Anti-windup Design for a Class of Nonlinear Control Systems

S. Tarbouriech ∗∗∗

Abstract: This paper focuses on the problem of static anti-windup design for a class of nonlinear systems subject to actuator saturation. Considering that a nonlinear dynamic output feedback controller has been designed to stabilize the nonlinear system, a method is proposed to compute a static anti-windup gain which ensures the local stability of the closed-loop system while providing an estimate of the region of attraction. The results are based on a differential-algebraic representation of rational systems and a modified sector bound condition to deal with the saturation effects. From these elements, LMI-based conditions are proposed to compute an anti-windup gain for enlarging the size of the closed-loop region of attraction. A numerical example is given to illustrate the proposed method.

Keywords: saturation control; nonlinear systems; stabilization methods.

1. INTRODUCTION

The general principle of the anti-windup technique is the introduction of modifications in a pre-designed controller in order to minimize the effects caused by saturation. The majority of the proposed methods has been focused on linear models and even in this case the anti-windup design is not an easy problem to solve (see, e.g., Tarbouriech and Turner [2009]). When the anti-windup is synthesized considered a linear approximation of a nonlinear system, the anti-windup compensator may lead to a poor behavior when applied to the original nonlinear system.

On the other hand, to characterize the stability of nonlinear systems, a key problem is to determine a non-conservative estimate of the system region of attraction. In general, the estimates are obtained from Lyapunov domains (see for instance references Khalil [1996], Gomes da Silva Jr. and Tarbouriech, [1999], Hu et al. [1999], Barreiro et al. [2006] and Coutinho and Gomes da Silva Jr. [2010]). Moreover, it is shown that through the design of anti-windup compensators, we can enlarge the region of attraction, as demonstrated in Cao et al. [2002], Gomes da Silva Jr. and Tarbouriech [2005] for linear systems.

Hence, it is important to consider systems with both nonlinear dynamics and actuator limitation. In this context, many methods for designing anti-windup compensators have been proposed in the past years. In spite of existing many different techniques and approaches, the majority of the results concentrates on linear models (see Gomes da Silva Jr. and Tarbouriech [2005], Teel and Kapoor [1997], Cao et al. [2002]). In particular, only few works have addressed nonlinear control systems such as Prempain et al. [2009], Wu et al. [2000] which consider anti-windup synthesis for linear-parameter varying systems, Morabito et al. [2004] which has proposed anti-windup methods for Euler-Lagrange systems and Kahveci et al. [2007] which considers adaptive control design. Moreover, we can also cite works dealing with an anti-windup architecture for systems with nonlinear dynamic inversion (NDI) as, for instance, Herrmann et al. [2009], Kendi and Doyle III [1997], Kapoor and Daoutidis [1999], Doyle III [1999], Menon et al. [2006]. More recently, in Valmorbidia et al. [2010] a dynamic anti-windup compensator is proposed for the class of quadratic systems, with the aim of enlarging an estimate of the region of attraction.

This paper aims at devising a numerical and tractable technique to design static anti-windup compensators for a class of nonlinear systems subject to actuator saturation. The class of systems to be considered covers all systems modeled by rational differential equations. We emphasize that a large class of systems can be embedded in this setup such as quadratic systems, polynomial systems and rational systems, and even more complex nonlinearities by means of additional algebraic constraints (Coutinho et al. [2004]) and/or change of variables (Coutinho et al. [2008]). The proposed approach, which applies a differential-algebraic representation (DAR) of nonlinear systems, allows to cast Lyapunov based stability conditions in terms of state-dependent linear matrix inequalities which are

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numerically solved at the vertices of a given polytope of admissible states. To deal with the saturation nonlinearity, we consider a modified version of the generalized sector bound condition ([Gomes da Silva Jr. and Tarbouriech 2005]) proposed in Loquen [2010]. From these elements, we derive regional stabilizing conditions for the closed-loop system. Namely, the stabilizing conditions are cast in terms of an LMI based optimization problem in order to compute an anti-windup gain which approximately provides the maximization of the basin of attraction of the closed-loop system.

This paper is organized as follows. In section 2, we introduce the problem to be addressed in the paper. Section 3 provides preliminary results concerning the system representation, the Lyapunov theory, and the modified sector bound condition. The main result is presented in section 4, while the computation of the anti-windup gain by means of an optimization problem is stated in section 5. A numerical example is given in section 6 demonstrating the potentialities of the proposed approach. Section 7 ends the paper with some concluding remarks.

