to $\mathbb{W} \subseteq \mathbb{R}^d$* ($\mathbb{W}$-RPI) for system (1) iff $\forall x \in \Gamma$ and $\forall w \in \mathbb{W}$, $f(x, w) \in \Gamma$.

**Definition 4.** Given a compact set $\Gamma \subseteq \mathbb{R}^n$ with $0 \in int(\Gamma)$, system (1) is said to be *Input-to-State practically Stable in $\Gamma$ with respect to $\mathbb{W} \subseteq \mathbb{R}^d$* ($\Gamma$, $\mathbb{W}$-IStPS) iff $\Gamma$ is a $\mathbb{W}$-RPI set and there exist functions $\beta \in \mathcal{K}_\mathcal{L}$, $\gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that, $\forall k \geq 0$, $\forall x \in \Gamma$ and $\forall w \in \mathbb{M}_\mathbb{W}$,

$$|x(k, x, w)| \leq \beta(\tilde{x}, k) + \gamma(|w||).$$

If $c = 0$, then the system is said to be *Input-to-State Stable in $\Gamma$ with respect to $\mathbb{W} \subseteq \mathbb{R}^d$* ($\Gamma$, $\mathbb{W}$-ISS).

**Definition 5.** A (not necessarily continuous) function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a ($\Gamma$, $\mathbb{W}$)-IStPS Lyapunov function for system (1) iff $\Gamma$ is a compact $\mathbb{W}$-RPI set and there exist a compact set $\Omega \subseteq \Gamma$ with $0 \in int(\Omega)$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and constants $c_1, c_2 \geq 0$ such that:

$$V(x) \geq \alpha_1(|x|) \quad \forall x \in \Gamma$$

(4a)

$$V(x) \leq \alpha_2(|x|) + c_1 \quad \forall x \in \Omega$$

(4b)

$$V(f(x, w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|) + c_2 \quad \forall x \in \Gamma \forall w \in \mathbb{W}.$$  

(4c)

If $c_1 = c_2 = 0$, $V$ is called a ($\Gamma$, $\mathbb{W}$)-ISS Lyapunov function.

**Proposition 2.** [9, 12] If (1) admits a ($\Gamma$, $\mathbb{W}$)-IStPS Lyapunov function, then it is ($\Gamma$, $\mathbb{W}$)-IStPS; if it admits a ($\Gamma$, $\mathbb{W}$)-ISS Lyapunov function, then it is ($\Gamma$, $\mathbb{W}$)-ISS.

3. INHERENT ROBUSTNESS ANALYSIS

The stability properties for the perturbed system (1) are analyzed by assuming that the origin is an asymptotically stable equilibrium for the nominal model (2). The idea is that the nominal decrease of $V$ translates into state norm decrease and that, for sufficiently small disturbances, such a decrease may be maintained. However, this rationale is sensible provided that the behavior of the state norm is characterized in terms of continuous functions. According to inequalities (3), this is achieved by properly combining the functions $\alpha_1$ and $\alpha_2$ (relating $V(x)$ with $|x|$), $\alpha_3$ (accounting for the decrease of $V$) and some information on the dependence of the perturbation $g(x, w)$ on $|x|$ and $|w|$. The results do not need the explicit knowledge of $V$.

**Definition 6.** Let $V$ be a Lyapunov function in $(\Gamma, \Omega)$ for system (2). For $s \geq 0$, let $\psi(s) = \max_{x \in [0, s]}(\alpha_2 - \alpha_3)(c)$, then $\Psi(s) = (I - \alpha_1^{-1} \circ \psi)(s)$ is called the $\Psi$-function associated to $V$.

Consider the perturbed system (1), let

$$\delta(s, \mu) = \sup_{|x| \leq s, |w| \leq \mu} g(x, w).$$

(5)

If $\varphi_x$ is an $\mathcal{N}_0$-function and $\varphi_w$ is a $\mathcal{K}_\infty$-function such that, $\forall s \geq 0$ and $\forall \mu \geq 0$, $\delta(s, \mu) \leq \varphi_x(s) + \varphi_w(\mu)$, then $\Theta(s) = (\Psi - \varphi_x)(s)$ is called the $\Theta$-function associated to $V$ and $g$.

