Inventory optimization of distribution networks with discrete-event processes by vendor-managed policies

Simona Sacone and Silvia Siri

* Department of Communications, Computer and Systems Science
University of Genova

e-mail: simona.sacone@unige.it, silvia.siri@unige.it

Abstract: The objective of this work is to study optimal Vendor-Managed Inventory policies in distribution systems in which demand and delivery processes are characterized by a discrete-event dynamics. In Vendor-Managed Inventory policies, the supplier organizes the deliveries to the customer in order to satisfy its demand by minimizing transportation costs and holding costs of the customer. In the considered case, a simple network composed of one supplier and one customer is studied, in which the supplier can use a limited number of capacitated vehicles; the objective is to plan the delivery policy over a finite horizon. The optimization problem resulting from the considered system characterized by discrete-event processes is then rewritten in a discrete-time form. For this problem different solution methods are proposed in the paper, based on the statement of mathematical programming problems and the definition of appropriate algorithms. These methods are then compared through some experimental results.

Keywords: Inventory control, transportation, discrete-event systems, optimization problems.

1. INTRODUCTION

This paper deals with distribution systems characterized by Vendor-Managed Inventory (VMI) policies. According to such a policy, the customer replenishment is determined by the supplier, that monitors the inventory of each customer and then acts as a central decision maker. The VMI policy has two main advantages. On one hand, the supplier can plan the replenishment policy (routes and delivery times) in a more efficient way by combining the deliveries to different customers. On the other hand, customers do not pay costs associated with inventory monitoring and ordering. Some research works are devoted to analyse VMI supply chain strategies in comparison with traditional managing policies, as in Disney et al. (2003) and Disney and Towill (2003).

When a VMI policy is applied, the supplier plans the replenishment to the customer by minimizing transportation costs and inventory costs for the customer. For this reason, the supplier must integrate two important decision aspects, i.e. inventory control and vehicle routing. In literature, the joined and coordinated problem of inventory management and vehicle routing is known as Inventory-Routing Problem (IRP). As highlighted in Campbell and Savelbergh (2004), most of the works on IRP can be divided in three research streams. First of all, the early works concerned single-day models, as in Federgruen and Zipkin (1984); with these myopic and simplified approaches all the deliveries are postponed except those necessary for the considered day. A second research direction regards multi-day models, sometimes faced by considering rolling-horizon frameworks, as in Jaillet et al. (2002). Thirdly, there are permanent routing approaches in which the objective is to find a schedule or a transportation frequency to be repeated indefinitely over an infinite horizon (see for instance Bertazzi and Speranza (2005) and Zhao et al. (2007)).

Most of the research works on IRP refer to time-driven systems, represented with continuous-time or discrete-time models, whereas few research works consider discrete-event modelling approaches. In this work the processes of demand and delivery are characterized by a discrete-event dynamics; this means that these processes are driven by specific events that occur in asynchronous time instants over the time horizon. By exploiting some features of the considered system, the discrete-event model is formulated in a discrete-time framework in which a discrete-time version of the planning problem is stated.

In this work we consider a simple case of distribution system, composed of one supplier and one customer, in which the decisions concern the delivery to the customer in order to meet his demand over a finite planning horizon and by minimizing transportation and inventory costs; a fleet of capacitated vehicles is also considered. Many research works can be found in the literature referring to VMI distribution systems with one supplier and one customer. One research stream refers to planning problems over an infinite horizon; for instance, in Bertazzi and Speranza (2001) the authors consider the problem of shipping a set of products from an origin to a destination, with an unlimited fleet of vehicles, over an infinite horizon, and their purpose is to determine a periodic shipping policy. A similar model is considered in Bertazzi and Speranza (2002) where both continuous and discrete shipping strategies are determined and discussed. In Anily and Tzur (2005) the same distribu-
tion system with one supplier and one customer is treated but over a finite horizon; the main difference of Anily and Tzur (2005) with respect to the present approach is that in Anily and Tzur (2005) the number of vehicles available in each time period is unlimited (whereas in this work we consider a limited fleet of vehicles) and the deliver of multiple items is considered (instead, in this work, only one type of products is treated).