**Notation:** \( I_n \) is the \( n \times n \) identity matrix and 0 may either denote the scalar zero or a matrix of zeros with appropriate dimensions. For a real matrix \( H, H' \) denotes its transpose and \( H > 0 \) means that \( H \) is symmetric and positive definite. For a matrix \( M \), the notation sim(M) represents \( M + M' \). The notation diag(M1, M2) denotes a diagonal matrix obtained from matrices M1 and M2. For a block matrix, the symbol \( \ast \) represents symmetric blocks outside the main diagonal block. For a given polytope \( \Phi, \Psi(\Phi) \) is the set of vertices of \( \Phi \). Matrix and vector dimensions are omitted whenever they can be inferred from the context.

### 2. PROBLEM STATEMENT

Consider the following class of nonlinear continuous-time control systems

\[
\dot{x}(t) = f(x(t)) + g(x(t))\text{sat}(v_c(t)) \\
y(t) = H_y x(t)
\]

where \( x \in \mathbb{R}^n \) denotes the state vector; \( y \in \mathbb{R}^m \) is the measured output; \( v_c \in \mathbb{R}^p \) is the control input; \( f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) are rational functions of \( x \) satisfying the conditions for the existence and uniqueness of solutions for all \( x \in \mathbb{B}_x \); and \( H_y \) is a constant matrix.

Considering system (1), we assume that a dynamic output stabilizing compensator

\[
\dot{\eta}(t) = f_\eta(\eta(t), y(t)) \\
v_c(t) = H_{\eta,v} \eta(t) + H_{\eta,y} y(t)
\]

is designed to guarantee some performance requirements and the stability of the closed-loop system in the absence of control saturation. In (2), \( \eta \in \mathbb{R}^{n_\eta} \) denotes the controller state; \( y(t) \) is the controller input; \( v_c(t) \) is the controller output; \( f_\eta : \mathbb{R}^{n_\eta} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_\eta} \) is a rational function of \( \eta \) and \( y \); and \( H_{\eta,v} \) and \( H_{\eta,y} \) are constant matrices.

In view of the undesirable effects of windup caused by input saturation, an anti-windup gain is added to the controller. Thus, considering the dynamic controller and the anti-windup strategy, the closed-loop system reads

\[
\dot{x}(t) = f(x(t)) + g(x(t))\text{sat}(v_c(t)) \\
y(t) = H_y x(t) \\
\dot{\eta}(t) = f_\eta(\eta(t), y(t)) + E_c(\text{sat}(v_c(t)) - v_c(t)) \\
v_c(t) = H_{\eta,v} \eta(t) + H_{\eta,y} y(t)
\]

where \( E_c : \mathbb{R} \rightarrow \mathbb{R}^{n_c} \) is a free matrix representing the anti-windup gain to be determined.

In the above setup, we aim at determining the anti-windup gain \( E_c \) such that the region of asymptotic stability of the closed-loop system is enlarged.

### 3. PRELIMINARIES

This section presents some basic results needed to derive an LMI-based method to address the anti-windup computation as stated in section 2.

#### 3.1 Differential Algebraic Representation - DAR

Firstly, define the deadzone nonlinearity as follows

\[
\psi(v_c(t)) = \begin{cases} v_c(t) - \text{sat}(v_c(t)) & \text{if } v_c(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}
\]

and rewrite system (3) as:

\[
\dot{x}(t) = f(x(t)) + g(x(t))\text{sat}(v_c(t)) - g(x(t))\psi(v_c(t)) \\
y(t) = H_y x(t) \\
\dot{\eta}(t) = f_\eta(\eta(t), y(t)) - E_c\psi(v_c(t)) \\
v_c(t) = H_{\eta,v} \eta(t) + H_{\eta,y} y(t)
\]