**Remark 1.** Notice that:

1. $(\alpha_1^{-1} \circ \psi)(s)$, and hence $\Psi(s)$, is well-defined because $\psi(s) \geq \varphi(0) = (\alpha_2 - \alpha_3)(c) = 0$ and $\alpha_1 \in \mathcal{K}_\infty$;
2. $\delta(s, \mu) < +\infty$ in view of the assumptions on $f$;
3. $\Psi$ and $\Theta$ are continuous functions;
4. The $\Psi$-function depends on the $\alpha_i$’s and it is not univocally fixed by $V$ but, for brevity, it is referred to as “associated to $V$”. Similar remarks hold for $\Theta$.

The existence of $\varphi_x$ and $\varphi_w$ so that relation (6) holds is guaranteed by the following mild assumption on the function $g$ defining the perturbed system:

**Assumption 1.** Consider the perturbed system (1) and let $\delta$ be as in equation (5). It is supposed that the function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\delta(s, \zeta) = \delta(s, c)$ is continuous in $0$.

**Lemma 1.** There exist an $\mathcal{N}_0$-function $\varphi_x$ and a $\mathcal{K}_\infty$-function $\varphi_w$ such that relation (6) holds if and only if Assumption 1 is satisfied. In this case, a feasible choice is $\varphi_x(s) = \varphi_w(s) = \varphi(2s)$, where

$$\varphi(s) = \begin{cases} 0 & \text{if } \zeta = 0 \\ \zeta + \frac{1}{2} \int_\zeta^{2\zeta} \delta(s, \zeta)ds & \text{if } \zeta > 0. \end{cases}$$

**Proof.** See in Appendix. ■
Remark 2. (Disturbance affine systems). Consider a disturbance affine system
$$x(k+1) = f(x(k)) + B(x(k))w(k),$$
where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^d$ and $B: \mathbb{R}^n \to \mathbb{R}^{n \times d}$. Then,
$$\delta(s, \mu) = \sup_{|x| \leq \mu} \sup_{|w| \leq \mu} |B(x)w| = \sup_{|x| \leq \mu} |B(x)\mu = \mathcal{B}(s)\mu,$$
where $\mathcal{B}(s) = \sup_{|x| \leq \mu} |B(x)|$ and $|B(x)|$ is the matrix norm induced by the considered vector norms in the state and in the disturbance spaces. From the assumptions on $f$, it follows that $\mathcal{B}(s)$ is bounded in a neighborhood of 0 so that Assumption 1 is satisfied. If, moreover, $\mathcal{B}(0) = 0$ and $\mathcal{B}(s)$ is continuous in 0, then there exists an $N_0$-function $\mathcal{B}$ such that $\forall s \geq 0, \mathcal{B}(s) \geq \mathcal{B}(s)$ (for instance, using Lemma 3a in Appendix, one can even take $\mathcal{B} \in K_\infty$) and $\varphi_x(s) = \frac{1}{2} \mathcal{B}(s)^2$ and $\varphi_w(\mu) = \frac{1}{2} \mu^2$. So that relation (6) is satisfied.

Theorem 1. (RPI analysis). Let $V$ be a Lyapunov function in the nominal system (2) and assume that Assumption 1 is satisfied. Let $\mathcal{B}_r \subseteq \Omega$ and $\Theta(\tau) > 0$, where $\Theta$ is the $\Theta$-function associated to $V$ and $g$, then $\mathcal{B}_r$ is a $\mathcal{B}_r$-RPI set for system (1) with
$$\mu = (\varphi_w^{-1} \circ \Theta)(r).$$

