The stable manifold approach for optimal swing up and stabilization of an inverted pendulum with input saturation

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Abstract: In this paper, the problem of swing up and stabilization of an inverted pendulum by a single feedback control law is considered. The problem is formulated as an optimal control problem including input saturation and is solved via the stable manifold approach which has been recently proposed to solve the Hamilton-Jacobi equation. In this approach, the problem is turned into an enhancement problem of the domain of validity to include the pending position. After a finite number of iterations, an optimal feedback control law is obtained and its effectiveness is verified by experiments. It is shown that the stable manifold approach can be applied for systems including practical nonlinearities such as saturation by directly deriving a controller satisfying the input limitation of the experimental setup.

1. INTRODUCTION

Inverted pendulums have attracted the attention of control researchers for several decades as a benchmark for the development of nonlinear control methods. Especially, the problem of swing up and stabilization of the inverted pendulum is challenging because the pendulum exhibits strong nonlinearities and the system is underactuated. Early research by Åström and Furuta [2000] solved this problem by switching between different laws, one that drives the pendulum to the upright region and another that stabilizes the pendulum around the upright position (see, also, Muskinja and Tovornik [2006] for experimental results and the comparison of several control strategies). From the theoretical as well as practical viewpoints, the design problem of a single feedback control law that performs the swing up and stabilization is considered to be important due to its simplicity and the robustness of the closed loop system. The authors of Angeli [2001], Åström et al. [2008], Shiriaev et al. [2001] consider this problem, especially from the global stabilization viewpoint, which has an intrinsic difficulty from a topological reason.

In this paper, we formulate the problem of swing up and stabilization as an optimal control problem including input saturations and, by solving it, derive a single feedback control law. It is well-known that the condition of the solvability of optimal control problems for nonlinear systems is given by the Hamilton-Jacobi equation. Recently, an approach for approximation of the solution to the Hamilton-Jacobi equation was proposed by one of the authors of the present paper using stable manifold theory (Sakamoto and van der Schaft [2008]) and extended to design a nonlinear optimal controller for systems with input saturations (Yuasa et al. [2010]). The method consists of computations of solutions for a Hamiltonian system that evolve on its stable manifold. By applying this method for the Hamilton-Jacobi equation derived from the optimal swing up and stabilization problem, the stable manifold is numerically approximated so that the stable region of the closed loop system contains the pending position. The feedback law is constructed from the computed solution on the stable manifold. The inclusion of input saturation into the model makes the problem more challenging but is essential because of the following reasons. First, in our experimental setup, the maximum input voltage is limited and the standard optimal control theory does not provide how to choose weighting matrices in order for the input to satisfy the limitation. Secondly, it is one of the main objectives of this paper to show that the stable manifold approach for solving the Hamilton-Jacobi equation is more useful than any other existing methods by showing, with experiments, that it can be applied to optimal control problems including practical non-analytic nonlinearities such as saturation.

The organization of the paper is as follows. In Section 2, the theories of the Hamilton-Jacobi equation, its stabilizing solution and stable manifold of a corresponding Hamiltonian system are described. In Section 3, the stable manifold approximation method is reviewed and its computational process is described. In Section 4, an inverted pendulum system on a cart is introduced and an optimal control problem with input saturation is defined for its swing up and stabilization. Both simulation and experimental results are presented after describing the feedback controller computation with the stable manifold algorithm. Finally, Section 5 concludes the paper with several remarks.
2. THE HAMILTON-JACOBI EQUATION AND
STABLE MANIFOLD

In this section, we start with the general Hamilton-Jacobi equation arising in nonlinear control theory and briefly review the theory of stable manifold by which we construct feedback laws for the corresponding optimal control problems. The equation takes the form of

\[ H(x, p) = p^\top f(x) - \frac{1}{4} p^\top R(x)p + q(x, p) = 0, \]

where \( p_i = \partial V / \partial x_1, \ldots, p_n = \partial V / \partial x_n \) with \( V(x) \) an unknown function, \( f : M \to \mathbb{R}^n, R : M \to \mathbb{R}^{n \times n}, q : M \times \mathbb{R}^n \to \mathbb{R} \) are all \( C^\infty \), and \( R(x) \) is a symmetric matrix for all \( x \in M \). We also assume that \( f \) and \( q \) satisfy \( f(0) = 0, q(0, 0) = 0 \) and \( q(x, p) = o(|x| + |p|^2) \). In what follows, we write \( f(x) = Ax + o(|x|), Q = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(0, 0) \) where \( A \) is an \( n \times n \) real matrix and \( Q \in \mathbb{R}^{n \times n} \) is a symmetric matrix.