We consider the following Differential Algebraic Representation (DAR) (Durola et al. [2008]) for the system (5):

\[
\dot{x}(t) = A_1 x(t) + A_2 \eta(t) + A_3 z(t) + A_4 \phi(v_c(t)) \\
\dot{\eta}(t) = C_1 x(t) + C_2 \eta(t) + C_3 z(t) + E_c \psi(v_c(t)) \\
0 = \Omega_1 x(t) + \Omega_2 \eta(t) + \Omega_3 z(t) + \Omega_4 \psi(v_c(t)).
\]

where \( z \in \mathbb{R}^{n_z} \) is an auxiliary nonlinear vector function of \( (x, \eta) \) containing rational and polynomial terms (having terms of order equal or larger than two) of \( f_\eta(x) + g(x)\text{sat}(v_c(t)) \) and of \( f_\eta(x) \); and \( A_1 \in \mathbb{R}^{n_x \times n_x}, A_2 \in \mathbb{R}^{n_x \times n_\eta}, A_3 \in \mathbb{R}^{n_x \times n_z}, A_4 \in \mathbb{R}^{n_x \times n_c}, C_1 \in \mathbb{R}^{n_y \times n_x}, C_2 \in \mathbb{R}^{n_y \times n_\eta}, C_3 \in \mathbb{R}^{n_y \times n_z}, C_4 \in \mathbb{R}^{n_y \times n_c}, \Omega_1 \in \mathbb{R}^{n_y \times n_x}, \Omega_2 \in \mathbb{R}^{n_y \times n_\eta}, \Omega_3 \in \mathbb{R}^{n_y \times n_z}, \Omega_4 \in \mathbb{R}^{n_y \times n_c} \) are affine linear matrix functions of \( (x, \eta) \).

Considering the definition \( \xi(t) = [x(t) \ \eta(t)]' \in \mathbb{B}_x \cup \mathbb{B}_\eta \), with \( n_\xi = n + n_c \), we can rewrite (6) as follows:

\[
\dot{\xi}(t) = A_\xi \xi(t) + A_\phi \phi(v_c(t)) + (A_\xi \xi(t) - \Omega_\psi(v_c(t))) \\
0 = \Omega_3 \xi(t) + \Omega_4 \xi(t) + \omega_4(\psi(v_c(t)))
\]

with

\[
A_\xi = \begin{bmatrix} A_1 & C_1 & C_2 & 0 \\ C_1 & A_2 & 0 & 0 \\ C_2 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_\phi = \begin{bmatrix} A_3 & C_3 \\ C_3 & 0 \end{bmatrix}, \quad \Omega_\psi = \begin{bmatrix} 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

Moreover, considering the augmented state \( \xi(t) \) we can also rewrite \( v_c(t) \) as follows

\[
v_c(t) = \begin{bmatrix} H_{\eta,v} & H_{\eta,y} \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} = K \xi(t),
\]

where \( K \in \mathbb{R}^{1 \times n_c} \) is a constant matrix.

Regarding system (7), we assume that: (A1) the origin is an equilibrium point; and (A2) the domain \( \mathbb{B}_x \) is a given polytope containing the origin with known vertices.
To guarantee that the DAR in (7) is well posed (i.e., the uniqueness of the solution $\xi(t)$ is ensured), we further consider that: (A3) the matrix function $\Omega_0(\xi)$ has full rank for all $\xi \in B_\xi$. Hence, $z(t)$ can be eliminated from (7) leading to the original system representation in (5) by means of $z = -\Omega_0(\xi)^{-1}(\Omega_0(\xi)\xi(t) + \Omega_2(\xi)(\psi(v_\xi(t)))$.

For further details on the above nonlinear decompositions, the reader may refer to the references Coutinho et al. [2004] and Durola et al. [2008].

3.2 Lyapunov Theory

In this section, we present the following basic result from the Lyapunov theory (Khalil [1996]).

Lemma 1. Consider a nonlinear system $\dot{\xi} = a(\xi)$ where $a : B_\xi \rightarrow B_\xi$, $B_\xi \subset \mathbb{R}^{n_\xi}$, is a locally Lipschitz function such that $a(0) = 0$. Suppose there exist positive scalars $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$, and a continuously differentiable function $V : B_\xi \rightarrow \mathbb{R}$ satisfying the following conditions:

$$
\epsilon_1 \xi' \xi \leq V(\xi) \leq \epsilon_3 \xi' \xi, \quad \forall \xi \in B_\xi,
$$

$$
V(\xi) \leq -\epsilon_1 \xi' \xi, \quad \forall \xi \in B_\xi,
$$

$$
\mathcal{R} \triangleq \{ \xi \in \mathbb{R}^{n_\xi} : V(\xi) \leq 1 \} \subset B_\xi,
$$

then, $V(\xi)$ is a Lyapunov function in $B_\xi$. Moreover, for all $\xi(0) \in \mathcal{R}$ the trajectory $\xi(t)$ belongs to $\mathcal{R}$ and approaches the origin as $t \rightarrow \infty$.