Other approaches consider for instance one supplier and multiple customers, as in Kang and Kim (2010), where again the number of vehicles is supposed to be large enough to handle all delivery requirements. In this paper we present the development of the work proposed in Sacone and Siri (2010); the model and the planning problem considered here is the same already described in Sacone and Siri (2010), while the main novelty of this paper stands in the definition of different solution methods for the proposed planning problem and the discussion of some experimental results in which these methods are compared and their performances are evaluated. Finally, it is important to point out that a VMI approach similar to the one proposed in this work is modelled and solved by the well-known Wagner-Whitin model (see Wagner and Whitin (2004)), typically used in inventory problems. In that case a discrete-time problem with a deterministic time-varying demand process is considered and, analogously to the present work, in each time instant it is possible to place a delivery or not. Anyway, the main difference between the two models stands in the fact that transportation operations are here explicitly considered, not only by including the transportation cost in the cost function, but also by modelling the use of a finite number of vehicles with bounded capacity.

The paper is organized as follows. In Section 2 the considered discrete-event model is described and the corresponding discrete-time formulation is provided and discussed. Some properties of the optimal solution are reported in Section 3 where different solution methods (both optimal and suboptimal) are proposed and analysed. In Section 4 some experimental results are reported and commented on. Finally, some conclusive remarks are drawn in Section 5.

2. MODEL DESCRIPTION

We consider a distribution system characterized by a single origin (one supplier) and a single destination (one customer), managed according to a Vendor-Managed Inventory policy. This means that the supplier must organize the deliveries (in terms of timing and size) to the customer, in order to satisfy its demand by minimizing transportation costs and the holding costs associated with the inventory of the customer. In the considered system, we suppose that the supplier can ship products by using a given fleet of vehicles, composed of a fixed number of capacitated vehicles. Let us denote with $I(t)$ the inventory level of the customer at time $t$, i.e. the system state variable. Moreover, the delivery process of goods (transported between the supplier and the retailer) is indicated as $q(t)$ whereas the demand process is denoted as $d(t)$.

The demand of the customer is supposed to be characterized by a discrete-event dynamics and to be deterministic and completely known. This demand is given by some instantaneous and asynchronous requests expressed as a set of required quantities $d_j$, $j = 1, \ldots, D$, and the corresponding time instants $t_j$, $j = 1, \ldots, D$. In other words, the $j$-th request at the customer corresponds to a quantity $d_j$ required at time $t_j$. The deliveries can be realized with $V$ vehicles, each one characterized by a maximum capacity $K$. Moreover, in this model we consider a minimum time interval $\Delta$ between one delivery and the next one; this time takes into account the travel time between the supplier and the customer and also possible agreements between them (this also implies that there is a minimum distance equal to $\Delta$ between two subsequent due dates). The delivery process is supposed to be characterized by a discrete-event dynamics as well, and it is represented as a sequence of transported quantities $q_i$, $i = 1, \ldots, Q$, and corresponding time instants $\tau_i$, $i = 1, \ldots, Q$. In this way, the $i$-th delivery corresponds to a quantity $q_i$ that arrives at the customer at time $\tau_i$.

As regards the optimization problem proposed in this work, the considered planning horizon coincides with the due date of the last demand, i.e. $t_D$. The decisions concern the delivery process, i.e. the values of the transported quantities $q_i$, $i = 1, \ldots, Q$, and corresponding time instants $\tau_i$, $i = 1, \ldots, Q$. As already described, in the proposed model it is assumed that the minimum time between two subsequent deliveries is $\Delta$. Then, without loss of generality, we fix $Q = t_D/\Delta$, that corresponds to treating $Q$ as the maximum number of deliveries to be realized within the considered time horizon.

The inventory level $I(t)$ is a piecewise linear function changing at asynchronous time instants, i.e. in $\tau_i$, $i = 1, \ldots, Q$, and in $t_j$, $j = 1, \ldots, D$. In particular, if a delivery is realized at time $\tau_i$, the inventory level increases of a certain quantity $q_i$, i.e. $I(\tau^+ - I(\tau^-) + q_i$. Analogously, if there is a certain request at time $t_j$, the inventory level decreases of a quantity $d_j$, i.e. $I(t^+ - I(t^-) - d_j$. Moreover, without loss of generality, it is assumed that the initial inventory is always null.