Proof. First, $\mu = (\varphi_w^{-1} \circ \Theta)(r)$ is well-defined because $\Theta(\tau) > 0$ and $\varphi_w \in K_\infty$. We have to show that, $\forall r \in \mathcal{B}_r$ and $\forall w \in \mathcal{B}_\mu$, $f(x, w) \in \mathcal{B}_r$. In view of (5) and (6), for $x \in \mathcal{B}_r$ and $w \in \mathcal{B}_\mu$, it holds that
$$|f(x, w)| \leq |f(x)| + |g(x, w)| \leq |f(x)| + \delta(r, \mu) \leq |f(x)| + \varphi_x(r) + \varphi_w(\mu).$$
For $x \in \mathcal{B}_r$, by inequalities (3c) and (3b), one has
$$V(f(x)) \leq V(x) - \alpha_3(|x|) \leq \alpha_2(|x|) - \alpha_3(|x|),$$
where the last inequality can be applied because $x \in \mathcal{B}_r \subseteq \Omega$. By the positive invariance of $\Gamma, f(x) \in \Gamma$ so that, by (3a) and (10), $\alpha_1(|f(x)|) \leq V(f(x)) \leq (\alpha_2 - \alpha_3)(|x|)$ whence
$$|f(x)| \leq (\alpha_1^{-1} \circ (\alpha_2 - \alpha_3))(|x|).$$
Thus, combining inequalities (9) and (11),
$$|f(x, w)| \leq (\alpha_1^{-1} \circ (\alpha_2 - \alpha_3))(|x|) + \varphi_x(r) + \varphi_w(\mu).$$
Therefore, $f(x, w) \in \mathcal{B}_r$, if $\forall r \in \mathcal{B}_r$,
$$\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)(|x|) + \varphi_x(r) + \varphi_w(\mu) \leq r,$$
that is
$$\varphi_w(\mu) \leq r - (\alpha_1^{-1} \circ (\alpha_2 - \alpha_3))(|x|) - \varphi_x(r)$$
which in turn is equivalent to
$$\varphi_w(\mu) \leq r - \max_{x \in \mathcal{B}_r} (\alpha_1^{-1} \circ (\alpha_2 - \alpha_3))(|x|) - \varphi_x(r) =
= r - \alpha_1^{-1} \max_{x \in \mathcal{B}_r} (\alpha_2 - \alpha_3)(|x|) - \varphi_x(r) =
= r - \alpha_1^{-1} \max_{x \in [0, r]} (\alpha_2 - \alpha_3)(x) - \varphi_x(r) =
= r - \alpha_1^{-1} \psi(\Theta) - \varphi_x(r) = \Theta(r)$$
and, by taking inverses, $\mu \leq (\varphi_w^{-1} \circ \Theta)(r)$.

Remark 3. Notice that, for $s > 0$ such that $\mathcal{B}_s \subseteq \Omega$, $(\alpha_2 - \alpha_3)(s) \geq 0$. In fact, for $x \in \Omega$, by (10) one has $(\alpha_2 - \alpha_3)(|x|) \geq V(f(x)) \geq 0$.

An ISpS property can be obtained under the following mild assumption on the $K_\infty$-function $\alpha_1$ associated with the Lyapunov function $V$ for the nominal model.

Assumption 2. Let $\alpha_1$ be a Lipschitz continuous function in $\Omega$ with Lipschitz constant $L_\alpha$, that is
$$|\alpha_1(|x|) - \alpha_1(|y|)| \leq L_\alpha |x - y| \quad \forall x, y \in \Omega. (12)$$

The ISpS result is based on the following properties:

Lemma 2. Let $V$ be a Lyapunov function in $(\Gamma, \Omega)$ for the nominal system (2) with $\Omega$ being a bounded set. Under Assumption 2 one has:

a. the function $V$ is such that
$$|V(x_1) - V(x_2)| \leq L\alpha |x_1 - x_2| + d \quad \forall x_1, x_2 \in \Omega, (13)$$
where
$$d = \sup_{x \in \Omega} (\alpha_2(|x|) - \alpha_1(|x|)); \quad (14)$$

b. suppose that also Assumption 1 is valid, if $x \in \Omega$ and $w \in \mathcal{W}$ are such that $f(x) \in \Omega$ and $f(x, w) \in \Omega$, then
$$V(f(x, w)) - V(x) \leq - (\alpha_3 - L\alpha \varphi_x)(|x|) + L\alpha \varphi_w(|w|) + d.$$