In this work, we consider a quadratic Lyapunov function:

$$
V(\xi) = \xi' P \xi,
$$

where $P = P' > 0$ and $P \in \mathbb{R}^{n_\xi \times n_\xi}$. From Lemma 1, if $V(\xi)$ as above defined satisfies the conditions (8)-(10) the set

$$
\mathcal{R} = \{ \xi \in B_\xi : \xi' P \xi \leq 1 \}
$$

is an estimate of the system region of attraction.

3.3 Polytope of Admissible States

Consider that region $B_\xi$ is given by a polytope containing the origin in its interior. Hence, $B_\xi$ can be described by a set of scalar inequalities as follows:

$$
B_\xi = \{ \xi \in \mathbb{R}^{n_\xi} : q_r \xi \leq 1, r = 1, \ldots, n_c \},
$$

where $n_c$ is the number of faces of $B_\xi$ and $q_r \in \mathbb{R}^{n_\xi}$. Alternatively, $B_\xi$ can be described as the convex hull of its vertices, where the notation $\mathcal{V}(B_\xi)$ denotes the set of vertices of $B_\xi$.

In light of (10), the set $\mathcal{R}$ has to be included in the region $B_\xi$. The above condition is satisfied if (Boyd et al. [1994]):

$$
\begin{bmatrix} P & q_r \\ q_r' & 1 \end{bmatrix} \geq 0, \quad \text{for } r = 1, \ldots, n_c.
$$

3.4 Generalized Sector Bound Condition

Consider a row vector $G \in \mathbb{R}^{1 \times n_\xi}$ and a positive scalar $u$. Define now the following set (Loquen [2010])

$$
S \triangleq \{ \xi \in \mathbb{R}^{n_\xi} : |(K - Gu^{-1})\xi| \leq 1 \}.
$$

From the deadzone nonlinearity $\psi(v_\xi)$ in (4) and the set $S$ as above defined, the following Lemma can be stated (Gomes da Silva Jr. and Tarbouriech [2005]).

**Lemma 2.** If $\xi \in S$ then the relation

$$
\psi(v_\xi)\gamma^{-1}[\psi(v_\xi) - Gu^{-1}\xi] \leq 0
$$

is verified for any positive scalar $u$.

In the sequel, in order to ensure that (16) is valid, we should ensure that $\mathcal{R} \subset S$. Similarly to the result of Section 3.3, this inclusion holds if:

$$
P K' - Gu^{-1} \geq 0.
$$

4. MAIN RESULT

In this section, an LMI framework to address the antiwindup synthesis problem stated in section 2 is presented.

In this case, considering $V(\xi)$ as defined in (11), it follows

$$
\dot{V}(\xi) = \xi' P \xi + \xi' P \xi.
$$

Considering the auxiliary vector $\zeta = [\xi(t)' z(t) \psi(v_\xi')]$, we can rewrite (18) as follows

$$
\dot{\zeta}(t) = A_\zeta \zeta + B \zeta + C \zeta + D \zeta + E \zeta + F \zeta + G \zeta + H \zeta + I \zeta + J \zeta + K \zeta + L \zeta + M \zeta + N \zeta + O \zeta + P \zeta + Q \zeta + R \zeta + S \zeta + T \zeta + U \zeta + V \zeta + W \zeta + X \zeta + Y \zeta + Z \zeta.
$$

Now, define the following state-dependent matrices

$$
N_0(\xi) \triangleq \begin{bmatrix} \Omega_0(\xi) \\ \Omega_1(\xi) \end{bmatrix},
N_1(\xi) \triangleq \begin{bmatrix} \Omega_0(\xi) \\ \Omega_1(\xi) \end{bmatrix}.
$$

Note that $N_1(\xi)\xi = 0$, that is, the matrix $N_1(\xi)$ is a linear annihilator of $\xi$ as proposed in Trofino [2000]. In view of (7), it follows that $N_0(\xi)\xi = 0$. Hence, by defining

$$
N(\xi) = \begin{bmatrix} N_0(\xi) \\ N_0(\xi) \end{bmatrix},
$$

we obtain $N(\xi)\xi = 0$. By the definitions above, from Lemma 1, the idea is to satisfy $\dot{V}(\xi) < 0$ along the trajectories of the system (7), that is $\zeta'N_1(\xi)\zeta < 0$ for any $\zeta$ such that $N(\xi)\zeta = 0$. Applying the Finsler’s Lemma (de Oliveira and Skelton [2001]), the condition (18) is satisfied if the following holds:

$$
\zeta'(A_\zeta \zeta + L N_0(\xi) + N_0(\xi))< 0, \forall \xi \in B_\xi,
$$

where $L$ is a free multiplier to be determined.