The cost terms to be considered are relevant to the transportation operations and the holding of goods in the inventory of the customer. In particular, the transportation cost $C_T$ is given by:

$$C_T = c \cdot \sum_{i=1}^{Q} v_i$$

being $c$ the unitary transportation cost (for each vehicle connecting the supplier with the customer) and $v_i$ the number of vehicles used for the $i$-th delivery. Of course, $v_i = \left\lceil \frac{q_i}{\Delta} \right\rceil$.

The inventory cost $C_I$ is defined as $H \int_{0}^{T} I(t)dt$, where $H$ is the unitary inventory cost. In other words, the cost $C_I$ is the area under $I(t)$ multiplied by $H$ and can be written as follows:

$$C_I = H \left[ \sum_{i=1}^{Q} q_i(t_D - \tau_i) - \sum_{j=1}^{D} d_j(t_D - t_j) \right]$$

In order to formalize the optimization problem, it is still necessary to define the state equations for the system state variable $I(t)$, for which the relative position of all $\tau_i$ and
$t_j$ over the time horizon must be known. Of course, this relative position of $\tau_i$ and $t_j$ is not known a priori since $t_j$, $j = 1, \ldots, D$, are problem data but $\tau_i$, $i = 1, \ldots, Q$, are decision variables. It is then necessary to introduce two sets of binary variables $y_{i,j}$ and $z_{i,j}$ as follows:

$$
y_{i,j} = \begin{cases} 
0, & \text{if } \tau_i \leq t_j \\
1, & \text{if } \tau_i > t_j 
\end{cases} \quad i = 1, \ldots, Q \quad j = 1, \ldots, D \quad (3)
$$

$$
z_{i,j} = \begin{cases} 
0, & \text{if } \tau_i < t_j \\
1, & \text{if } \tau_i \geq t_j 
\end{cases} \quad i = 1, \ldots, Q \quad j = 1, \ldots, D \quad (4)
$$

The equations providing the values of $I(t)$ in time instants $\tau_i$ and $t_j$ (immediately after the relative event, i.e. in $\tau_i^+$ and $t_j^+$) are then:

$$I(\tau_i) = \sum_{h=1}^{i} q_h - \sum_{k=1}^{D} z_{i,k} d_k \quad i = 1, \ldots, Q \quad (5)$$

$$I(t_j) = \sum_{h=1}^{j} (1 - y_{h,j}) q_h - \sum_{k=1}^{j} d_k \quad j = 1, \ldots, D \quad (6)$$

The optimization problem can then be stated, resulting in a nonlinear mixed-integer formulation (the interested reader can find the problem statement in Sacone and Siri (2010)). This kind of problem can be addressed with mathematical programming techniques even if no optimal solution algorithm is available and the computational complexity of the existing solution approaches limits its practical applicability. In the following, we derive a discrete-time version of the model, leading to a linear problem.

By adopting $\Delta$ as a time discretization interval, it is possible to redefine the discrete-event processes (demand and delivery) as discrete-time processes in which a time step $h$ is considered with $h = 1, \ldots, Q$, and the time length between two consequent time steps is just equal to $\Delta$. Note that, due to the model assumptions (and specifically to the fact that no event can happen in the system within a time interval of length $\Delta$), this does not introduce any approximation neither in the model nor in the optimization problem.

The discrete-event demand process $d(t)$ can be represented as a discrete-time process, i.e. $d(h)$, $h = 1, \ldots, Q$, as:

$$d(h) = \begin{cases} 
d_i, & \text{if } \exists i : (h-1) \Delta \leq t_i < h\Delta \\
0, & \text{otherwise} 
\end{cases} \quad (7)$$

Moreover, the discrete-time process $\bar{q}(h)$, $h = 1, \ldots, Q$, is defined: a positive value of $\bar{q}(h)$ means that a delivery is executed during the $h$-th time interval. Analogously, the variable $v_i$, $i = 1, \ldots, Q$, representing the number of vehicles used in the $i$-th delivery, is substituted by a discrete-time variable $\bar{v}(h)$, $h = 1, \ldots, Q$, indicating the number of vehicles used at time $h$.