The stabilizing solution of (HJ) is defined as follows.

**Definition 1.** A solution \( V(x) \) of (HJ) is said to be the stabilizing solution if \( p(0) = 0 \) and \( 0 \) is an asymptotically stable equilibrium of the vector field \( \frac{\partial}{\partial p}(x, p(x)) \), where \( p(x) = (\partial V / \partial x)^T(x) \).

Suppose \( V(x) \) is a solution of (HJ). Then, the set

\[ \Lambda_V = \{(x, p) | p = \partial V / \partial x(x)\} \]

is invariant under the flow of the associated Hamiltonian system derived from (HJ):

\[
\begin{align*}
\dot{x} & = \frac{\partial}{\partial p}(x, p) \\
p & = \frac{\partial^2}{\partial x^2} p(x, p).
\end{align*}
\]

Conversely, if an \( n \)-dimensional manifold \( \Lambda \) in the \( (x, p) \)-space is invariant under (1) and, at a point \( (x_0, p_0) \), the projection \( \pi \) of \( \Lambda \) to the \( x \)-space is surjective, then, \( \Lambda \) possesses the Lagrangian submanifold property and there exists a solution \( V(x) \) of (HJ) around \( x_0 \). Let \( (x_0, p_0) = (0, 0) \). If, in addition to the above conditions, the Hamiltonian flow of (1) on \( \Lambda \) is convergent to the origin, then, \( \Lambda \) is a stable manifold of (1) and there exists a function \( V(x) \) defined on a neighborhood \( U \) of the origin such that

\[ \Lambda \cap \pi^{-1}(U) = \{(x, p) | p = \partial V / \partial x(x), x \in U\}. \]

A sufficient condition for the local existence of the stable manifold, or, equivalently, the local stabilizing solution for (HJ) is obtained in van der Schaft [1991]. It is a natural condition based on a linearization argument. Let us consider the Riccati equation

\[ PA + AP^\top - PR(0)P + Q = 0, \]

which is the linearization of (HJ). A symmetric matrix \( P \) is said to be the stabilizing solution of (RIC) if it is a solution of (RIC) and \( A - R(0)P \) is stable.

**Theorem 2.** (van der Schaft, 1991). If (RIC) has the stabilizing solution \( P \), there exists, locally around the origin, the stabilizing solution \( V(x) \) to (HJ) with \( (\partial^2 V / \partial x^2)(0) = P \).

The theorem also says that if the stabilizing solution to (RIC) exists, an \( n \)-dimensional stable manifold, from which the stabilizing solution to (HJ) can be derived, locally exists around the origin.

3. THE STABLE MANIFOLD APPROACH TO APPROXIMATE THE STABILIZING SOLUTION

Throughout the paper, we assume that the stabilizing solution to (RIC), denoted by \( \Gamma \), exists. As is described in the previous section, an \( n \)-dimensional stable manifold \( \{(x, p) | p = p(x)\} \) exists for (1) around the origin. The function \( p(x) \) is used in feedback laws and our problem in this section is how to approximate \( p(x) \).

Let

\[ T = \begin{pmatrix} I & S \\ 0 & \Gamma S + I \end{pmatrix}, \]

where \( S \) is the solution of the Lyapunov equation \( (A - R(0)\Gamma)S + (S(A - R(0)\Gamma)^T = R(0)). \) Using the linear coordinate transformation

\[ \begin{pmatrix} \dot{x}' \\ p' \end{pmatrix} = T^{-1} \begin{pmatrix} \dot{x} \\ p \end{pmatrix}, \]

the Hamiltonian system (1) is written

\[ \begin{pmatrix} \dot{x}' \\ p' \end{pmatrix} = \begin{pmatrix} A - R(0)\Gamma & 0 \\ 0 & -(A - R(0)\Gamma)^T \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} + \text{higher order terms}. \]

**3.1 The stable manifold algorithm (Sakamoto and van der Schaft [2008])**

Let us consider the following \( 2n \)-dimensional system.

\[
\begin{align*}
\dot{x} &= Fx + n_s(t, x, y) \\
y &= -F^Ty + n_u(t, x, y).
\end{align*}
\]

We assume that \( F \) is an asymptotically stable \( n \times n \) real matrix and \( n_s, n_u \) are higher order terms with sufficient smoothness.