Proof. In view of Theorem 1, $\mathcal{B}_r$ is a (compact) $\mathcal{B}_\mu$-RPI set. Let us show that $V$ satisfies inequalities (4) with $\Omega = \Gamma = \mathcal{B}_r$ and suitable choices for $\alpha_1$’s ($i = 1, 2, 3$), $\sigma$ and $c_i$’s ($j = 1, 2$). The thesis then follows from Proposition 2. Indeed, since $\mathcal{B}_r \subseteq \Omega \subseteq \Gamma$, by inequalities (3a) and (3b) one has that properties (4a) and (4b) hold in the form
$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathcal{B}_r$$
(in particular, $c_1 = 0$). Thanks to Lemma 2b, which can be applied because $\mathcal{B}_r$ is a $\mathcal{B}_\mu$-RPI set contained in $\Omega$, property (4c) holds in the form
$$V(f(x, w)) - V(x) \leq - (\alpha_3 - L\alpha \varphi_x)(|x|) + L\alpha \varphi_w(|w|) + d$$
$\forall x \in \mathcal{B}_r$ and $w \in \mathcal{B}_\mu$, where $d$ is given in equation (14).

Finally, the most interesting property of ISS is analyzed.

Theorem 3. (ISS analysis). Let $V$ be a Lyapunov function in $(\Gamma, \Omega)$ for the nominal system (2) and suppose that Assumption 1 is satisfied. If $\mathcal{B}_r \subseteq \Omega$ and $\Theta \in K_\infty$ such that, $\forall \tau \leq r, \Theta(\tau) \geq \overline{\Theta}(r)$, where $\Theta$ is the $\Theta$-function associated to $V$ and $g$, then, $\forall \tau \in [0, r]$ system (1) is $(\mathcal{B}_r, \mathcal{B}_\mu)$-ISS with $\mu = (\varphi_w^{-1} \circ \Theta)(r)$ and $V(x) = |x|$ is a $\mathcal{B}_r$-ISS Lyapunov function.

Proof. By Theorem 1, since $\Theta(\tau) \geq \overline{\Theta}(r) > 0$, then $\mathcal{B}_r$ is a $\mathcal{B}_\mu$-RPI set. It is shown that $V$ satisfies inequalities (4) with $\Omega = \Gamma = \mathcal{B}_r,$ $\alpha_1 = \alpha_2 = \alpha_3 = \overline{\Theta},$ $\sigma = \varphi_w$ and $c_1 = c_2 = 0$. The thesis then follows from Proposition 2. Inequalities (4a) and (4b) are trivial, let us then consider inequality (4c). For $x \in \mathcal{B}_r$ and $w \in \mathcal{B}_\mu$, one has...
\[ \tilde{V}(\tilde{f}(x,w)) - \tilde{V}(x) = |\tilde{f}(x,w)| - |x| \leq |f(x)| - |x| + |g(x,w)| \leq (\alpha^{-1} \circ (\alpha_2 - \alpha_3))(|x|) - |x| + \cdots \] for \( \delta, \phi_x \) and \( \phi_w \) by considering the Euclidean vector norm \( |\cdot|^2 \). Then, we derive the analogous quantities for \( \delta, \phi_x \) and \( \phi_w \) by considering the Euclidean vector norm \( |\cdot|^2 \). This is a consequence of the Bellman principle.

Notice that \( \tilde{V} \) is a continuous function while \( V \) is not necessarily continuous.

**Remark 4.** (Additive disturbance) For systems perturbed by an additive disturbance, namely \( g(x,w) = w \), one has \( \delta(s, \mu) = \mu \). Hence, the proposed results can be restated with \( \varphi_x \) and \( \varphi_w \) replaced with the null and the identity function, respectively, and by considering the \( \Psi \)-function, rather than the \( \Theta \)-function, associated to \( V \).

An example of application of the proposed results to a system having a discontinuous model function, which admits only discontinuous Lyapunov functions and subject to an additive disturbance, can be found in [17].

