Designing Distributed Control Gains for Consensus in Multi-agent Systems with Second-order Nonlinear Dynamics

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Abstract: This paper discusses the design of distributed control gains for consensus in multi-agent systems with second-order nonlinear dynamics. An effective distributed adaptive strategy on the control gains is developed for reaching consensus based only on local information of the network structure. A simulation example is given to illustrate the theoretical analysis.

1. INTRODUCTION

Consensus, a typical collective behavior in networked systems with a group of autonomous mobile agents, has received increasing attention recently (Cao et al. [2008], Hong et al. [2008], Jadabaie et al. [2003], Olfati-Saber & Murray [2004], Ren [2008], Ren & Atkins [2007], Ren & Beard [2005], Vicsek et al. [1995], Yu et al. [2010a,b]), due to its broad applications in biological systems, sensor networks, Unmanned Air Vehicle (UAV) formations, robotic teams, underwater vehicles, etc. In a typical multi-agent system, each agent shares information only with its neighboring agents under a distributed protocol while the whole group of agents can coordinate so as to achieve a certain global behavior of common interest. Leader-follower consensus is a special case of consensus where there are one virtual leader and some informed agents who have the leader’s information, together with the other uninformed agents who only follow their neighbors, and the task is for all the agents to follow the leader, asymptotically.

In the literature, much progress has been devoted to the study of the conditions for reaching consensus among a group of autonomous agents in a dynamical network (Cao et al. [2008], Jadabaie et al. [2003], Olfati-Saber & Murray [2004], Ren & Beard [2005], Reynolds [1987], Vicsek et al. [1995], Yu et al. [2010a,b]). In particular, second-order dynamics (Hong et al. [2008], Ren [2008], Ren & Atkins [2007], Yu et al. [2010a,b]) have received a great deal of interest due to many real-world applications where agents are governed by both position and velocity states. It is now known that in most cases, second-order consensus can be reached in multi-agent systems if the coupling control gains and the spectra of the Laplacian matrix satisfy some additional conditions (Hong et al. [2008], Ren [2008], Ren & Atkins [2007], Yu et al. [2010a,b]), which are somewhat different from those in multi-agent systems with first-order dynamics (Cao et al. [2008], Jadabaie et al. [2003], Olfati-Saber & Murray [2004], Ren & Beard [2005], Vicsek et al. [1995]). Note that a typical consensus protocol in multi-agent systems with second-order dynamics is distributed, and yet some centralized information depending on the spectra of the Laplacian matrix is required a priori for the control gains design, which actually did not take full advantage of the powerful distributed protocol. The main motivation of this paper is to design a fully distributed consensus protocol for multi-agent systems with second-order nonlinear dynamics to reach global consensus.

Furthermore, observe that many derived conditions for reaching second-order consensus in multi-agent systems were only sufficient and somewhat conservative (Hong et al. [2008], Ren [2008], Ren & Atkins [2007], Yu et al. [2010b]). In the studies of synchronization in complex networks (Wang & Chen [2002]), on the other hand, many works have been devoted to using adaptive strategies to adjust network parameters to derive better conditions for reaching network synchronization. For example, in Chen et al. [2007], Yu et al. [2009], Zhou et al. [2006], adaptive laws were applied on the control gains from the leader to the nodes, while in Chen et al. [2007], Yu et al. [2009], the adaptive schemes were actually designed for the coupling strength of the whole network using the information of the states of all agents, which are therefore centralized algorithms. In Lu [2007], Zhou & Kurths [2006], an algorithm was proposed for updating the coupling strengths for network synchronization. Recently, a general distributed adaptive strategy on the coupling

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weights was proposed and some theoretical conditions were derived for reaching network synchronization in DeLellis et al. [2009]. However, not much work has been devoted to the distributed control gains design for synchronization in complex networks or consensus in multi-agent systems with second-order dynamics. One contribution of this paper is to develop a simple distributed adaptive strategy for the control gains of multi-agent systems with second-order dynamics, and the derived conditions are very easy to verify and apply.

2. PRELIMINARIES

In this section, some basic concepts and results on algebraic graph theory (Godsil & Royle [2001]) are first introduced.