We define the sequences \( \{x_k(t, \xi)\} \) and \( \{y_k(t, \xi)\} \) by

\[
\begin{align*}
x_{k+1} &= e^{Ft} \xi + \int_0^t e^{F(t-s)} n_s(s, x_k(s), y_k(s)) \, ds \\
y_{k+1} &= -\int_0^t e^{-F(t-s)} n_u(s, x_k(s), y_k(s)) \, ds
\end{align*}
\]

for \( k = 0, 1, 2, \ldots \), and

\[
\begin{align*}
x_0 &= e^{Ft} \xi \\
y_0 &= 0
\end{align*}
\]

with arbitrary \( \xi \in \mathbb{R}^n \).

**Theorem 3.** (Sakamoto and van der Schaft, 2008). The sequences \( x_k(t, \xi) \) and \( y_k(t, \xi) \) are convergent to zero for sufficiently small \( |\xi| \), that is, \( x_k(t, \xi), y_k(t, \xi) \to 0 \) as \( t \to \infty \) for all \( k = 0, 1, 2, \ldots \). Furthermore, \( x_k(t, \xi) \) and \( y_k(t, \xi) \) are uniformly convergent to a solution of (3) on \( [0, \infty) \) as \( k \to \infty \) for sufficiently small \( |\xi| \). Let \( x(t, \xi) \) and \( y(t, \xi) \) be the limits of \( x_k(t, \xi) \) and \( y_k(t, \xi) \), respectively. Then, \( x(t, \xi) \) and \( y(t, \xi) \) are the solution on the stable manifold of (3), that is, \( x(t, \xi), y(t, \xi) \to 0 \) as \( t \to \infty \).

**3.2 Optimal Control for Systems with Input Saturations.**

In this subsection, we briefly show that the stable manifold approach can be extended to approximate the exact solution of Hamilton-Jacobi equations for systems with input saturations. See Yuasa et al. [2010] for detail. Let us consider nonlinear optimal regulation problems to
minimize quadratic form cost functions for nonlinear time-invariant systems with state nonlinearities and input saturations. The state equation $\Sigma$ and the cost function $J$ are given as (5) with $R$ being defined as a diagonal matrix.

\[
\begin{aligned}
\Sigma : \dot{x} &= f(x) + g(x) \cdot \text{sat}(u), \ x(0) = x_0 \\
J &= \int_0^\infty (x^TQx + u^TRu) \, dt,
\end{aligned}
\]

where $Q \geq 0, \ R > 0, \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, $\text{sat}(u) = [\text{sat}_1(u_1), \ldots, \text{sat}_m(u_m)]^T$

and

\[
\text{sat}(u_i) = \begin{cases} 
\overline{u}_i & (\overline{u}_i \leq u_i) \\
\underline{u}_i & (\underline{u}_i < u_i < \overline{u}_i) \\
\underline{u}_i & (\underline{u}_i \leq u_i),
\end{cases}
\]

$\overline{u}_i < 0, \ \underline{u}_i > 0, \ (i = 1, 2, \ldots, m)$.

The Hamilton-Jacobi equation to be solved is

\[
\left( \frac{\partial V}{\partial x} \right)^T \left( f(x) + g(x) \cdot \text{sat} \left( \hat{u} \left( x, \left( \frac{\partial V}{\partial x} \right) \right) \right) \right) \\
+ x^TQx + \text{sat} \left( \hat{u} \left( x, \left( \frac{\partial V}{\partial x} \right) \right) \right)^T \tilde{R} \cdot \text{sat} \left( \hat{u} \left( x, \left( \frac{\partial V}{\partial x} \right) \right) \right) = 0.
\]

Defining a new function $\overline{\text{sat}}(u) = \text{sat}(u) - u$, the Hamilton-Jacobi equation can be re-written as

\[
H(x, p) = p^Tf(x) - \frac{1}{4} (g(x)^Tp)^T\tilde{R}^{-1}g(x)p \\
+ x^TQx + \overline{\text{sat}} \left( \hat{u}(x, p) \right)^T\tilde{R} \cdot \overline{\text{sat}} \left( \hat{u}(x, p) \right) = 0.
\]

4. OPTIMAL SWING UP AND STABILIZATION OF AN INVERTED PENDULUM

4.1 Inverted pendulum system

Fig. 1. Inverted pendulum system.

The inverted pendulum system (Figs. 1, 2) consists of a cart that is moving on a rail and a pole attached to the cart. The pole is constrained to rotate within the vertical plane. The cart is moved by an external force from a DC-motor. By letting the effect of viscous friction zero, the

Fig. 2. Inverted pendulum model.

Lagrange’s equations are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = f
\]

and

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0
\]

where $x$ is the cart position, $\theta$ is the angle of pendulum and $f$ is the external force.