### 4. LINEAR SYSTEMS

The results derived in the previous section are now specialized to perturbed stable linear systems described by

\[ x(k+1) = \tilde{f}(x(k),w(k)) = Ax(k) + \tilde{E}w(k), \quad (15) \]

where \( A \in \mathbb{R}^{n \times n} \) is a Schur matrix and \( E \in \mathbb{R}^{n \times d} \). It is well-known that asymptotically stable linear systems enjoy ISS properties (see [5]). Let us show that such properties do follow from Theorem 3.

A Lyapunov function for the nominal system \( x(k+1) = Ax(k) \) is \( V(x) = x'Px \), where \( P > 0 \) is such that \( A'PA - P = -Q < 0 \). (16)

Let us derive feasible choices for the \( \Theta \)-functions \( \alpha_1 \)'s so that inequalities (3) hold. Consider the vector norm \( \lVert \cdot \rVert \). Then \( \alpha_1(s) = \alpha_2(s) = s^2 \) are so that inequalities (3a) and (3b) are satisfied \( \forall x \in \mathbb{R}^n \). As far as \( \alpha_3 \) is concerned, following [17], one can consider \( \alpha_3(s) = cs^2 \) with

\[ c = \lambda_{\min}(Q) / \lambda_{\max}(P). \]

Other feasible choices for \( c \) are

\[ c = \lambda_{\min}\left((T^{-1})'QT^{-1}\right) / \lambda_{\max}\left((T^{-1})'PT^{-1}\right), \]

where \( T \in GL(n, \mathbb{R}) = \{ T \in \mathbb{R}^{n \times n} \mid \det T \neq 0 \} \) is a free parameter. Moreover, the maximal feasible value of \( c \) is obtained by taking \( T \) such that \( T'P = P \); in this case,

\[ c = \lambda_{\min}\left((T^{-1})'QT^{-1}\right) / \lambda_{\max}\left((T^{-1})'PT^{-1}\right) = 1 - |A|^2_p, \]

with \( |A|^2_p = \max_{x, \lVert x \rVert = 1} |Ax|_p \).

As for the disturbance term \( g(x,w) = \tilde{E}w \), denoting by \( \lVert \cdot \rVert_x \), the considered norm in the disturbance space \( \mathbb{R}^d \),

\[ \delta(s, \mu) = \sup_{|w|_x \leq \mu} |\tilde{E}w|_p = |E|_p \cdot \mu, \]

where \( |E|_p = \max_{|w|_x = 1} |\tilde{E}w|_p \) is the corresponding induced matrix norm. Thus, relation (6) holds with \( \varphi_x(s) \equiv 0 \) and \( \varphi_w(\mu) = |E|_p \cdot \mu \). In particular, the \( \Theta \)-function associated to system (15) is reduced to the \( \Psi \)-function.

Let us compute the \( \Psi \)-function: since \( (\alpha_2 - \alpha_3)(s) = (1 - c)s^2 \) and, according to Remark 3, \( (\alpha_2 - \alpha_3)(s) \geq 0 \forall s \geq 0 \), then \( c \leq 1 \) and \( \psi(s) = (1 - c)s^2 \). Thus,

\[ \Psi(s) = (Id - \alpha_1^{-1} \circ \psi)(s) = (1 - \sqrt{1-c})s \]

and \( \Psi \in K_{\infty} \) because, by \( c > 0 \), one has \((1 - \sqrt{1-c}) > 0 \). Hence, by Theorem 3, the perturbed system (15) is \((B_x, B_w)\)-ISS \( \forall r > 0 \), with

\[ \mu = (\varphi_w \circ \Theta)(r) = 1 - \sqrt{1-c}. \quad (17) \]

**Remark 5.** The \( \Psi \)-function (as well as the \( \Theta \)-function) depends on the considered vector norm. The choice of a proper norm is crucial for the applicability of the proposed results: if we consider \( V(x) = x'Px \) and the Euclidean norm \( \lVert \cdot \rVert_2 \), then inequalities (3) are satisfied with \( \alpha_1(s) = \lambda_{\min}(P)s^2, \alpha_2(s) = \lambda_{\max}(P)s^2 \) and \( \alpha_3(s) = \lambda_{\min}(Q)s^2 \). Denoted by \( \Psi_2 \) the corresponding \( \Psi \)-function, one has

\[ \Psi_2(s) = \left(1 - \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}}\right)s. \]