Moreover, the inventory level $I(h)$, $h = 1, \ldots, Q$, at time $h$ can be easily written as follows:

$$I(h) = \sum_{k=1}^{h} \bar{q}(k) - \bar{d}(k) \quad h = 1, \ldots, Q \quad (8)$$

The discrete-time version of the planning problem is the following.

**Problem 1.** Given the minimum time between two subsequent deliveries $\Delta$, the number $V$ of vehicles, the maximum capacity $K$ of each vehicle, the unitary costs $c$ and $H$, and given the external demand $\bar{d}(h)$, $h = 1, \ldots, Q$, find:

$$\min_{\bar{q}(h), h = 1, \ldots, Q} c \sum_{h=1}^{Q} \bar{v}(h) + H \Delta \sum_{k=1}^{Q} \bar{q}(k) - \bar{d}(k) \quad (9)$$

subject to constraints:

$$\sum_{k=1}^{h} \bar{q}(k) - \bar{d}(k) \geq 0 \quad h = 1, \ldots, Q \quad (10)$$

$$\bar{v}(h) \geq \frac{\bar{q}(h)}{K} \quad h = 1, \ldots, Q \quad (11)$$

$$\bar{v}(h) \leq V \quad h = 1, \ldots, Q \quad (12)$$

$$\bar{q}(h) \geq 0 \quad h = 1, \ldots, Q \quad (13)$$

$$\bar{q}(h) \leq KV \quad h = 1, \ldots, Q \quad (14)$$

$$\bar{v}(h) \in \mathbb{N} \quad h = 1, \ldots, Q \quad (15)$$

In Problem 1 the cost function is given by transportation and inventory costs, and is the discrete-time version of (1) and (2). Constraints (10) impose that the inventory level in each time instant is non-negative. Constraints (11) and (15) assure that $\bar{v}(h) = \left\lfloor \frac{\bar{q}(h)}{K} \right\rfloor$, while constraints (12) bound the number of vehicles used in each time step. Finally, constraints (13) and (14) impose lower and upper bounds for the delivered quantity $\bar{q}(h)$.

Problem 1 is a linear mixed-integer mathematical programming problem which can be solved by standard techniques. However, the presence of a set of integer variables makes it cumbersome when large problem instances are considered. For this reason, the objective of this work is the development of two solution methods, alternative to the solution of Problem 1, in order to solve this planning problem with acceptable computational times.

3. DEFINITION OF DIFFERENT SOLUTION METHODS FOR THE PROBLEM

Before proposing the solution methods for Problem 1, some important results are reported in the following. For a sketch of the proof and some more comments on these results, refer to Sacone and Siri (2010).

**Proposition 1.** If, for a certain $h$, $h = 1, \ldots, Q$, it is $\bar{d}(h) \geq nK$, with $n \in \mathbb{N}$, $n \leq V$, the optimal solution of Problem 1 includes the delivery $\bar{q}^o(h) \geq nK$, with $\bar{q}^o(h) \leq KV$.

**Corollary 1.** If, for a certain $h$, $h = 1, \ldots, Q$, it is $\bar{d}(h) \geq KV$, the optimal solution of Problem 1 includes the delivery $\bar{q}^o(h) = KV$.

The meaning of Proposition 1 and Corollary 1 can be summarized in the fact that, in the present system with the considered cost function, the demand corresponding to full vehicles is delivered as late as possible. The remaining decisions concern the part of the demand that is lower than the vehicle capacity. In this case, the choice regards whether to aggregate several demands (to minimize the number of vehicles used, i.e. the transportation cost) or to let them travel separately (to minimize the inventory cost). This choice of course consists in finding the best trade-off between the two cost terms.

As a consequence, the discrete-time delivery process $\bar{q}(h)$ can be considered as the sum of two components that
are denoted as $\bar{q}^1(h)$ and $\bar{q}^2(h)$ and can be computed separately:

- $\bar{q}^1(h)$ is the delivery associated with the part of the demand multiple of the vehicle capacity and is obtained by considering the results of Proposition 1 and Corollary 1;
- $\bar{q}^2(h)$ is the delivery associated with the residual part of the demand, i.e. lower than the vehicle capacity.