In view of Lemma 2, if $\xi \in S$, then the relation

$$
\psi(v_\xi)\gamma^{-1}[\psi(v_\xi) - Gu^{-1}\xi] \leq 0
$$

is verified for any positive scalar $u$. Hence, if

$$
\zeta'(A_\zeta \zeta + L N_0(\xi) + N_0(\xi))< 0
$$

is verified, then (22) is satisfied for all $\xi \in S \cap B_\xi$. Note that we can rewrite (23) as follows

$$
\zeta'(A_\zeta \zeta + L N_0(\xi) + N_0(\xi))< 0
$$

where

$$
A_\zeta \zeta = \begin{bmatrix} A_\zeta P + P A_\zeta P + P A_\zeta - W E \xi + G u^{-1} \end{bmatrix}.
$$

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In light of the above, we state the following result.

**Theorem 1.** Consider system (7) satisfying A1, A2 and A3. If there exist constant matrices $P = P^T > 0$, $L$, $E_c$ and $G$ of appropriate dimensions and a positive scalar $u$, satisfying the following matrix inequalities for all $\xi \in \mathcal{B}_L$

$$\Lambda_4(\xi) + LN(\xi) + N'(\xi)'L' < 0, \quad (25)$$

$$\begin{bmatrix} Q & u \\ u^T & 1 \end{bmatrix} \geq 0, \quad (26)$$

$$\begin{bmatrix} Q & K'u - G' \end{bmatrix} \geq 0, \quad (27)$$

where $Q = Pu^2$, $\Lambda_3(\xi) = \Lambda_2(\xi)u^2$ and $\bar{L} = Lu^2$. Then, the gain matrix $E_c$ is such that for all $\xi(0) \in \mathcal{R}$ the trajectory $\xi(t)$ belongs to $\mathcal{R}$ and approaches the origin as $t \to \infty$, where $\mathcal{R}$ is as given in (12).

**Proof.** If the inequalities (25)-(27) are feasible for each $\xi \in \mathcal{V}(\mathcal{B}_L)$, then, by convexity, they are also satisfied for all $\xi \in \mathcal{B}_L$. Consider now $Qu^{-2} = P > 0$ and define $V(\xi) = \xi^TP\xi$. Pre- and post-multiplying the resulting inequality by $\xi'u^{-2}$ and $u^{-1}$, respectively, it follows that $V(\xi) - 2\psi(\xi)^T\psi(\xi) - G\xi < 0$. Hence, if $\mathcal{R} \subseteq \mathcal{S} \cap \mathcal{B}_L$, from the discussion in section 3.2, the condition (25) ensures that for all $\xi(0) \in \mathcal{R}$ the trajectory $\xi(t)$ belongs to $\mathcal{R}$ and approaches the origin as $t \to \infty$. Now, define $\Pi = diag(I_nu^{-1}, 1)$ and consider the relations (26) and (27). Pre- and post-multiplying (26) and (27) by $\Pi$, lead to (14) and (17), respectively. It follows that the inclusion $\mathcal{R} \subseteq \mathcal{B}_L \cap \mathcal{S}$ is satisfied, which concludes the proof. ☐

5. ANTI-WINDUP COMPUTATION

Observe that relation (25) in Theorem 1 is a bilinear matrix inequality (BMI), due to the products $PWE_c$ and $G'u^{-2}$. However, it can be solved with an adaptation of a coordinate-descent algorithm (see Peaucelle and Arzelier [2001]). Firstly, note that we can obtain an equivalent expression for the BMI (25) as follows

$$E_1^T\Lambda_4(e)E_1 + LN(e) + N'(e)'L' < 0, \quad (28)$$

where

$$\Lambda_4(e) = \begin{bmatrix} A_4(e)Q + QA_4(e) & QA_4(e) & QA_4(e)^T + G^TQN^T \\ * & 0 & 0 \\ * & * & -2u \\ * & * & 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} I_{n_e} & 0 \\ 0 & I_{n_e} \\ 0 & 0 \\ 0 & E_c \end{bmatrix}.$$

In addition, by defining $N_\mathcal{R} = [0 \ 0 \ E_c - I_{n_e}]$ with $N_\mathcal{R} \in \mathbb{R}^{n_e \times (n_c + n_e + 1)}$, notice from (28) that $N_\mathcal{R}E_1 = 0$.