In this case, however, it is not guaranteed that \( \Psi_2 \in K_{\infty} \). This is a consequence of the following fact: if, for a certain norm, \( \Psi \in K_{\infty} \), then robust positive invariance properties of the balls in that norm hold. Therefore, a necessary condition in order that \( \Psi \in K_{\infty} \) is that the system is contractive with respect to the considered norm. In this sense, the choice of considering the norm \( \lVert \cdot \rVert \) is natural, whilst a necessary condition in order that \( \Psi_2 \in K_{\infty} \) is that the system is 2-norm contractive (namely, \( A'PA - I < 0 \)).

### 5. EXAMPLE

The following discrete-time model is obtained by an Euler approximation of a mass-spring-damper system with an unknown damping parameter \( h_\delta = 1.1 + w \) and controlled by a linear law (see [11, 18] for more details):

\[ x(k+1) = Ax(k) + B(x(k))w(k) = \]

\[ \begin{bmatrix} 1 & 0.4 & -1.57 & -0.63 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -0.4x_2(k) \end{bmatrix} w(k) \]

The nominal system is linear and asymptotically stable (\( A \) is a Schur matrix). With \( Q = I \), the Lyapunov equation (16) is solved by

\[ P = \begin{bmatrix} 5.02 & 1.61 \\ 1.61 & 1.64 \end{bmatrix} \]

so that \( V(x) = x'P x \) is a Lyapunov function for the nominal model. According to the developments presented in Section 4, let us consider the vector norm \( \lVert \cdot \rVert \). Hence, \( \alpha_1(s) = \alpha_2(s) = s^2 \) and \( \alpha_3(s) = cs^2 \), with

\[ c = \lambda_{\min}(Q) / \lambda_{\max}(P) = 0.18 \]

are so that inequalities (3) are satisfied and

\[ \Psi(s) = (1 - \sqrt{1-c})s = 0.09s. \]

The system is disturbance affine with \( g(x,w) = B(x)w, B(x) \) being a linear function of \( x \). Let us compute the function \( \delta \) as defined in equation (5). Notice that such a function, and hence \( \varphi_x \) and \( \varphi_w \) satisfying relation (6), depends on the considered vector norm in the state space. To simplify the computations, we first determine the expressions for \( \delta, \varphi_x \) and \( \varphi_w \) by considering the Euclidean vector norm \( \lVert \cdot \rVert_2 \). Then, we derive the analogous quantities.
associated with the vector norm $| \cdot |_P$. With obvious
meaning of the notation, it holds that, $\forall s \geq 0$ and $\forall \mu \geq 0$,
\[
\delta_p(s, \mu) \leq \sup_{|x|_P \leq s} \sqrt{\lambda_{\text{Max}}(P)}|g(x, w)|_2 \leq \sqrt{\lambda_{\text{Max}}(P)} \cdot \delta_2 \left( \frac{s}{\sqrt{s_{\text{min}}(P)}} \right) \mu,
\]
where both inequalities follow from
\[
\sqrt{s_{\text{min}}(P)} |x|_2 \leq |x|_P \leq \sqrt{\lambda_{\text{Max}}(P)} |x|_2.
\]
Then, feasible choices for $\varphi^{(p)}_x$ and $\varphi^{(p)}_w$ are
\[
\begin{align*}
\varphi^{(p)}_x(s) &= \sqrt{\lambda_{\text{Max}}(P)} \cdot \varphi^{(s)}_x \left( \frac{s}{\sqrt{s_{\text{min}}(P)}} \right) \\
\varphi^{(p)}_w(\mu) &= \sqrt{\lambda_{\text{Max}}(P)} \cdot \varphi^{(s)}_w(\mu).
\end{align*}
\] (18)

