Simulated Annealing with Partial Evaluation of Objective Function Applied to Electrical Impedance Tomography

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Abstract: Electrical Impedance Tomography (EIT) is an imaging technique that attempts to reconstruct the conductivity distribution inside an object from electrical currents and potentials measured at its surface. The EIT reconstruction problem can be approached as an optimization problem where one tries to maximize the matching between a simulated impedance domain and the observed one. This optimization problem may be approached by Simulated Annealing (SA), but at a large computational cost due to the expensive evaluation process of the objective function. We propose here a variation of SA applied to EIT where the objective function is evaluated only partially, while ensuring upper boundaries on the deviation on the behavior of the modified SA. Copyright © 2011 IFAC.

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1. INTRODUCTION

Electrical impedance tomography (EIT) is an imaging modality that estimates the electrical conductivity distribution within the body when a low amplitude current pattern is applied to a body surface and the potential at determined points of that surface is measured through electrodes or, alternatively, when a potential is applied and the current flowing through the surface is measured (Trigo et al., 2004).

The fundamental problem in the image reconstruction is that, in a general case, the electric current cannot be forced to flow linearly (or even along a known path) in an inhomogeneous conductor; the current flow is determined by the electrical conductivity distribution within the body. The model of the body is based on an elliptic partial differential equation obtained from Maxwell’s equations and the electrical conductivity distribution represents the solution of an ill-posed nonlinear inverse problem, meaning that large changes in conductivity at the interior of the body may result in only small potential or current changes at the surface. The main problems with EIT are its sensitivity to electrode positioning, its rather low image resolution due to noise in the measurements and numerical effects of the discretization, and its sensitivity to the boundary geometry.

The two main forms of EIT are dynamic imaging and static imaging yielding differential and absolute images respectively. The images produced by differential imaging represent the conductivity changes of a region between two time intervals (Barber and Brown, 1984). Imaging physiological function within the body largely relies on this technique. This work, is mainly concerned with the reconstruction of static conductivity images which requires more advanced numerical algorithms.

The spatial resolution of EIT is not comparable to other imaging techniques such as magnetic resonance, computerized tomography or ultrasonic imaging. However, EIT presents some advantages over these techniques, such as being harmless to the patient, low cost and portable. EIT also has faster time-response characteristics, which enables it to monitor cyclic changes in the living tissues better than conventional imaging modalities. Many researchers have been making continuous efforts in the pursuit of algorithms that are, at the same time, fast and capable of providing images of good spatial resolution.

EIT has a wide range of medical applications, detection of acute cerebral stroke (Clay and Frree, 2002), breast cancer (Kao et al., 2006), monitor cardiac activity (Eyübürgü et al., 1989) and monitor lung aeration imposed by mechanical ventilation in critically ill patients (Hua et al., 1993; Trigo et al., 2004). The last application is the main interest of our group, where EIT can be used to implement a protective lung strategy that can result in an improved lung function and survival (Amato et al., 1998).

2. FORMULATION

The typical forward problem in EIT is given the conductivity distribution \( \sigma \) and the current \( J \) injected through boundary electrodes, find the potential distribution \( \phi \)
within $\Omega$ and in particular the resulting potentials at the measurement electrodes $\phi_m$. The frequencies used in EIT are low enough so that the quasi-static approximation hold, and thus we can ignore capacitive and inductive effects. Under such quasi-static conditions, the solution of the forward problem is rather simple as it only requires solving the Laplace equation

$$\nabla (\sigma \nabla \phi) = 0 \quad (1)$$

At the boundary, currents are injected through electrodes; thus the current density through the $l$-th electrode surface $J_l$ is given by

$$\sigma \frac{\partial \phi}{\partial n} = J_l \quad (2)$$

where $\hat{n}$ is the external normal versor and zero elsewhere at the boundary.

### 2.1 Current Patterns

Data is collected by injecting current with a single source and measuring voltage. Current is injected sequentially to the body using a pair of electrodes. There are several ways in which the pair of electrodes is switched and the voltage measurements are collected in the literature.

Brown and Seagar (1987) suggested a method whereby the EIT sequentially applies electrical currents to the body using a pair of adjacent electrodes. While current is applied, voltages between adjacent non current carrying electrodes are measured. This procedure is repeated, applying current between each pair of adjacent electrodes to obtain a voltage data set. Hua et al. (1988) suggested the cross method, where current is injected between a pair of electrodes which are more separated from each other. This method obtains a more uniform current distribution.