From the transmission model of the DC-motor, $f$ is described as

\[
f = C_1 \dot{x} + C_2 u,
\]

where $u$ is the input voltage and $C_1, C_2$ consist of the motor characteristic parameters such as the moment and inductive constants. Then, (7) and (8) are described by the form

\[
\begin{pmatrix}
ml \cos \theta & M + m \\
ml^2 + J & ml \cos \theta
\end{pmatrix}
\begin{pmatrix}
\dot{\theta} \\
\dot{x}
\end{pmatrix}
= \begin{pmatrix}
ml^2 \sin \theta + C_1 \dot{x} + C_2 u \\
mgl \sin \theta
\end{pmatrix}
\]

The mechanical parameters of the system are listed in Table 1.

<table>
<thead>
<tr>
<th>$m$</th>
<th>mass of pendulum</th>
<th>0.06(kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>mass of cart</td>
<td>0.96(kg)</td>
</tr>
<tr>
<td>$l$</td>
<td>length of pendulum / 2</td>
<td>9.5x10^{-2}(m)</td>
</tr>
<tr>
<td>$J$</td>
<td>moment of pendulum / 2</td>
<td>1.81x10^{-3} (kg m²)</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
<td>9.80(m/s²)</td>
</tr>
<tr>
<td>$C_1$</td>
<td>actuator parameter</td>
<td>-9.10(N/m)</td>
</tr>
<tr>
<td>$C_2$</td>
<td>actuator parameter</td>
<td>0.95(N/V)</td>
</tr>
</tbody>
</table>
4.2 State equation and optimal control problem

Setting the states \( x = [x_1, x_2, x_3, x_4]^T = [\theta, \dot{\theta}, x, \dot{x}]^T \), the system (9) is expressed as follows

\[
\dot{x} = f(x) + g(x) \cdot \text{sat}(u),
\]

(10)

where

\[
f(x) = \begin{bmatrix}
-x_2 \\
-ml \cos x_1 c_1 x_4 - ml^2 \sin x_1 x_2 + (m + M) ml g \sin x_1 \\
(m+M)ml^2 + ml^2 \sin^2 x_1 \\
2 \sin^2 x_1 \sin x_1 c_1 x_4 + (m+M)ml^2 \sin x_1 x_2 + (m+M) ml g \sin x_1 \\
(m+M)ml^2 + ml^2 \sin^2 x_1
\end{bmatrix},
\]

\[g(x) = \begin{bmatrix}
0 \\
-(m+M) c_2 \\
0 \\
(m+M) c_2
\end{bmatrix}.
\]

Our swing up and stabilization control is designed by an optimal control problem including input saturations for the cost function

\[
J = \int_0^{\infty} x^T Q x + u^T R u \, dt,
\]

(11)

where

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 40
\end{bmatrix}, \quad R = 1.
\]

The maximum and minimum values of saturation are \( \bar{\pi} = 18 \), \( \underline{\pi} = -18 \).

4.3 Controller computation via the stable manifold iteration

First, the Riccati equation which is the linearization of the Hamilton-Jacobi equation for the optimal control problem (10)-(11) is solved. The corresponding Hamiltonian matrix is

\[
\text{Ham} = \begin{bmatrix}
A & -\frac{1}{2} B R^{-1} B^T \\
-2Q & -\frac{1}{2} A^T
\end{bmatrix},
\]

where \( A = \frac{\partial f}{\partial x}(0) \), \( B = g(0) \). The stabilizing solution of the Riccati equation is

\[
\Gamma = \begin{bmatrix}
249 & 28.1 & 21.2 & 126 \\
28.1 & 3.17 & 2.41 & 14.2 \\
21.2 & 2.41 & 48.3 & 14.9 \\
126 & 14.2 & 14.9 & 67.7
\end{bmatrix},
\]

and the matrix \( F \) in (3), or \( A - R(0)F \) is

\[
F = \begin{bmatrix}
283 & -40.9 & -15.4 & -93.2 \\
0.00 & 1.00 & 0.00 & 0.00 \\
-283 & 45.6 & 5.19 & 1.96 & 11.8
\end{bmatrix}.
\]

After a number of tests, the radius of the convergence is determined as \( |\xi| \leq 0.2 \). For each \( \xi \) within the convergence domain, the stable manifold iteration is applied around 25 times. A Matlab program is specially made to perform the iterations by adaptively extending the time interval of (4) to the negative direction so that the domain of validity is enhanced while regulating the value of Hamiltonian less than 1 (see, Fig. 3).