In the following, some methods are proposed to determine $\bar{q}^1(h)$ and $\bar{q}^2(h)$, separately.

The first part of the delivery $\bar{q}^1(h)$ can be obtained by applying Algorithm 1, shown hereinafter. Algorithm 1 also provides the following quantities: the residual demand at time $h$, i.e. $d^h(h)$, $h = 1, \ldots, Q$, and the residual number of vehicles at time $h$, i.e. $\bar{v}^h(h)$, $h = 1, \ldots, Q$. Note that in Algorithm 1 quantities $\bar{d}^h_{alg}(h)$ and $d^0_{alg}(h)$ are simply used for computations within the algorithm itself.

**Algorithm 1 Computation of $\bar{q}^1(h)$.**

1. Initialize $\bar{d}^h_{alg}(h) = d(h)$, $h = 1, \ldots, Q$
2. Initialize $d^0_{alg}(Q + 1) = 0$
3. for $h = Q$ to 1 do
   1. $\bar{d}^h_{alg}(h) = d^h_{alg}(h) + d^0_{alg}(h + 1)$
   2. if $\bar{d}^h_{alg}(h) \geq KV$ then
      1. $\bar{q}^1(h) = KV$
      2. $d^h_{alg}(h) = \bar{d}^h_{alg}(h) - KV$
      3. $\bar{v}^h(h) = 0$
   3. else if $\bar{d}^h_{alg}(h) \geq K$ then
      1. $\bar{q}^1(h) = \left\lceil \frac{\bar{d}^h_{alg}(h)}{K} \right\rceil \cdot K$
      2. $d^h_{alg}(h) = 0$
      3. $\bar{d}^h_{alg}(h) = \bar{d}^h_{alg}(h) - \left\lceil \frac{\bar{d}^h_{alg}(h)}{K} \right\rceil \cdot K$
      4. $\bar{v}^h(h) = V - \left\lceil \frac{\bar{d}^h_{alg}(h)}{K} \right\rceil$
   4. else
      1. $\bar{q}^1(h) = 0$
      2. $d^h_{alg}(h) = 0$
      3. $\bar{d}^h_{alg}(h) = \bar{d}^h_{alg}(h)$
      4. $\bar{v}^h(h) = V$
   end if
   end if
end for

Note that the quantity $d^0_{alg}(1)$ obtained in Algorithm 1 provides a condition for verifying the problem feasibility. In particular, if $d^0_{alg}(1) > 0$, then Problem 1 is not feasible.

After applying Algorithm 1, the residual demand $d^h_{alg}(h)$ and the residual number of vehicles $\bar{v}^h(h)$ are obtained. The residual demand represents the demand that has not been served with the first part of the delivery $\bar{q}^1(h)$ and so it must be served with $\bar{q}^2(h)$. The residual number of vehicles is the number of vehicles remaining available, i.e. obtained by subtracting from $V$ the number of vehicles used in the first part of the delivery. In the following an important result is given (the proof is not reported here for space limitations).

**Proposition 2.** If, for a certain $h, h = 1, \ldots, Q$, it is $d^h_{alg}(h) = 0$, then in the optimal solution of Problem 1 the second part of the delivery associated with time instant $h$ is null, i.e. $\bar{q}^2(h) = 0$.

Proposition 2 states that the second part of the delivery $\bar{q}^2(h)$ can be null only in the time instants in which the residual demand is positive. Taking into account this result, a first possibility to compute $\bar{q}^2(h)$ corresponds to solving another mathematical programming problem, that is a partial version of Problem 1, in which the decisions on the delivery only concern the time instants $h$ in which $d^h_{alg}(h) \neq 0$. To this end, let us define the set $R$ as the subset containing the indexes of the time instants corresponding to a positive residual demand, i.e. $R = \{ h = 1, \ldots, Q : d^h_{alg}(h) \neq 0 \}$. Moreover, let us denote with $\bar{v}^2(h)$ the number of vehicles used in the delivery $\bar{q}^2(h)$.

The partial version of Problem 1 for computing $\bar{q}^2(h)$ and $\bar{v}^2(h)$ can then be stated as follows.

