Stochastic Receding Horizon Control: Stability Results

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Abstract: We present new results on stochastic stability of linear discrete-time systems with additive uncertainty under several receding horizon control policies. We guarantee bounded variance of the closed-loop system under each policy via appending a stability constraint to the existing optimization routine of the corresponding policy.

Keywords: receding horizon, predictive control, constrained control, stochastic control

1. INTRODUCTION

We study the problem of synthesizing bounded controllers for linear systems affected by additive disturbances. On the quantitative side of this problem, one typically seeks to minimize a finite-horizon cost. While the principle of dynamic programming applies, analytical solutions are extremely difficult to find, and it is well-known that the method scales poorly with the number of dimensions. An alternative suboptimal solution in this constrained setting is a static optimization-based control technique such as receding horizon (RH) control (Maciejowski, 2001; Mayne et al., 2000). Fixing an optimization horizon \( N \) and a control horizon \( N_c \), the RH control proceeds as follows: measure the current state, solve a constrained optimal control problem for the next \( N \) stages over a class of policies, apply the first \( N_c \) steps of the obtained policy, resample the state, and recurse the preceding steps. While being suboptimal, RH control enjoys the desirable property of being tractable for a wide class of dynamics, constraints, and policy classes.

On the qualitative side of this problem, we face nontrivial issues of stability in the presence of bounded controls and additive disturbances with non-compact support. Regulation to a reference point being impossible due to the additive nature of the disturbance, we have to consider weaker forms of stability as shown by the following example. Consider the scalar system 
\[ x_{t+1} = x_t + u_t + w_t, \quad x_0 = \bar{x} \text{ (given)}, \quad t \in \mathbb{N}_0, \]
where \( x_t \) is the state, \( u_t \) the control, and \( w_t \) the disturbance at stage \( t \). We assume that \( w_t \in [-\gamma, \gamma] \) and \( u_t \in [-U_{\max}, U_{\max}] \) for each \( t \). On the one hand, we observe that if \( U_{\max} < \gamma \), then there exists a disturbance sequence \((w_t)_{t\in\mathbb{N}_0}\) such that \((x_t)_{t\in\mathbb{N}_0}\) diverges for all initial states and for all controls \((u_t)_{t\in\mathbb{N}_0}\). Therefore, \( U_{\max} \geq \gamma \) is enough to ensure boundedness of the state under all possible disturbance realizations. On the other hand, if we assume that the \((w_t)_{t\in\mathbb{N}_0}\) is a mean-zero i.i.d. random process with bounded fourth moment, it follows from (Ramponi et al., 2010, Proposition 3.9) that with \( U_{\max} > \mathbb{E}[|w_0|] = C_1 \), for every initial state it is possible to ensure a uniform bound on the closed-loop variance of the state. Since \( C_1 \) is at most equal to \( \gamma \) for all distributions, we see at once that one needs lesser control authority to ensure a bounded variance of the state than that needed to ensure a bounded state. From the preceding discussion it is clear that the amount of control authority plays a central role in determining the qualitative behavior of systems. In typical applications both the qualitative and the quantitative aspects of control need to be addressed simultaneously; therefore, RH control techniques that guarantee certain desirable qualitative properties from the closed-loop system are preferable. However, since the interactions of optimality and stability under bounded controls are difficult to analyze, such results are difficult to get.

We focus on establishing new results on stochastic stability under bounded RH control. Stability under bounded controls in the presence of disturbances has a vast history, see e.g., (Ramponi et al., 2010) and the references therein. If the system matrix is Schur stable, and the controls and the disturbances are both bounded, then for every initial state the closed-loop system states stay bounded. If at least one eigenvalue of the system matrix is outside the closed unit disc, then two cases are relevant: First, if the disturbances are bounded, then input-to-state stability (ISS) results indicate that there exist bounded controllers with nonempty robust control invariant regions; the sizes of these regions depend on the control bound, and there is no guarantee of boundedness if the initial state is outside these regions. Second, every bounded control leads to unbounded variance of the states if the disturbance is stochastic in nature and has unbounded support. If the system matrix is marginally stable and the controls are bounded, global stability results are rare, and a rigorous treatment has appeared only recently (Ramponi et al., 2010).