In this method, two adjacent electrodes for current and voltage reference are chosen. Hua et al. (1988) also introduced the opposite method where current is injected through two diametrically opposed electrodes, and uses the electrode adjacent to the current injecting electrode as the voltage reference.

### 2.2 Finite Element Model

The inverse problem is formulated as given the injected currents $\mathbf{J}$ and the potentials at measurement electrodes $\phi_m$, find the electrical conductivity distribution $\sigma$ within $\Omega$. In practice only a finite number of potential measurements is made through the electrodes, so the Dirichlet boundary condition is incomplete (Moura et al., 2010). For an irregular domain and isotropic media, analytical solution to the Laplace equation (1) with boundary condition (2) are unknown; thus, the partial differential equations were approximated by the finite element method (FEM), the domain is discretized with triangular linear elements with constant conductivity and both problems, forward and inverse, are solved numerically. The virtual potential principle associated with the Laplace equation provides the local element matrices.

When the local element matrices are stated in terms of the global coordinates of the mesh, the global conductivity matrix (Trigo et al., 2004) which includes electrode contact impedance effects, is obtained; then the following relation holds:

$$\mathbf{K} \cdot \mathbf{\Phi} = \mathbf{C} \quad (3)$$

where $\mathbf{K}(\sigma) \in \mathbb{R}^{n \times n}$ is the conductivity matrix calculated at a given particular distribution $\sigma_p$, $\mathbf{\Phi}$ is a matrix containing nodal potentials corresponding to each applied current pattern, and $\mathbf{C}$ represents $p$ linearly independent current patterns.

### 2.3 The Inverse Problem as an Optimization Problem

Since there are known methods for efficiently solve the forward problem (such as FEM), one possible approach to the inverse problem is to look at it as an optimization problem, where the optimization variables are the parametrization of the conductivity inside the domain (for instance, the domain is divided in continuous segments and the conductivity inside each segment — presumed to be constant — becomes a parameter) and the optimization function is some measure of how the solution of the forward problem applied to the conductivity distribution produced by the optimization variables matches the measured data.

One possible objective function $E(\sigma)$ is the Euclidean distance between the measured electric potentials and the calculated potentials for all the applied current patterns for a given conductivity distribution. Let $\phi^i_m$ be the measured potential vector at the electrodes and $\hat{\phi}^i(\sigma)$ the corresponding calculated electrode potentials obtained from the solution of the forward problem for a conductivity distribution $\sigma$ (notice the calculated inner potential values are discarded) at the $i$-th current pattern:

$$E(\sigma) = \sqrt{\sum \vert \phi^i_m - \hat{\phi}^i(\sigma) \vert^2} \quad (4)$$

An example of such approach is in (Mello et al., 2008), where the objective function (4) is minimized by Sequential Linear Programming yielding estimations of the conductivity distribution.

Mello et al. (2008) pointed that it is difficult to solve this problem by methods based on gradients of the objective function due to the fact that the problem is often ill-posed. Numerical errors in the calculation of the objective function are greatly amplified in its derivatives.

That is why the interest on Simulated Annealing (SA) applied to EIT is increasing, as it requires no evaluation of objective function derivatives (in fact, as we will show, it does not even require a complete computation of the objective function).

### 3. SIMULATED ANNEALING

SA is a hill-climbing local exploration optimization heuristic, which means it can skip local minima by allowing the exploration of the space in directions that lead to a local increase on the cost function. It sequentially applies random modifications on the evaluation point of the cost function. If a modification yields a point of smaller cost, it is automatically kept. Otherwise, the modification also can be kept with a probability obtained from the Boltzmann distribution:

$$P(\Delta E) = e^{-\frac{\Delta E}{kT}} \quad (5)$$

where $P(\Delta E)$ is the probability of the optimization process to keep a modification that incurs an increase $\Delta E$ of
the cost function. \( k \) is a parameter of the process (analogous to the Stefan–Boltzman constant) and \( T \) is the instantaneous “temperature” of the process. This temperature is defined by a cooling schedule, and it is the main control parameter of the process. The probability of a given state decreases with its energy, but as the temperature rises, this decrease (the slope of the curve \( P(\Delta E) \)) diminishes.