Noting that $E_1^T\begin{bmatrix} LN(e) \\ 0 \end{bmatrix}E_1 = LN(e)$, we can apply Finsler’s Lemma in (28) leading to

$$\Lambda_4(e) + \text{sym}\left(\begin{bmatrix} [L] & [N(\xi)] \end{bmatrix}L_\alpha N_\mathcal{R}\right) < 0, \quad (29)$$

with $L_\alpha$ being a free multiplier.

Then, if (29) holds the condition in (25) is satisfied. Moreover, we can rewrite (29) as follows

$$\Lambda_4(e) + \text{sym}\left(\begin{bmatrix} [L] & [N(\xi)] \end{bmatrix}L_\alpha N_\mathcal{R}\right) < 0. \quad (30)$$

Note that the term $L_\alpha N_\mathcal{R}$ is bilinear (because of the product involving the anti-windup gain $E_c$ and the free multiplier $L_\alpha$). To overcome this problem, we propose to use the algorithm presented in Peaucelle and Arzelier [2001]. In this case, we can choose $L_\alpha = [F^T(a) F']$, $K_s = F, F^{-1}$ and $R = F E_c$, with $F \in \mathbb{R}^{n_c \times n_e}$ and $E_c \in \mathbb{R}^{n_c + n_e + 1 \times n_c}$. Applying the above in (30) leads to:

$$\begin{bmatrix} L \bar{K} \\ 0 \end{bmatrix} N_\mathcal{R} + N_\mathcal{R}'L' < 0, \quad (31)$$

with $L_\alpha = \begin{bmatrix} L \bar{K} \\ 0 \end{bmatrix} N_\mathcal{R}$ and $N_\mathcal{R}' = \begin{bmatrix} N(\xi) \end{bmatrix} 0 \ 0 R - F \end{bmatrix}.$

Observe that the matrices $K_s, R$ and $F$ are unknown. Then, the following result can be stated.

**Theorem 2.** Consider system (7) satisfying A1, A2 and A3. Suppose there exist constant matrices $P = P^T > 0$, $L$, $K_s, R, F$ and $G$ of appropriate dimensions, and a positive scalar $u$, satisfying the matrix inequalities (31), (26) and (27), for all $\xi \in \mathcal{V}(\mathcal{B}_L)$. Then, the gain $E_c = F^{-1}R$ is such that for all $\xi(0) \in \mathcal{R}$ the trajectory $\xi(t)$ belongs to $\mathcal{R}$ and approaches the origin as $t \to \infty$, where $\mathcal{R}$ is as given in (12).

5.1 Algorithm and Optimization Problems

In this section, we show how to apply the result proposed in Theorem 2 for computing the anti-windup gain while providing an estimate of the region of attraction of the system in (1).

To handle this problem, we firstly introduce the following optimization problem:

$$\min \text{trace}(Q) - \epsilon u : (26), (27), (31), \quad \forall \xi \in \mathcal{V}(\mathcal{B}_L) \quad (32)$$

where $\epsilon_1$ is a chosen weight factor of $u$.

However, observe that the condition (31) is a BMI if we let $K_s, R$ and $F$ be free in $L_\alpha N_\mathcal{R}$. In this case, we can apply a similar algorithm to the one proposed in Peaucelle and Arzelier [2001] to solve the optimization problem in (32) as given below.

**Algorithm 1:**

1. (Step $k = 1$ – initialization) The solution proposed in this step is to fix the terms $R$ and $F$ to determine a solution to (31). Hence, the initialization can be done by fixing $R = 0$ and $F = I_{n_e}$. Since $\mathcal{R}$ is an ellipsoidal domain, the optimization problem (32) can be considered yielding the determination of $K_s$. In this step, note that $E_c = 0$. Hence, we determine $\mathcal{R}$ without the anti-windup gain.