Let $G = (V, E, G)$ be a weighted undirected graph of order $N$, with $V = \{v_1, v_2, \ldots, v_N\}$ representing the set of nodes, $E \subseteq V \times V$ the set of undirected edges, and $G = (G_{ij})_{N \times N}$ the underlying weighted adjacency matrix with nonnegative elements. An undirected edge $e_{ij}$ in graph $G$ is denoted by the unordered pair of nodes $(v_i, v_j)$, meaning that nodes $v_i$ and $v_j$ can exchange information with each other. In this paper, only positively weighted undirected graphs are considered, thus, $G_{ij} = G_{ji} > 0$ if and only if there is an edge $(v_i, v_j)$ between nodes $v_i$ and $v_j$ in $G$; otherwise, $G_{ij} = G_{ji} = 0$. The degree of node $v_i$ is defined by

$$k_i = \sum_{j=1, j \neq i}^{N} G_{ij}, \quad i, j = 1, 2, \ldots, N. \quad (1)$$

A path between nodes $v_i$ and $v_j$ in $G$ is a sequence of edges $(v_i, v_k), (v_k, v_l), \ldots, (v_l, v_j)$ in the graph with distinct nodes $v_k, k = 1, 2, \ldots, l$ (Godsil & Royle [2001], Horn & Johnson [1985]).

**Definition 1.** (Horn & Johnson [1985]) A graph $G$ is connected if between any pair of distinct nodes $v_i$ and $v_j$ in $G$, there exists a path between $v_i$ and $v_j, i, j = 1, 2, \ldots, N$.

The Laplacian matrix $L = (L_{ij})_{N \times N}$ is defined by

$$L_{ii} = -\sum_{j=1, j \neq i}^{N} L_{ij} = k_i, \quad L_{ij} = -G_{ij}, \quad i \neq j, \quad (2)$$

which ensures the diffusion property that $\sum_{j=1}^{N} L_{ij} = 0$.

Moreover, let $R(u)$ and $I(u)$ be the real and imaginary parts of a complex number $u$, respectively, $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the $N$ eigenvalues of the Laplacian matrix $L$, $I_m \in R^{m \times m} (O_m \in R^{m \times m})$ be the $m$-dimensional identity (zero) matrix, and $I_m \in R^{m \times m} (0_m \in R^{m \times m})$ be the vector with all entries being $1 (0)$.

**Lemma 1.** (Horn & Johnson [1985])

(1) The Laplacian matrix $L$ in an undirected graph $G$ is semi-positive definite. It has a simple zero eigenvalue and all the other eigenvalues are positive if and only if the graph $G$ is connected.

(2) The second smallest eigenvalue $\lambda_2(L)$ of the Laplacian matrix $L$ in the undirected graph $G$ satisfies

$$\lambda_2(L) = \min_{x^T 1_N = 0, x \neq 0_N} \frac{x^T L x}{x^T x}.$$ 

The second-order consensus protocol in multi-agent dynamical systems is described by Ren [2008], Ren & Atkins [2007], Yu et al. [2010a]

$$\dot{x}_i(t) = v_i(t),$$

$$\dot{v}_i(t) = u_i(t),$$

$$u_i(t) = \alpha \sum_{j=1, j \neq i}^{N} G_{ij} (x_j(t) - x_i(t)) + \beta \sum_{j=1, j \neq i}^{N} G_{ij} (v_j(t) - v_i(t)), \quad i = 1, 2, \ldots, N,$$

where $x_i \in R^n$ and $v_i \in R^d$ are the position and velocity states of the $i$th agent (node), respectively, $u_i$ is the control input, $\alpha > 0$ and $\beta > 0$ are the coupling strengths (gains), and $G = (G_{ij})_{N \times N}$ is the weighted adjacency matrix of the network.

The second-order consensus protocol in multi-agent system (3) is distributed since each agent only uses local information of neighboring agents in the control input. In most cases, second-order consensus can be reached if the coupling gains and the spectra of the Laplacian matrix satisfy some additional conditions (Hong et al. [2008], Ren [2008], Ren & Atkins [2007], Yu et al. [2010a,b]), for example, second-order consensus in multi-agent system (3) can be reached if and only if

$$\frac{\beta^2}{\alpha} > \max_{2 \leq i \leq N} \frac{R(\lambda_i)|R^2(\lambda_i) + I^2(\lambda_i)|}{R(\lambda_i)} \quad (\text{Yu et al. [2010a]).}$$