4. APPLYING SIMULATED ANNEALING TO EIT

As seen in section 2.3, the EIT inverse problem can be formulated as an optimization problem, and as such, can be approached with SA.

Herrera et al. (2007) minimized objective function (4) with SA and by doing so, managed to reconstruct very accurate conductivity distributions of the body, but at a very high computational cost. This is unsurprising, as each step of the SA involves the solution of a full FEM problem in order to evaluate the objective function.

4.1 Incomplete Evaluation of the Objective Function

As the evaluation of the objective function is responsible for the bulk of the SA computational cost, it is interesting to look for means to reduce its cost. In particular, we will study how the SA process behaves in the presence of a partial evaluation of the objective function. An example of Simulated Annealing with incomplete evaluation of the objective function can be seen in Lakshmanan and Derin (1989), where an estimate of the probability distribution of the objective function is used instead of the actual value. In this work, we’ll presume the exact distribution of the objective function unknown, but we’ll attempt to create an iterative computation process that yields boundaries for its values. As we’ll see, there’s a relationship between those boundaries and the probability of deviation of the algorithm from an exact SA. Using those relationships and imposing upper limits for those probabilities we can obtain stopping criteria for the interactive calculation of the objective function.

We will use the objective function given by (4). The calculated potentials \( \bar{\phi}_c \) for each current pattern are obtained from a FEM algorithm whose kernel is a solver for the linear system posed by (3). Notice that there are multiple current patterns \( C^i \), leading to multiple calculated potentials \( \Phi^i \).

The Conjugate Gradients (CG) method is usually applied to such linear systems (Shewchuk, 1994) as matrix \( K \) is sparse, symmetric and positive–definite. Meurant (2005) proposed a method to estimate an upper bound for the \( l_2 \) norm of the error of the CG method at each iteration. Such bound can be used to estimate a bound on the error of the objective function (4) when the exact FEM solutions \( \phi_c \) are replaced by partial solutions \( \bar{\phi}_c \) obtained by stopping the CG method before the final convergence. This is done by taking the conservative assumption that all the error is concentrated on the electrode nodes.

Then, if \( d_i = |\phi_c^i - \bar{\phi}_c^i| \) is the component of the objective function for a given current pattern,

\[
\bar{d}_i = |\phi_c^i - \bar{\phi}_c^i| \quad (6)
\]

is its estimate obtained with the partial solution \( \bar{\phi}_c \) and

\[
e^i \geq |\phi_c^i - \bar{\phi}_c^i| \quad (7)
\]
is an upper boundary on the error of the CG method, we have

\[
D^i \leq d_{\text{max}}^i = \bar{d}_i + e^i \\
D^i \geq d_{\text{min}}^i = \max (0, \bar{d}_i - e^i)
\]

and of course,

\[
\tilde{E} = \sqrt{\sum (d_i^2)} \\
E \leq E_{\text{max}} = \sqrt{\sum (d_{\text{max}}^i)^2} \\
E \leq E_{\text{min}} = \sqrt{\sum (d_{\text{min}}^i)^2}
\]

The upper and lower boundaries for \( E \) converge to the exact value as the number of iterations of the multiple CG algorithms increase.

As SA is sensible only to relative variations of the objective function those boundaries must be converted to boundaries of the variation of \( E \). If \( \Delta E = E^j - E^i \) is the variation of the objective function when the SA process moves from one solution \( x^j \) to another \( x^{j+1} \) then, by intervalar arithmetic,

\[
\Delta \tilde{E} = \tilde{E}^j - \tilde{E}^{j+1} \\
\Delta E \leq \Delta E_{\text{max}} = E_{\text{max}}^j - E_{\text{min}}^{j+1} \\
\Delta E \geq \Delta E_{\text{min}} = E_{\text{min}}^j - E_{\text{max}}^{j+1}
\]

Again, those boundaries for \( \Delta E \) converge to the exact value, but this time, it is necessary to increase the number of iterations for both the evaluation of \( x^j \) and \( x^{j+1} \).