2. (Step $k$ – first part) Once $K_s$ is given, the optimization problem (32) can be considered to find $R$ and $F$. Note that, here we obtain the anti-windup gain from $E_c = F^{-1}R$.

3. (Step $k$ – second part) Fix $R$ and $F$. Solve the optimization problem (32) to compute $L_\alpha$ and $K_s$.

4. (Termination step) Let $\epsilon > 0$ be a given scalar defining the precision of the solution. If $\|E_c(k) - E_c(k - 1)\| < \epsilon$, then stop, otherwise $k \leftarrow k + 1$ and go to step 2.
Remark 2. Note that the minimization of $\text{trace}(Q) - e_1 u$ is a multiobjective criteria that leads to an implicit maximization of the size of $\mathcal{R}$. Other classical size criteria of ellipsoidal sets such as volume maximization, minor axis maximization, minimization of trace of $P$ and the maximization in certain directions (see, e.g., Gomes da Silva Jr. and Tarbouriech [2005], Boyd et al. [1994]) can be also applied.

Remark 3. By using Algorithm 1, note that we avoid the iteration between the unknown variables $P$ and $E_c$. The main advantage in this case is that matrix $P$ is a free variable in each iteration step. This fact tends to reduce the conservatism associated to the size of the ellipsoidal region of stability $\mathcal{R}$.

6. NUMERICAL EXAMPLE

Consider the following nonlinear closed-loop system:

\[
\begin{align*}
x(t) &= (x^2(t) - 1)x(t) + sat(v_c(t)) \\
y(t) &= x(t),
\end{align*}
\]

and the controller

\[
\begin{align*}
\dot{y}(t) &= -x(t) \\
v_c(t) &= \eta(t) - 2y(t).
\end{align*}
\]

We suppose that the state-space domain is bounded by the set: $\mathcal{B}_2(\alpha_1, \alpha_2) := \{x \in \mathbb{R}^2 : |\xi| \leq \alpha_1, |\eta| \leq \alpha_2\}$, where $\alpha_1 = 1.3$ and $\alpha_2 = 2.3$.

Considering the DAR representation given in (7) with $z(t) = x^2$, we get for (33)-(34) the following:

\[
\begin{align*}
\mathcal{A}_u @ \mathcal{G} &= \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{A}_v @ \mathcal{G} &= \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \mathcal{A}_c @ \mathcal{G} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
\Omega_u @ \mathcal{G} &= \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \Omega_v @ \mathcal{G} = -1, \quad \Omega_c @ \mathcal{G} = 0, \quad K = \begin{bmatrix} -2 & 1 \end{bmatrix}.
\end{align*}
\]

Based on Algorithm 1 of Section 5.1, we have determined the estimate $\mathcal{R}_1$ of the region of attraction for $E_c = 0$ (the initialization step). Figure 1 shows the estimate $\mathcal{R}_1$ obtained considering $e_1 = 3$ in (32). In this case the matrix $P$ is given by:

\[
P = \begin{bmatrix} 0.8455 & -0.1666 \\ -0.1666 & 0.3033 \end{bmatrix}.
\]

Applying Algorithm 1, with $e_1 = 3$, we obtain:

\[
P = \begin{bmatrix} 0.6195 & -0.0808 \\ -0.0808 & 0.2353 \end{bmatrix}, \quad E_c = -1.0396.
\]

Figure 4 shows the trajectory of the state $x$ and the control signal obtained from the initial condition $\xi(0) = [-1.3 , -0.45]$ considering the cases with and without anti-windup strategy. In this case, note that, with anti-windup strategy (dashed-line), the control remains less time in saturation.
7. CONCLUDING REMARKS

This paper has presented an approach to compute anti-windup gains for a class of nonlinear systems subject to actuator saturation. The approach relies on a differential-algebraic representation of rational systems, which can model a broad class of nonlinear systems. To deal with the saturation, we consider a modified version of the generalized sector bound condition. From these elements, this paper has focused on the problem of enlarging the region of attraction of the closed-loop system by means of an anti-windup gain. This problem has been indirectly addressed through an algorithm that allows to design an anti-windup gain leading to the maximization of an estimate of the basin of attraction. This approach has been illustrated by a numerical example.

REFERENCES