To satisfy this kind of conditions, a centralized information scheme depending on the spectra of the Laplacian matrix must be known a priori for the control gains design. In order to design a fully distributed consensus protocol, the following consensus protocol with updated coupling control gains is considered in this paper, even with nonlinear dynamics:

$$\dot{x}_i(t) = v_i(t),$$

$$\dot{v}_i(t) = f(x_i(t), v_i(t), t) + \alpha c_i(t) \sum_{j=1, j \neq i}^{N} G_{ij} (x_j(t) - x_i(t))$$

$$+ \beta c_i(t) \sum_{j=1, j \neq i}^{N} G_{ij} (v_j(t) - v_i(t)), \quad i = 1, 2, \ldots, N,$$

where $f : R^n \times R^n \times R^+ \rightarrow R^n$ is a continuously differentiable vector-valued nonlinear function, $\alpha > 0$ and $\beta > 0$ are coupling strengths, and $c_i(t)$ is the time-varying distributed control gain of the $i$th agent. The objective of introducing the time-varying control gains here is to find some adaptive distributed laws acting on the control gains such that second-order consensus can be reached in the multi-agent system (4) without requiring the knowledge of the spectra of the Laplacian matrix of the graph.
Definition 2. Second-order consensus in multi-agent system (4) is said to be achieved if, for any initial conditions,
\[ \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0, \]
\[ \forall i, j = 1, 2, \ldots, N. \]

Clearly, since \( \sum_{j=1}^{N} L_{ij} = 0 \), if consensus can be achieved, any solution \((x_i^0(t), v_i^0(t))\) \(\in \mathbb{R}^{2n} \) of system (4) is a trajectory of an isolated node satisfying
\[
\dot{x}_0(t) = v_0(t), \quad v_0(t) = f(x_0(t), v_0(t), t).
\]
Here, \((x_i^0, v_i^0)^T \in \mathbb{R}^{2n} \) will be used to describe a group leader, followed by agents satisfying (4).

Because of (2), system (4) can be equivalently rewritten as follows:
\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= f(x_i(t), v_i(t), t) - \alpha c_i(t) \sum_{j=1}^{N} L_{ij} x_j(t) \\
&\quad - \beta c_i(t) \sum_{j=1}^{N} L_{ij} v_j(t), \quad i = 1, 2, \ldots, N.
\end{align*}
\]

Lemma 2. (Boyd et al. [1994]) (Schur Complement) The following linear matrix inequality (LMI)
\[
\begin{pmatrix}
Q(x) & S(x) \\
S(x)^T & R(x)
\end{pmatrix} > 0,
\]
where \(Q(x) = Q(x)^T, R(x) = R(x)^T \), is equivalent to one of the following conditions:
(i) \(Q(x) > 0, R(x) - S(x)^T Q(x)^{-1} S(x) > 0\);
(ii) \(R(x) > 0, Q(x) - S(x) R(x)^{-1} S(x)^T > 0\).

Lemma 3. (Horn & Johnson [1991]) For matrices \(A, B, C\) and \(D\) with appropriate dimensions, the Kronecker product \(\otimes\) satisfies
(i) \((\phi A) \otimes B = A \otimes (\phi B)\), where \(\phi\) is a constant;
(ii) \((A + B) \otimes C = A \otimes C + B \otimes C\);
(iii) \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\);
(iv) \((A \otimes B)^T = A^T \otimes B^T\).

Lemma 4.
(i) Let \(L\) be the Laplacian matrix of the undirected graph \(G\). Then, the matrix \(L^2\) is semi-positive definite. It has a simple eigenvalue 0 and all the other eigenvalues satisfy \(0 < \lambda_2^2 \leq \cdots \leq \lambda_N^2\) if and only if the graph \(G\) is connected.
(ii) The second smallest eigenvalue \(\lambda_2(L^2)\) of the matrix \(L^2\) satisfies \(\lambda_2(L^2) = \lambda_2(L) = \min_{x^T 1_N = 0, x \neq 0} \frac{x^T L^2 x}{x^T x}\).