From that, it can be seen that a SA process that would use the partial FEM solutions \( \bar{\phi}_c \) instead of the exact ones \( \phi_c \) would have a limited probability of diverging from the process that uses the exact solutions. Indeed, by imposing \( P_{\text{err}} \) as an upper limit on the probability of the process taking a “wrong” decision (rejecting a solution when it should accept it or accepting when it should reject), we can create conditions for the boundaries in (10):

\[
e^{-\Delta E_{\text{max}}/k_t} \geq \frac{1 - P_{\text{err}}}{e^{-\Delta E/\kappa_t} - P_{\text{err}}} \quad \text{if } \Delta E \leq 0 \quad (11)
\]

\[
e^{-\Delta E_{\text{min}}/k_t} \leq \min(1, P_{\text{err}} + e^{-\Delta E/\kappa_t}) \quad (12)
\]

The conditions given by (11) and (12) do not translate directly into stopping criteria for the CG algorithm, as \( \Delta E_{\text{max}} \) and \( \Delta E_{\text{min}} \) are the partial solutions of two separated sets of FEM problems, and in general, they are not reachable by tightening the boundaries in just one set. As such, every solution must have its objective function those boundaries must be converted to.

In fact, it is often computationally more efficient to improve the boundaries on the evaluation of the previous SA solution than the current one. Since CG has an asymptotically geometric convergence and presuming that both the previous SA solution and the current one have similar convergence rates (a reasonable assumption considering that the \( K \) matrix do varies very little from one iteration to another and a preconditioner equals even more the convergence rates), a simple heuristic for picking which
solution to evaluate further is to just pick the one with looser boundaries.

As for which FEM problem (corresponding to a current pattern) to evaluate further within the selected solution, another simple heuristic is to just pick the one with the larger value $d_i e_i$.

5. RESULTS

To evaluate the viability of SA with partial evaluation applied to the EIT problem, a simple implementation was built using a simulated domain. Our simulated domain shown in Fig. 1 is a square, measuring 16 × 16 units, composed of 76 triangular elements with 320 nodes, of whose 32 are electrodes. The electric conductance domain was divided in 64 $2 \times 2$ squares (notice the potential discretization and the conductance discretization is not the same). The conductance in each element is defined in the interval $[1, 2]$.

The electrodes are modeled as $2 \times 1$ rectangles, the larger edge being in contact with the domain. The opposite edge is modeled as being in contact with an infinite conductivity element and consequently the potential across it is constant (the triangular shapes seen in Fig. 1 are thus misleading, they just reflect the fact that the outer edge is collapsed into a single potential node in the FEM model).

The first electrode (the topmost electrode in the left column) was defined as the ground. We applied 31 simulated current patterns, corresponding to an unitary current entering each electrode – but the ground – and leaving through the other ground node.

The FEM problems were solved using a preconditioned CG method, with Incomplete Choleski decomposition preconditioning (Kershaw, 1978). As suggested by (Meurant, 2005), the pre–conditioner $l_\infty$ norm was used for estimation of the error $l_2$ norm.

The 64 conductivity parameters are generated by SA. The neighborhood heuristic used was taken from (Martins and Tsuzuki, 2008), changing only a single conductivity parameter at each iteration and reducing the modifications on parameters that lead to rejected solutions.

The divergence probability $P_{err}$ was arbitrarily defined as 1/100. The whole reconstruction process took 90 minutes on a 2.8 Ghz i7 processor, but one should be weary of using those figures as a measure of the process performance, as the boundaries for the SA temperature parameter are intentionally overblown (both the initial temperature is too high and the final temperature is too low) in order to get a wider view of the convergence process. Perhaps more relevant to the performance process is the discussion on section 5.5 where it is shown how few iterations of the CG algorithm are required per SA iteration.

5.1 Central Dot Problem

The first evaluated problem was a simple domain with conductivity 2 in all its extension but the 4 central squares, which have conductivity 1.

Figure 2 shows the final result of the SA process. It can be see that, despite the use of partial evaluations of the objective function, it still converges to the correct solution.

5.2 Box Problem

This is a non–convex conductance distribution, with a high impedance (conductivity equal 1) $8 \times 8$ box containing a $4 \times 4$ low impedance box (conductivity equal 2). As the electric current naturally avoid the box interior, this is a highly ill–conditioned problem, posing a challenge to a partial evaluation approach as ours.

Figure 3 shows the final result of the optimization process. As it can be seen, while the external shape of the high–impedance region was correctly determined, the process got stuck on a local minimum while determining the conductivity of the box interior. We suspected that our initial neighborhood heuristic influenced the result, and adopted a less restrictive heuristic where, in addition to modifications to single conductivity values, a “conductivity transport” between two adjacent cells was allowed. The amount of conductivity to be transferred from one cell to another also followed the adaptive heuristic proposed by Martins and Tsuzuki (2008).
Fig. 4. Final result for the “Box” problem with modified neighborhood heuristic.