According to Remark 2, $\delta_2(s, \mu) = 0.4s\mu$. Thus, $\delta_2(s, \mu)$ takes the form
\[
\delta_2(s, \mu) = \ell s \mu
\] (19)
which, for any choice of $\ell_1 > 0$ and $\ell_2 > 0$ such that $\ell_1 \cdot \ell_2 = \ell$, can be rewritten as $\delta_2(s, \mu) = (\ell_1)(s)(\ell_2)\mu$.
Therefore, $\delta_2(s, \mu) \leq \frac{\ell^2}{2} s^2 + \frac{\ell^2}{2} \mu^2$ and, introducing the free parameter $\eta = \frac{\ell^2}{2} > 0$, inequality (6) is satisfied with
\[
\varphi^{(s)}_x(s) = \eta s^2 \quad \text{and} \quad \varphi^{(s)}_w(\mu) = \frac{\ell^2}{4} \eta \mu^2.
\] (20)

Combining equations (18) and (20), one has:
\[
\varphi^{(p)}_x(s) = C_p \eta s^2 \quad \text{and} \quad \varphi^{(p)}_w(\mu) = \frac{C_w}{\eta} \mu^2,
\]
where $C_p = \sqrt{\lambda_{\text{Max}}(P)} = 2.38$ and $C_w = \frac{\ell^2}{4} \sqrt{\lambda_{\text{Max}}(P)} = 0.1$. The corresponding $\Theta$-function is
\[
\Theta(s) = (\Psi - \varphi^{(p)}_x)(s) = C_c s - C_p \eta s^2,
\]
where $C_c = 1 - \sqrt{1 - \ell} = 0.09$.
Let $\theta_{\text{max}} > 0$ be such that $\Theta(\theta_{\text{max}}) = 0$, namely
\[
\theta_{\text{max}} = \frac{C_c}{C_p \eta} = \frac{0.04}{\eta}.
\] (21)

Then, $\forall \eta < \theta_{\text{max}}, \Theta(\eta) > 0$ (see Figure 1.a) and $\exists \tilde{\Theta} \in K_{\infty}$ such that $\forall r \leq \tilde{r}, \Theta(r) \geq \tilde{\Theta}(r)$. Theorem 3 can hence be applied and,

for $r < \theta_{\text{max}}$, the system is $(B_r, B_{\mu(r)})$-ISS, (22)

where $\mu(r) = (\varphi^{(w)}_w)^{-1}(\Theta)(r)$ (see equation (8)). Since $\varphi^{(w)}_w^{-1}(y) = \sqrt{\frac{y}{C_w}}$, then
\[
\mu(r) = \sqrt{\frac{\eta(C_r - C_p \eta r^2)}{C_w}}.
\]
Finally, it remains to study how the free parameter $\eta > 0$ affects the above analysis. In this respect, let us show that

$\forall r > 0$, the system is $(B_r, B_{\mu^*})$-ISS, (23)

where, remarkably,
\[
\mu^* = \frac{C_c}{\ell \sqrt{\kappa(P)}} = 0.097
\]
is a constant independent of $r$ and $\eta$.
In fact, the maximum value of the function $\mu(r)$ is taken for $r^*(\eta) = \frac{C_c}{\ell \sqrt{\kappa(P)}} = \eta_{\text{max}}$ and it is equal to $C_c/\ell \sqrt{\kappa(P)}$ (see Figure 1.b). Therefore, for any fixed $r > 0$, letting $\eta$ be so that $r^*(\eta) = r$ (namely, $\eta = \frac{C_c}{\ell \sqrt{\kappa(P)}}$), one has $\mu(r) = \mu^*$ and, according to equation (22), property (23) holds true.

Remark 6. For systems with additive disturbances, the maximum norm $\mu(r)$ of $w$ so that the positive invariance of $B_r$ is preserved is such that $\lim_{r \to 0^+} \mu(r) = 0$. In the example, instead, it has been shown the independence of such a maximum norm (i.e., of $\mu^*$) from the size of the region $B_r$. This is a consequence of the bilinear structure of the function $g$, where the disturbance $w$ is multiplied by a term $B(x)$ such that $\lim_{|x| \to 0^+} |B(x)|$. Taking notice that the proposed analysis has been capable of properly taking such a phenomenon into account although it is based on the decomposition (6) of the function $\delta$ associated with $g$ which is suggestive of an additive disturbance-like analysis.