Proof. Let \(A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)\) be the diagonal form associated with the Laplacian matrix \(L\), i.e., there exists an unitary matrix \(P = (p_1, p_2, \ldots, p_N)\) such that \(P^T LP = A\). It is easy to verify that \(p_1\) is the eigenvector of \(L\) associated with the eigenvalue \(\lambda_1\), i.e., \(L p_1 = \lambda_1 p_1\). Then, one has \(L^2 p_1 = \lambda_2^2 p_1\). Therefore, \(p_1\) is the eigenvector of \(L^2\) associated with eigenvalue \(\lambda_2^2\). From Lemma 1, (i) is proved.

Let \(x = Py\) and \(y = (y_1, \ldots, y_N)^T\). From \(L p_1 = 0\), it is easy to get that \(p_1 = 1_N/\sqrt{N}\). If \(x^T 1_N = 0\), then \(y_1 = p_1^T x = 0\). Since \(L^2 = P A^2 P^T\), one obtains
\[
\lambda_2(L^2) = \min_{x^T 1_N = 0, x \neq 0} \frac{x^T L^2 x}{x^T x}.
\]

Lemma 5. For any vectors \(x, y \in \mathbb{R}^n\) and positive definite matrix \(G \in \mathbb{R}^{n \times n}\), the following matrix inequality holds:
\[
2x^T y \leq x^T G x + y^T G^{-1} y.
\]

Lemma 6. (Chen et al. [2007]) If \(L\) is irreducible, \(L_{ij} \leq 0\) for \(i \neq j\), and \(\sum_{j=1}^{N} L_{ij} = 0\), \(i = 1, 2, \ldots, N\), then, for any constant \(\varepsilon > 0\), all eigenvalues of the matrix
\[
\begin{pmatrix}
L_{11} + \varepsilon & L_{12} & \cdots & L_{1N} \\
L_{21} & L_{22} & \cdots & L_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1} & L_{N2} & \cdots & L_{NN}
\end{pmatrix}
\]
are positive.

Lemma 7. (Slotine & Li [1991]) (Barbalat’s lemma) If \(\dot{V}(x, t)\) satisfies the following conditions:
(i) \(\dot{V}(x, t)\) is lower bounded;
(ii) \(\dot{V}(x, t)\) is negative semi-definite;
(iii) \(\dot{V}(x, t)\) is uniformly continuous in time or \(\ddot{V}(x, t)\) is bounded,
then \(V(x, t) \to 0\) as \(t \to \infty\).

3. DISTRIBUTED CONTROL GAINS DESIGN

In this section, distributed control gains design for system (6) is considered.

Assumption 1. There exist nonnegative constants \(\rho_1\) and \(\rho_2\) such that
\[
\|f(x, v, t) - f(y, z, t)\| \leq \rho_1 \|x - y\| + \rho_2 \|v - z\|, \\
\forall x, y, v, z \in \mathbb{R}^n.
\]
Let \( \hat{x}(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(t) \) and \( \hat{v}(t) = \frac{1}{N} \sum_{j=1}^{N} v_j(t) \) be the average position and velocity states of all agents, and \( \hat{x}_i(t) = x_i(t) - \hat{x}(t) \) and \( \hat{v}_i(t) = v_i(t) - \hat{v}(t) \) represent the position and velocity vectors relative to the average position and velocity of agents in system (6), respectively. Then, the following error dynamical system can be obtained by a simple subtraction:

\[
\begin{align*}
\dot{\hat{x}}_i(t) &= \hat{v}_i(t), \\
\dot{\hat{v}}_i(t) &= f(x_i(t), v_i(t), t) - \frac{1}{N} \sum_{k=1}^{N} f(x_k(t), v_k(t), t) \\
&\quad - \alpha c_i(t) \sum_{j=1}^{N} L_{ij} \hat{x}_j(t) - \beta c_i(t) \sum_{j=1}^{N} L_{ij} \hat{v}_j(t) \\
&\quad + \alpha \sum_{k=1}^{N} \sum_{j=1}^{N} c_k(t) L_{kj} \hat{x}_j(t) \\
&\quad + \beta \sum_{k=1}^{N} \sum_{j=1}^{N} c_k(t) L_{kj} \hat{v}_j(t), \\
&\quad i = 1, 2, \ldots, N.
\end{align*}
\]