Fig. 5. Evaluating the effect of FEM mesh errors.

The result with the new neighborhood heuristic is shown in Fig. 4. As it can be seen, the initial blame on the neighborhood heuristic seems justified, as the conductivity on the interior of the box is accurately determined.

5.3 Effects of the FEM mesh

As our goal is to estimate the impact of a partial objective function evaluation on the SA, for all the above problems the observations of current patterns were simulated using the same mesh used by the SA to recreate the impedance (thus compensating FEM errors). It is still interesting to see how the process behaves in the presence of errors introduced by the FEM discretization. For such, a new simulated domain was created using a new mesh with twice the resolution of the old one, composed of 2432 triangular elements.

A quick simulation shows that the FEM errors can be quite significant. Estimating the electric impedance between two adjacent electrodes in the middle of one side of the domain (with conductivity 1 in all its extension) using the coarse mesh yields a result of 1.42, while the same simulation using the fine mesh yields a result of 1.56, suggesting that the coarse mesh tends to underestimate the impedance of the electrodes by almost 10%.

A problem with an 8 × 8 square high impedance region was simulated using the fine mesh then the coarse mesh was used by the SA to attempt to reconstruct the conductivity distribution.

The results are shown in Fig. 5. While the image was reasonably reconstructed, there are – as predicted – impedance artifacts in front of the electrodes (caused by the underestimation of electrode impedance by the coarse mesh). Also there is visible noise across the border of the high-impedance region, particularly on its corners.

Fig. 6. Convergence behavior for the “Checkerboard” problem.

Fig. 7. Energy (Objective Function) for the “Checkerboard” problem (error bars represent the standard deviation at a fixed temperature).

5.4 Checkerboard Problem and a qualitative observation of the convergence

To retrieve more information from the convergence of the process, we proposed a complex problem – a 4 × 4 “Checkerboard” conductivity pattern, alternating high conductivity (2) and low conductivity (1).

The SA process has accurately reconstructed the conductivity distribution. The examination of intermediary results, displayed on Fig. 6 and 7 show that the impedance distribution is reconstructed from the outside towards the interior of the domain. While further studies are required, it is reasonable to presume that the objective function is more sensible to modifications on external cells.

5.5 Impact on the evaluation of the objective function

Those results suggest the partial evaluation objective function can be as effective as the full evaluation for the SA, but how efficient is it? Fig. 8 shows the average number of CG iterations used by the process at each temperature. Considering that the system has 320 nodes, it is remarkable that the system is able to achieve those results while using in average less than 20 iterations of CG.
interesting is the evolution of the iteration number as the optimization progresses. At high temperatures, the high $k_t$ and $\Delta \tilde{E}$ values lead to relatively loose conditions in (11) and (12), reachable by few iterations of the CG algorithm. As the temperature diminishes, lower $k_t$ and $\Delta \tilde{E}$ values (as the evaluated solutions become closer) impose through (11) and (12) tighter conditions, leading to a higher number of CG iterations. It may be then a bit surprising that as the process converges towards the final solution that the number of CG iterations reduces again. This is explained by two factors. First, the “initial guess” for the CG algorithm is obtained from the solution obtained at the previous solution of the SA algorithm. Second, the adaptive neighborhood heuristic reduces dramatically the modifications on the conductivity distribution as the process converges towards a global optimum (Martins and Tsuzuki, 2008). Combining those two factors, it can be seen that in the final convergence of the optimization process, the previous FEM solutions are an initial guess good enough to compensate for the increasingly severer conditions on the boundaries of $\Delta E$.

6. CONCLUSIONS

We proposed a new approach to the application of SA to the EIT inverse problem, where the optimization process is allowed to proceed with just a partial evaluation of the objective function. We showed how in the particular case of EIT it is possible to enforce upper bounds on the probability of the modified SA process to deviate from a regular one (with full evaluation of the objective function) by converting those bounds in stopping criteria for the inner CG algorithms used to solve the forward EIT problem. Initial results show that while the partial evaluation of objective functions does not compromise the convergence of the SA algorithm, it has a great potential of improving its efficiency.

REFERENCES