**Problem 2.** Given the minimum time between two subsequent deliveries $\Delta$, the maximum capacity $K$ of each vehicle, the unitary costs $c$ and $H$, the residual number of vehicles $\bar{v}^0(h), h = 1, \ldots, Q$, the residual demand $d^h(h), h = 1, \ldots, Q$, and given set $R$, find:

$$\min \sum_{h=1}^{Q} c \sum_{k=1}^{R} \bar{q}^2(k) - d^h(k)$$

subject to constraints:

$$\sum_{h=1}^{Q} \bar{q}^2(k) - d^h(k) \geq 0 \quad h = 1, \ldots, Q$$

$$\bar{v}^2(h) \geq \frac{\bar{q}^2(h)}{K} \quad h \in R$$

$$\bar{v}^2(h) \leq \bar{v}^0(h) \quad h \in R$$

$$\bar{q}^2(h) = 0 \quad h \notin R$$

$$\bar{q}^2(h) \leq K \bar{v}^0(h) \quad h \in R$$

$$\bar{q}^2(h) = 0 \quad h \notin R$$

$$\bar{v}^2(h) \in N \quad h = 1, \ldots, Q$$

Problem 2 is a MIP problem, as Problem 1, but characterized by a lower number of variables. As a matter of fact, in Problem 1 the number of variables is $2 \cdot Q$, whereas in Problem 2 it is $2 \cdot |R|$.

Besides Problem 2, another possibility to compute $\bar{q}^2(h)$ is by applying Algorithm 2, through the definition of an appropriate graph. Let $N$ be the number of instants such that $d^h_{alg}(h) \neq 0$, i.e. $N = |R|$. Then, a graph with $N + 1$ nodes is built; for each node $n = 1, \ldots, N$, the function $\eta(n)$ identifies the corresponding time instant. One arc $(n, m)$ means that in $\eta(n)$ there is a delivery equal to the residual demand relevant to nodes from $n$ to $m - 1$, i.e. $\sum_{l=n}^{m-1} d^l(\eta(l))$. The arc must exist only if the number of vehicles resulting from this delivery is not greater than the residual number of vehicles in $\eta(n)$. This graph is built in order to represent all the possible alternatives (in terms of delivery process) to satisfy the residual demand. Moreover, a cost is associated with each arc (representing the sum of transportation and inventory costs relevant to
the considered choice). Then, the optimal delivery process \( q^2(h), h = 1, \ldots, Q \), is obtained by finding the shortest path in the graph.

Algorithm 2 Computation of \( q^2(h) \).

- Insert \( N + 1 \) nodes
- Insert all the edges \((n,m)\) such that \( m > n \) and 
  \[
  \sum_{i=1}^{m-1} \frac{d^k(\eta(l))}{K} \leq \bar{v}^k(\eta(n))
  \]
- Associate to each edge \((n,m)\) a cost given by 
  \[
  c = \sum_{i=1}^{m-1} \frac{d^k(\eta(l))}{K} + H \sum_{i=1}^{m-1} d^k(\eta(l))(\eta(l) - \eta(n))
  \]
- Find the shortest path and let \( \mathcal{P} \) denote the set of edges belonging to the shortest path
- For each edge \((n,m)\) \( \in \mathcal{P} \) define 
  \[
  q^2(\eta(n)) = \sum_{l=m}^{n} d^k(\eta(l))
  \]

Once \( q^1(h) \) and \( q^2(h) \) have been computed, respectively, the overall deliveries are obtained as their sums, as specified in Algorithm 3. Then, the deliveries are realized in the time instants in which \( q(h) \neq 0 \) and the delivered quantities correspond to \( q(h) \). Moreover, the overall number \( v(h) \) of vehicles used can be obtained as the sum of the vehicles used in the first and second part of the delivery.