RH control is presently dominated by nominal and robust synthesis methods in the deterministic setting, which have a rich array of rigorous results concerning feasibility, stability, etc.; see e.g., (Maciejowski, 2001; Mayne et al., 2000). Their counterparts in the stochastic setting, however, are comparatively recent developments, see e.g., (Chatterjee et al., 2009) and the references therein. The series of articles (Hokayem et al.,...
Second, we define another objective function that takes into account the worst case cost with respect to the noise sequence, i.e.,

\[ V^2_N(x_t) := \max_{W_t \in \mathcal{W}} \tilde{J}(x_t, U_t, W_t). \] (5)

We are interested in RH controllers, and accordingly, for a control horizon \( N_c \in \mathbb{N} \), at each stage \( N_c \), \( t \in \mathbb{N}_0 \), we are interested in the minimization of the objective function (4) or (5) over the class of causal state feedback strategies \( \Pi \) containing measurable functions \( \pi_{N_c}, \ldots, \pi_{N_c-N+1}, \) such that

\[ u_{N_C+t} := \pi_t(x_{N_C+t}, x_{N_C+t-1}, \ldots, x_t), \quad t = 0, \ldots, N-1. \] (6)

and each \( u_t \) satisfies the control constraints in (2). For a generic \( t \in \mathbb{N}_0 \) we let \( \pi_{t,N-1} \) denote the sequence \( \left[ \pi_{t}, \pi_{t+1}, \ldots, \pi_{t+N-1} \right]^T \).

Below we consider several setups depending on the policy and the structure of \( \mathcal{W} \). At each stage \( N_C \), we consider the constrained robust LQ problem

\[
\minimize \left\{ V^2_N(x_{N_C}) \right\} \quad \text{(1),(2)}, \quad \mathcal{W} = \{ \varepsilon \in \mathbb{R}^n \left| \| \varepsilon \|_{2} \leq \gamma \} \right.,
\]

\[
\pi_t(x_{N_C}, \ldots, x_{N_C+t}) = u_{N_C+t} \quad \text{for } t = 0, \ldots, N-1; \] (7)

the affine noise feedback problem \(^2\)

\[
\minimize \left\{ V^2_N(x_{N_C}) \right\} \quad \text{(1),(2)}, \quad \mathcal{W} = \{ \varepsilon \in \mathbb{R}^n \left| \| \varepsilon \|_{\infty} \leq \gamma \} \right.,
\]

\[
\pi_t(x_{N_C}, \ldots, x_{N_C+t}) = \eta_t + \sum_{j=0}^{t-1} \theta_{N_C+t} w_{N_C+t+j} \quad \text{for } t = 0, \ldots, N-1; \] (8)

the affine saturated-noise feedback problem

\[
\minimize \left\{ V^2_N(x_{N_C}) \right\} \quad \text{(1),(2)}, \quad \mathcal{W} = \{ \varepsilon \in \mathbb{R}^n \left| \| \varepsilon \|_{\infty} \leq \gamma \} \right.,
\]

\[
\pi_t(x_{N_C}, \ldots, x_{N_C+t}) = \eta_t + \sum_{j=0}^{t-1} \phi_{N_C+t} \varphi(w_{N_C+t+j}) \quad \text{for } t = 0, \ldots, N-1; \] (9)

with \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) a component-wise saturation function; and the affine state feedback problem

\[
\minimize \left\{ V^2_N(x_{N_C}) \right\} \quad \text{(1),(2)}, \quad \mathcal{W} = \{ \varepsilon \in \mathbb{R}^n \left| \| \varepsilon \|_{\infty} \leq \gamma \} \right.,
\]

\[
\pi_t(x_{N_C}, \ldots, x_{N_C+t}) = \eta_t + K_{N_C+t} x_{N_C+t} \quad \text{for } t = 0, \ldots, N-1. \] (10)