Let \( \hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)^T, \hat{v} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_N)^T, \hat{y} = (\hat{x}^T, \hat{v}^T)^T, \) and \( f(x, v, t) = (f^T(x_1(t), v_1(t), t), \ldots, f^T(x_N(t), v_N(t), t))^T. \) Then, system (7) can be recast into a compact matrix form as follows:

\[
\dot{\hat{y}}(t) = F(x, v, t) + (\hat{\Lambda} \otimes I_N) \hat{y}(t) + G(x, v),
\]

where \( F(x, v, t) = \left( \left( (I_N - L_N^T N/N) \otimes I_N \right) f(x, v, t) \right), \)
\[
\hat{\Lambda} = \left( \begin{array}{ccccc}
O_N & I_N \\
-\alpha C(t)L - \beta \hat{v}(t)L & 0_{N_N} \\
\end{array} \right),
\]
\[
G(x, v) = \left( \begin{array}{ccccc}
0_{N_N} \\
1_N \otimes g(x, v) \\
\end{array} \right),
\]
\[
ge(t) = \alpha \sum_{k=1}^{N} \sum_{j=1}^{N} c_k(t) L_{kj} x_j(t) + \beta \sum_{k=1}^{N} \sum_{j=1}^{N} c_k(t) L_{kj} v_j(t).
\]

**Theorem 1.** Suppose that the graph \( \mathcal{G} \) is connected and Assumption 1 holds. Then, second-order consensus in system (4) can be reached under the following distributed adaptive laws:

\[
\dot{c_i}(t) = \xi_i \left( \alpha \sum_{j=1}^{N} L_{ij} x_j(t)^T \sum_{j=1}^{N} L_{ij} x_j(t) + \beta \sum_{j=1}^{N} L_{ij} v_j(t)^T \sum_{j=1}^{N} L_{ij} v_j(t) \right)
\]

\[
+ \left( \beta + \alpha \gamma \right) \left( \sum_{j=1}^{N} L_{ij} x_j(t)^T \sum_{j=1}^{N} L_{ij} v_j(t) \right),
\]

where \( \xi_i > 0 \) and \( \gamma > 0 \) are constants.

**Proof.** Consider the following Lyapunov function candidate:

\[
V(\hat{y}, c, t) = \frac{1}{2} \hat{y}^T(t)(\hat{\Omega} \otimes I_N) \hat{y}(t) + \sum_{i=1}^{N} \frac{\varepsilon}{2} (c_i(t) - \hat{c}_i(t))^2,
\]

where \( \hat{\Omega} = \left( \begin{array}{ccccc}
\mu L^2 & \varepsilon L \\
\varepsilon L & \eta L \\
\end{array} \right), \]

\[
c = (c_1, \ldots, c_N)^T, \hat{c}_i > 0 \text{ are constants to be determined, } \mu \gg \varepsilon > 0, \text{ and } \eta > 0. \]

It will be shown that \( V(\hat{y}, c, t) \geq 0 \) and \( V(\hat{y}, c, t) = 0 \) if and only if \( \hat{y}(t) = 0 \) and all \( c_i(t) = \hat{c}_i. \) From Lemmas 1 and 4, one has

\[
V(\hat{y}, c, t) = \frac{1}{2} \hat{y}^T(t)(L^2 \otimes I_N) \hat{y}(t) + \frac{\varepsilon}{2} \hat{y}^T(t)(L \otimes I_N) \hat{y}(t)
\]

\[
+ \frac{\varepsilon}{2} \hat{y}^T(t)I_N \hat{y}(t) + \sum_{i=1}^{N} \frac{\varepsilon}{2} (c_i(t) - \hat{c}_i)^2,
\]

where \( \hat{y}^T(1_N = 0, \hat{v}^T 1_N = 0, \) and

\[
\hat{\Omega} = \left( \begin{array}{ccccc}
\mu L^2 & \varepsilon L \\
\varepsilon L & \eta L \\
\end{array} \right).
\]

By Lemma 2, \( \hat{\Omega} > 0 \) is equivalent to that \( \eta \lambda_2(L) > 0 \) and \( \mu L^2 \eta \lambda_2(L) - \varepsilon^2 > 0, \) which are satisfied since \( \mu \gg \varepsilon > 0. \) Therefore, \( V(\hat{y}, c, t) \geq 0 \) and \( V(\hat{y}, c, t) = 0 \) if and only if \( \hat{y}(t) = 0 \) and \( c_i(t) = \hat{c}_i, i = 1, 2, \ldots, N. \)