Algorithm 3 Computation of \( q(h) \) and \( v(h) \).

for \( h = 1 \) to \( Q \) do 
  \[
  q(h) = q^1(h) + q^2(h)
  \]
  \[
  v(h) = \left[ \frac{q^1(h)}{K} \right] + \bar{v}(h)
  \]
end for

It is now possible to propose three different solution methods in order to face the discrete-time planning problem in the considered VMI system:

- Method A: solving Problem 1 to compute directly \( q(h) \);
- Method B: applying Algorithm 1 to compute \( q^1(h) \), solving Problem 2 to compute \( q^2(h) \) and, then, applying Algorithm 3 to compute \( q(h) \);
- Method C: applying Algorithm 1 to compute \( q^1(h) \), applying Algorithm 2 to compute \( q^2(h) \) and, then, applying Algorithm 3 to compute \( q(h) \).

Method A corresponds to computing the optimal solution of the overall planning problem (Problem 1) by a specific solver; as already pointed out, this is a MIP problem that can be solved by mathematical programming techniques (branch-and-bound algorithm) with exponential complexity, therefore the computational times can become very large for large problem instances (as shown in Section 4).

Method B allows to find the optimal solution of the planning problem as well (since it can be proven that Algorithm 1 provides the optimal solution as regards the first part of the delivery). In this case, again, a MIP problem must be solved (Problem 2) but this problem is characterized by a lower number of variables, as already analysed; then the computational times are expected to be lower than with method A (as shown in Section 4).

Finally, Method C does not provide the optimal solution (since it can be shown that Algorithm 2 provides a suboptimal solution as regards the computation of \( q^2(h) \)), but the solution is obtained in computational times that are polynomial in the number of time steps. Therefore this method can be useful in case of large problem instances; in addition, another strength of Method C refers to the fact that no MIP solver is necessary and the solution is simply obtained by applying three algorithms.

4. EXPERIMENTAL RESULTS

In order to evaluate the effectiveness of the proposed approach and to compare the different solution methods, an extensive experimental campaign has been performed. To do that, we have implemented the different methods with Matlab, by using Cplex MIP solver for the solution of Problem 1 and Problem 2.

<table>
<thead>
<tr>
<th>( H/c = 100 )</th>
<th>( H/c = 10 )</th>
<th>( H/c = 1 )</th>
<th>( H/c = 0.1 )</th>
<th>( H/c = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = 30 )</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>( Q = 50 )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.09</td>
</tr>
<tr>
<td>( Q = 80 )</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Table 1. Computational times for Problem 1 (Part I).

<table>
<thead>
<tr>
<th>( H/c = 100 )</th>
<th>( H/c = 10 )</th>
<th>( H/c = 1 )</th>
<th>( H/c = 0.1 )</th>
<th>( H/c = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = 100 )</td>
<td>0.02</td>
<td>0.01</td>
<td>0.03</td>
<td>0.31</td>
</tr>
<tr>
<td>( Q = 200 )</td>
<td>0.03</td>
<td>0.03</td>
<td>0.06</td>
<td>0.69</td>
</tr>
<tr>
<td>( Q = 500 )</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
<td>140.62</td>
</tr>
</tbody>
</table>

Table 2. Computational times for Problem 1 (Part II).