Remark 7. The proposed approach applies to any system with $\delta_2(s, \mu)$ as in equation (19). In particular, property (23) holds for any system where the effect of the disturbance is mediated by a bilinear function $g(x, w)$.

6. CONCLUSIONS

A number of results have been derived on the analysis of inherent robustness enjoyed by nominally stable equilibria in nonlinear discrete-time systems. The case of systems with a not necessarily continuous model function and with possibly non additive disturbances is considered. The presented analysis results can be used for control synthesis purposes: the formulation of a nominal model predictive controller capable of ensuring inherent robustness for the resulting closed-loop dynamics is proposed in [18].

Acknowledgments: We thank Prof. L. Magni and D. Limon for useful discussions on the subject of this paper.

APPENDIX

In order to prove Lemma 1, we need the following result:

Lemma 3. (see [10], cf. [23])
a. Let $\Xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function, continuous in $0$ and such that $\Xi(0) = 0$. Then there exists a $K_{\infty}$-function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall s \geq 0, \Xi(s) \leq \xi(s)$.
b. Let $\delta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\delta(s) = \delta(s, s)$ is continuous in $0$ and $\delta(0) = 0$. Then there exists a $K_{\infty}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall (s, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+, \delta(s, \mu) \leq \varphi(s) + \varphi(\mu)$.

Proof. Part a. The function
\[
\xi(s) = \left\{ \begin{array}{ll}
0 & \text{if } s = 0 \\
\frac{1}{s} & \text{if } s > 0
\end{array} \right.
\]
is well-defined (indeed any monotonic function is integrable) and, by the monotony of $\Xi, \xi(s) \leq \xi(2s)$, $\forall s \geq 0$. The required properties then immediately follow.

Part b. One has $\delta(s, \mu) \leq \delta(s + \mu)$ because $\delta$ is non-decreasing. By part a., we know that there exists a $K_{\infty}$-function $\delta$ such that $\delta(s) \leq \delta(s, s)$. Hence, $\delta(s, \mu) = \delta(2s) + \delta(2\mu) = \varphi(s) + \varphi(\mu)$.

Proof of Lemma 1. The necessity is trivial since
\[
0 \leq \delta(s) \leq \varphi_w(s) + \varphi_w(s) + \varphi_x + \varphi_w + \varphi = \mathcal{N}_0-
\]
function. As for sufficiency and the possibility to let $\varphi_w(s) = \varphi_w(s) = \varphi_w(2s)$, it is an immediate consequence of the above Lemma 3.

Proof of Lemma 2. Part a. For $x_1, x_2 \in \Omega$, by inequalities (3a) and (3b), it holds that
\[
V(x_1) - V(x_2) \leq \alpha_2(|x_1|) - \alpha_1(|x_2|) = \alpha(|x_1|) + \alpha_1(|x_1|) - \alpha_1(|x_2|).
\]
where \( \delta(s) = \alpha_2(s) - \alpha_1(s) \). Then, by Assumption 2 and equation (14), one has
\[
V(x_1) - V(x_2) \leq d + LV|x_1 - x_2|.
\]
On the other hand, by inequalities (3a) and (3b), Assumption 2 and equation (14), one also deduces that
\[
V(x_1) - V(x_2) \geq \alpha_1(|x_1|) - \alpha_2(|x_2|) = \alpha_1(|x_1|) - \alpha_2(|x_2|) - \delta(|x_2|) \geq -LV|x_1 - x_2| - d.
\]
Part (b). Indeed it holds that
\[
V(f(x, u)) - V(x) = \frac{1}{2} (|f(t)|^2 - f(t)^2) + V(f(x)) - V(x) \leq \frac{1}{2} (|f(t)|^2 - f(t)^2) + \frac{1}{2} (|f(x)|^2 - f(x)^2) + 2\eta_1|x| \eta_2|w| \leq \frac{1}{2} (|f(t)|^2 - f(t)^2) + \frac{1}{2} (|f(x)|^2 - f(x)^2) + \eta_1|x| \eta_2|w|,
\]
where inequality (a) follows from inequalities (13) and (3c), inequalities (b) and (c) follow from relations (5) and (6), respectively.

**REFERENCES**