By Lemma 3, taking the derivative of \( V(t) \) along the trajectories of (8) yields

\[
\dot{V}(\hat{y}, c, t) = (\varepsilon \hat{y}^T(t) + \eta \hat{v}^T(t))[L(I_N - L_N^T N/N) \otimes I_N] f(x, v, t)
\]

\[
+ (\varepsilon \hat{v}^T(t) + \eta \hat{y}^T(t))[L^2 \otimes I_N] \hat{y}(t)
\]

\[
+ \sum_{i=1}^{N} \frac{\varepsilon}{2} (c_i(t) - \hat{c}_i)^2,
\]

\[
= \left[ \varepsilon \hat{v}^T(t) + \eta \hat{y}^T(t) \right] \left[ L \otimes I_N \right] f(x, v, t)
\]

\[
- \eta \lambda_2(L) \hat{y}(t)
\]

\[
+ \sum_{i=1}^{N} \frac{\varepsilon}{2} (c_i(t) - \hat{c}_i)^2.
\]

By simple calculation, one has

\[
\frac{1}{2} \left( \Omega \hat{L} + \hat{L}^T \Omega \right)
\]

\[
= \left( -\alpha \epsilon LC(t)L \beta \epsilon L^2 - (\beta \epsilon + \alpha \gamma)LC(t)L^2 \right).
\]

By Assumption 1 and Lemma 5, one obtains
\[
\dot{\hat{x}}(t)(L \otimes I_n)[f(x, v, t) - 1_N \otimes f(\overline{x}, \overline{v}, t)] \\
\leq \rho_1 L_{\max} N \sum_{i=1}^{N} \|\hat{x}_i\|^2 + \frac{\rho_2 L_{\max} N}{2} \sum_{i=1}^{N} (\|\hat{x}_i\|^2 + \|\hat{v}_i\|^2),
\]

(14)

and

\[
\dot{\hat{v}}(t)(L \otimes I_n)[f(x, v, t) - 1_N \otimes f(\overline{x}, \overline{v}, t)] \\
\leq \rho_2 L_{\max} N \sum_{i=1}^{N} \|\hat{v}_i\|^2 + \frac{\rho_1 L_{\max} N}{2} \sum_{i=1}^{N} (\|\hat{x}_i\|^2 + \|\hat{v}_i\|^2),
\]

where \( L_{\max} = \max_{i,j=1,\ldots,N} |L_{ij}|. \)

Since \( \sum_{j=1}^{N} L_{ij} = 0 \), one obtains \( \sum_{i=1}^{N} L_{ij} \hat{x}_j = \sum_{i=1}^{N} L_{ij} x_j \) and \( \sum_{i=1}^{N} L_{ij} \hat{v}_j = \sum_{i=1}^{N} L_{ij} v_j \). Combining adaptive laws in (9), (12)-(15), and letting \( \gamma = \eta/\varepsilon \), one has

\[
\dot{V}(t) \leq \left( \varepsilon \rho_1 + (\eta \rho_1 + \varepsilon \rho_2)/2 \right) L_{\max} N \sum_{i=1}^{N} \|\hat{x}_i\|^2 \\
+ \left( \eta \rho_2 + (\eta \rho_1 + \varepsilon \rho_2)/2 \right) L_{\max} N \sum_{i=1}^{N} \|\hat{v}_i\|^2 \\
- \alpha \varepsilon \hat{x}^T (L \hat{C} L \ I_n) \hat{x} \\
+ \hat{v}^T (\varepsilon L - \beta \eta L \hat{C} L \ I_n) \hat{v} \\
+ \hat{x}^T (\mu L^2 - (\beta \varepsilon + \alpha \eta)L \hat{C} L \RowAt I_n) \hat{v},
\]

(16)

where \( \hat{C} = \text{diag}(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_N) \). Let \( \hat{c}_i = \hat{c} \) and \( \mu = (\beta \varepsilon + \alpha \eta) \hat{c} \). From Lemmas 4 and 5, one has

\[
\dot{V}(\hat{y}, c, t) \\
\leq \left( \alpha \varepsilon \hat{x}^T \hat{L} \hat{x} - \rho_1 L_{\max} N - (\eta \rho_1 + \varepsilon \rho_2)L_{\max} N/2 \right) \dot{\hat{x}}^T \hat{x} \\
+ \hat{x}^T \left( \beta \varepsilon \hat{x}^T \hat{L} \hat{x} - \rho_2 L_{\max} N - \varepsilon \lambda N - (\eta \rho_1 + \varepsilon \rho_2) L_{\max} N/2 \right) \hat{x} \\
+ \hat{v}^T(\mu L^2 - (\beta \varepsilon + \alpha \eta) L \hat{C} L \RowAt I_n) \hat{v}.
\]