First of all, it is interesting to analyse the computational times for solving Problem 1 (corresponding to Method A). At this purpose we have tested the MIP solver on some randomly generated instances and the main result is that the computational times for solving Problem 1 depend on the problem sizes (i.e. the planning horizon \( Q \)) but also depend on the trade-off between the cost terms (i.e. \( H \) and \( c \)). In Table 1 and Table 2 the average computational times (for 5 random instances per group) are reported when \( Q \) and \( H/c \) vary. The time limit for the solver has been fixed to 900 seconds. Note that the computational times increase if \( Q \) increases, but above all they increase when the unitary transportation cost \( c \) is much larger than the unitary inventory cost \( H \). This behavior can be partly motivated by the following consideration: for high values of \( H \) (with respect to \( c \)), the optimal solution corresponds to deliver as late as possible, then leading to rather trivial solutions; when instead \( H \) is small, the optimal solution concerns the choice of the best aggregation of demands in a small number of deliveries, corresponding to not trivial solutions, more difficult to find. Of course this is only
an empirical consideration, that we are trying to prove mathematically. Referring again to Table 1 and Table 2, the instances characterized by \( H/c = 0.01 \) and \( Q \geq 100 \) are not solved optimally after the time limit of 900 seconds. These results motivate the importance of using Methods B and C, that are characterized by lower computational times, as shown just in the following.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Objective A</th>
<th>Objective B</th>
<th>Objective C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>46005(*)</td>
<td>46005</td>
<td>46255</td>
</tr>
<tr>
<td>2</td>
<td>23384</td>
<td>23384</td>
<td>23542(*)</td>
</tr>
<tr>
<td>3</td>
<td>64590(*)</td>
<td>64590</td>
<td>64810</td>
</tr>
<tr>
<td>4</td>
<td>31516</td>
<td>31516</td>
<td>31548</td>
</tr>
<tr>
<td>5</td>
<td>29430</td>
<td>29430</td>
<td>29615</td>
</tr>
<tr>
<td>6</td>
<td>900</td>
<td>0.67</td>
<td>0.06</td>
</tr>
<tr>
<td>7</td>
<td>29430</td>
<td>29430</td>
<td>29615</td>
</tr>
<tr>
<td>8</td>
<td>900</td>
<td>0.54</td>
<td>0.11</td>
</tr>
<tr>
<td>9</td>
<td>900</td>
<td>1.23</td>
<td>0.13</td>
</tr>
<tr>
<td>10</td>
<td>31516</td>
<td>31516</td>
<td>31548</td>
</tr>
</tbody>
</table>

Table 3. Comparison of objective functions - (*)=solver stopped by time limit.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Comp. time A</th>
<th>Comp. time B</th>
<th>Comp. time C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>900</td>
<td>0.34</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>900</td>
<td>0.40</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>900</td>
<td>1.23</td>
<td>0.13</td>
</tr>
<tr>
<td>4</td>
<td>30.41</td>
<td>0.71</td>
<td>0.03</td>
</tr>
<tr>
<td>5</td>
<td>4.47</td>
<td>0.35</td>
<td>0.03</td>
</tr>
<tr>
<td>6</td>
<td>900</td>
<td>0.39</td>
<td>0.02</td>
</tr>
<tr>
<td>7</td>
<td>900</td>
<td>0.67</td>
<td>0.06</td>
</tr>
<tr>
<td>8</td>
<td>34.91</td>
<td>0.33</td>
<td>0.02</td>
</tr>
<tr>
<td>9</td>
<td>900</td>
<td>1.41</td>
<td>0.18</td>
</tr>
<tr>
<td>10</td>
<td>900</td>
<td>0.91</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 4. Comparison of computational times.

In order to compare the three solution methods, we have considered 10 random instances characterized by \( Q \sim U[100, 400] \) and \( H/c \sim U[0.1, 0.05] \), by imposing again a time limit for the MIP solver equal to 900 seconds. The values of the objective functions in the three cases are reported in Table 3, whereas the computational times are shown in Table 4. Analysing these results, the effectiveness of Methods B and C is shown. By firstly comparing Methods A and B, that are both optimal, we can observe that Method A is stopped after the time limit in 7 over 10 instances (in some cases not finding the optimal solution), whereas Method B always finds the optimal solution after a very small time (much smaller than Method A). Method C finds a suboptimal solution but in very small times; note that the solution found with Method C is always less than 1% worse than the optimal one.

Considering the experimental results reported in this section, it is then possible to conclude that when large instances must be considered, Method B and Method C are very effective, because both are characterized by small computational times. Method B finds the optimal solution but it can be applied only if a mathematical programming solver is available, whereas Method C (finding a suboptimal but good solution) can be adopted by implementing simple algorithms.

5. CONCLUSIONS

In this work, we have considered a model of a simple distribution system managed by a VMI policy. We have proposed an optimization problem in which the decisions regard the quantities and the time instants of the delivery process in order to meet the discrete-event demand and to minimize transportation and inventory costs. A discrete-time version of this problem has been defined in order to obtain an easier formulation; three solution methods have been proposed, based on the statement of mathematical programming problems or the definition of appropriate algorithms. Present and future research is devoted to studying an analytical formulation (or another algorithm) in order to find the optimal solution of the planning problem without using any mathematical programming solver. Moreover, a future development of this research regards the extension of the considered network to the case of multiple customers and multiple suppliers.

REFERENCES