Within the convergence domain, a number of \( \xi \) is selected and the iteration is applied (see, Fig. 4). We have successfully found several \( \xi \)'s, from which the corresponding functions \( x_2(t, \xi) \) pass through the pending position (the red line in Fig. 4). From Fig. 3, one can see the number of iterations required to enhance the domain of validity to include the pending position is more than 25 iterations. The analytic approach for the stable manifold iteration can never handle this number of iterations because an astronomical number of terms appear even if third or fourth order approximations are used for trigonometric functions. The same can be said about the Taylor expansion approach (see, e.g., Lukes [1969]) for the Hamilton-Jacobi equation in this problem. Also, Fig. 3 shows that the negative time extension needs special care because the solutions rapidly diverge from the stable manifold after \( t < 0 \) exceeds a certain value. In the figure, the line for \( k = 6 \) is calculated to \( t = -1.06 \) while that for \( k = 25 \) is calculated to \( t = -1.50 \). Finally, polynomials are fitted to all the data by the least square method to relate \( p_j \) with \( x = [x_1, \ldots, x_4] \) for \( j = 1, \ldots, 4 \). The maximal orders of the polynomials are 8th for \( x \).

Fig. 3. Errors \( H(x_2(t, \xi), p_j(t, \xi)) \) along the swing up trajectory.

Fig. 4. Closed loop trajectories in the \( (\theta, \dot{\theta}) \)-space.
From a practical viewpoint, the effort that is required in the design needs to be discussed. Simple comparison is not possible because no optimal swing up and stabilization are reported and the existing results by single feedback controls are mostly for two-dimensional systems (see, e.g., Angeli [2001], Åström et al. [2008]). Also, the global stabilization is not aimed at in this paper. The most time-consuming task in our approach is to calculate trajectories in \((x, p)\) space (such as in Fig. 4). They should be uniformly distributed in the \(x\)-space, for which the feedback law is defined and some of them must pass through the pending position. This process of computation took us a couple of days. However, we remark that the trajectories depicted in Fig. 4 correspond to closed loop trajectories and therefore, once desired trajectories are computed, the designer can be sure that the controller based on this data will work as expected before the actual construction of the controller. The subsequent subsections confirm this feature of our approach from the fact that no trial and error in choosing the weighting matrices was necessary and the computed controller did the required job with little modification.

4.4 Simulation result

Fig. 5 shows that the controller designed in the previous subsection drives the pendulum from the pending position to the upright position and the cart to the original position. The corresponding input response is depicted in Fig. 6. It is seen that the control input satisfies the input voltage limitation. It can also be observed that the closed loop trajectory in Fig. 5 is no different from the one in Fig. 4 drawn with red.

Fig. 5. Responses for \(x(0) = [−\pi, 0, 0, 0]^T\) (simulation).

Fig. 6. Control input (simulation).

4.5 Experimental result

Fig. 7 shows the experimental result. The corresponding input response is depicted in Fig. 8. The pendulum swings into the upright position in two swings as well as the simulation result. The effectiveness of the obtained state feedback control is verified. This controller is a state feedback and therefore, it also stabilizes the pendulum from different initial positions. Fig. 9 shows the responses for the pendulum with initial position \(θ(0) = −184/180\pi\) (rad) and the control input is shown in Fig. 10. Also, the robustness against parameter variation is confirmed for the increase of the moment of inertia of the pendulum up to 20%.

Fig. 7. Responses for \(x(0) = [−\pi, 0, 0, 0]^T\) (experiment).

Fig. 8. Control input (experiment).

Remark 1. With the solution of the standard optimal control problem without saturation, the controller drives the pendulum from the pending position into the upright position in one swing (see, Fig. 11). But this controller cannot be applied on the real inverted pendulum because the maximum input voltage exceeds the input limitation (see, Fig. 12). Different weighting matrices did not give controllers satisfying the input limitation and the swing up control was always achieved in one swing. On the other hand, considering input saturation directly with the form of (5), the controller drives the pendulum from the pending position into the upright position in two swings in order for the input to satisfy the input limitation.

5. CONCLUDING REMARKS

In this paper, the problem of swing up and stabilization of an inverted pendulum by a single feedback control law is solved as an optimal control problem including input saturation. The stable manifold algorithm for the approximation of the stabilizing solution of the Hamilton-Jacobi equation is applied, enhancing the domain of validity to include the pending position. The effectiveness of the obtained state feedback control is verified by both simulations and experiments. The optimal control with
input saturation directly gives a controller that satisfies the input voltage limitation of the experimental setup.

The result of the present paper proves that the stable manifold approach for solving the Hamilton-Jacobi equation is more effective and practical than other existing methods such as the Taylor expansion approach as this is the first result, in both simulation and experiment, of the optimal swing up and stabilization problem for an inverted pendulum.

REFERENCES