(17)

Choose \( \hat{c} \) sufficiently large such that \( \alpha \varepsilon \hat{x}^T \hat{L} \hat{x} > \varepsilon \rho_1 L_{\max} N + (\eta \rho_1 + \varepsilon \rho_2) L_{\max} N/2 + 1 \) and \( \beta \varepsilon \hat{x}^T \hat{L} \hat{x} > \eta \rho_2 L_{\max} N + \varepsilon \lambda N + (\eta \rho_1 + \varepsilon \rho_2) L_{\max} N/2 + 1 \). Then, one has

\[
\dot{V}(\hat{y}, c, t) \leq -\hat{x}^T \hat{x} - \hat{v}^T \hat{v}.
\]

(18)

It is easy to see that \( V(\hat{y}, c, t) \) satisfy conditions (i) and (ii) in Lemma 7. From (18), one obtains that \( \hat{y} \) and \( c \) are bounded, and thus from (8), (9) and Assumption 1, \( \hat{y} \) and \( \hat{c} \) are bounded. By \( \dot{V}(\hat{y}, c, t) \) in (12), one finally has that \( V(\hat{y}, c, t) \) is bounded. Then, the proof can be completed by using the well-known Barbalat’s Lemma (Slotine & Li [1991]) (see Lemma 7).

Remark 1. Second-order consensus in system (4) can be reached under the distributed adaptive laws given in (9) without requiring any additional centralized conditions, unlike most results in the literature (Hong et al. [2008], Ren [2008], Ren & Atkins [2007], Yu et al. [2010a,b]). In the design of distributed adaptive laws for the control gains here, each agent only uses local information of neighboring agents. Given the parameters \( \alpha, \beta, \) and \( \gamma \), one can choose a small \( \xi \) such that \( c_i(t) \) changes very slowly.

4. SIMULATION EXAMPLE

Example 1. Distributed control gains design for consensus in general multi-agent systems with second-order nonlinear dynamics

Consider the second-order consensus protocol with time-varying velocities in system (4), where \( \alpha = \beta = \ldots \)
1. the Laplacian matrix of the graph $G$ is given by
\[
\begin{pmatrix}
5 & -2 & -3 & 0 \\
-2 & 6 & 0 & -4 \\
-3 & 0 & 3 & 0 \\
0 & -4 & 0 & 4
\end{pmatrix},
\]
and the nonlinear function $f$ is described by Chua's circuit (Chua [1992])
\[
f(x_i(t), v_i(t), t) = \left( \begin{array}{c}
\zeta(-v_{i1} + v_{i2} - l(v_{i1})), \\
v_{i1} - v_{i2} + v_{i3}, \\
-\theta v_{i2},
\end{array} \right),
\] (19)

with $l(v_{i1}) = bv_{i1} + 0.5(a - b)(|v_{i1} + 1| - |v_{i1} - 1|)$. System (19) is chaotic when $\zeta = 10$, $\theta = 18$, $a = -4/3$, and $b = -3/4$. It is easy to verify that Assumption 1 is satisfied. The distributed adaptive laws given in Theorem 1 are designed, where $\xi = 0.1$ and $\gamma = 1$. By Theorem 1, one knows that second-order consensus in system (4) can be reached under the distributed adaptive laws (9). The position and velocity states of all the agents are shown in Fig. 1. A distinct feature is that the designed distributed control gains are very small, as shown in Fig. 2, which can be hardly achieved by other control schemes proposed in the literature.

5. CONCLUSIONS

In this paper, the distributed control gains design for leaderless consensus in multi-agent systems with second-order nonlinear dynamics has been investigated. An distributed adaptive strategy for the control gains has been developed based on local information. It has been found that consensus can be reached under the designed distributed control gains only if the undirected network is connected.

However, it is still an unsolved challenging problem about designing distributed control gains in multi-agent systems with second-order dynamics and general directed topologies, leaving an important and interesting topic for future research.

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REFERENCES