3. MEAN-SQUARE BOUNDEDNESS

To our knowledge, none of the policies described in §2, besides the one in (9), can be synthesized via their respective optimization programs while requiring that \( \mathcal{U} \) and \( \mathcal{W} \) in (2) are bounded. In order to ensure that the resulting controls are bounded, we shall mostly restrict to the case of \( \mathcal{W} \) being bounded, and point out cases when this assumption can be lifted; see the discussion following Hypothesis 3, Remark 7, and Remark 10 below. Two distinct features of the results presented here are:

- the optimization programs that synthesize the RH policies are feasible for all states, in contrast to robust optimization techniques which are only feasible in a strict subset of the state-space,
- the effect of the random noise appears as the constant \( C_1 \), which represents the statistics of the random noise, in the constraints; since \( C_1 \) is at most equal to the size of the set \( \mathcal{W} \), the constraints turn out to be less conservative than the size of the set \( \mathcal{W} \) that typically features in the robust counterparts, and

\(^2\) We apply the standard convention that \( \sum_{k=0}^{t} \varepsilon(k) \) is zero if \( k > t \).

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1 Here \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \) and \( \mathbb{N} := \mathbb{N}_0 \setminus \{0\} \).

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Dynamics

We consider the discrete-time linear stochastic system:

\[ x_{t+1} = Ax_t + Bu_t + w_t, \quad t \in \mathbb{N}_0, \] (1)

where \( x_t \in \mathbb{R}^n \) is the state, \( u_t \in \mathbb{R}^m \) is the control, \( (w_t)_{t \geq 0} \) is an \( \mathbb{R}^n \)-valued i.i.d. sequence of stochastic noise vectors with mean zero, and \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are given fixed matrices.\(^1\)

We assume that at each \( t \) the state \( x_t \) is observed exactly. For the purposes of this article we require that the control and noise vectors satisfy

\[ u_t \in \mathcal{U} \quad \text{and} \quad w_t \in \mathcal{W}, \] (2)

where \( \mathcal{U} \) and \( \mathcal{W} \) are nonempty sets; their precise structure will differ according to the particular settings considered in the sequel, and will be defined as we progress.

Cost

We fix an optimization horizon \( N \in \mathbb{N} \), and given the state \( x_t \) at time \( t \), we define the quadratic cost function

\[ J(x_t, U_t, W_t) = \sum_{i=0}^{N-1} \left( \| x_{i+1} - Q_x x_i \|_{Q_x}^2 + \| u_t \|_{Q_u}^2 + \| w_t \|_{Q_w}^2 \right), \] (3)

where \( Q_x = Q_x^T > 0 \), \( R_t = R_t^T > 0 \) are given matrices, \( U_t := [u_t^T, \ldots, u_{t+N-1}^T]^T \), and \( W_t := [w_t^T, \ldots, w_{t+N-1}^T]^T \). The quantity \( J(x_t, U_t, W_t) \) is uncertain as it depends on the realization of the noise sequence \( W_t \). To arrive at a well defined optimization problem, we consider two scenarios: First, we assume that \( w_t \) possesses certain distributional properties, and we define an objective function at time \( t \) given \( x_t \) as

\[ V^1_t(x_t) := \mathbb{E}_x[J(x_t, U_t, W_t)]. \] (4)
by relaxing the usual terminal constraint that ensures stability in robust MPC programs to a “drift constraint”, the feasible region of our programs becomes the entire state-space.

Guided by our earlier work in (Hokayem et al., 2010a), we show that it is possible to augment all the previous approaches with an extra drift constraint that guarantees bounded variance in closed-loop. We have the following standing assumption.

**Hypothesis 1.**

(i) The system matrices in (1) satisfy the following:

- \((A, B)\) is stabilizable.
- \(A\) is discrete-time Lyapunov stable, i.e., the eigenvalues \(\{\lambda_i(A)\}_{i=1,\ldots,n}\) lie in the closed unit disc, and those eigenvalues \(\lambda_i(A)\) with \(\lambda_i(A) = 1\) have equal algebraic and geometric multiplicities.

(ii) The controls take values in the control constraint set

\[
\{u_t \in U : u_t \in \mathbb{R}^n \|w_t\|_o \leq U_{\text{max}} \text{ for } t \in \mathbb{N}_0\}.
\]

(iii) \(C_t = \sup_{t \in \mathbb{N}_0} \mathbb{E}[\|w_t\|^2] < \infty\) and let \(C^*_t = \sqrt{C_t}\).

Under Hypothesis 1-(i) there exists a coordinate transformation (Ramponi et al., 2010, §3.4) on the state-space that brings the system (1) to the form

\[
x_{t+1} = A x_t + B u_t + w_t, \quad z_t = x_{t+1} + c, \quad w_t = B \tilde{w}_t,
\]

where \(A_t\) is Schur stable and \(A_0\) is orthogonal, and for some \(k \leq n_0\) we have \(\text{rank}(\mathcal{R}_t(A_0, B_0)) = n_0\) with \(\mathcal{R}_t(A_0, B_0) := [A_0^{-1} B_0, \ldots, B_0]\). This positive integer \(k\) is fixed throughout the rest of this article.

We state the following immediate adaptation of (Pemantle and Rosenthal, 1999, Theorem 1).

**Proposition 2.** Let \((\xi_t)_{t \in \mathbb{N}_0}\) be a sequence of nonnegative random variables on some probability space \((\Omega, \mathcal{F}, P)\), and let \((\bar{\xi}_t)_{t \in \mathbb{N}_0}\) be any filtration to which \((\xi_t)_{t \in \mathbb{N}_0}\) is adapted. Suppose that there exist finite, positive constants \(\varepsilon, J, M\), such that \(\xi_0 \leq J\), and for all \(t\), we have:

\[
\mathbb{E}[\bar{\xi}_{t+1} - \xi_t] = \frac{\varepsilon}{2} \quad \text{on the event } \{\xi_t > J\}, \quad \text{and}
\]

\[
\mathbb{E}[\bar{\xi}_{t+1} - \xi_t^4] \leq \kappa \mathbb{E}[\xi_t^4] \quad \text{for all } t \in \mathbb{N}_0.
\]

Then there exists a constant \(\gamma = \gamma(\varepsilon, J, M) > 0\) such that \(\sup_{t \in \mathbb{N}_0} \mathbb{E}[\bar{\xi}_t] < \gamma\).

We are now ready to state the stability results pertaining to the various control techniques.

### 3.1 Constrained robust LQ

**Hypothesis 3.** In addition to Hypothesis 1, we assume that the noise sequence \((w_t)_{t \in \mathbb{N}_0}\) takes values from the bounded set \(W = \{\xi \in \mathbb{R}^n \ | \ \|\xi\|_o \leq \gamma\}\) for some \(\gamma > 0\) and that

\[
U_{\text{max}} = U_{\text{max}}^{*} := \frac{\min_{t \in \mathbb{N}_0} \mathbb{E}[\|\bar{w}_t\|^2]}{\sigma_{\text{min}}(\mathcal{R}_t(A_0, B_0))} \quad \text{for some } \varepsilon > 0.
\]

The restriction of the noise to the set \(W\) is necessary for feasibility of the optimization problem in Theorem 4—note that boundedness of the noise plays no role in any constraint except the first, which we inherit from (Bertsimas and Brown, 2007). Due to this particular reason, although the analysis carried out in (Bertsimas and Brown, 2007) involves Gaussian (unbounded) noise, the control synthesis is actually performed with a “truncated distribution”. 3

**Theorem 4.** Consider the system (12), suppose that the Hypothesis 3 holds, and let \(N_t = k\). Then the controls \(u_t = \tilde{B}^{-1} y_t - \tilde{B}^{-1} b\), constructed by solving the following feasible and tractable optimization program:

minimize \(z\)

subject to

\[
\begin{bmatrix}
I & Y_{t+1}^T & F \\
Y_{t+1} & -\gamma^2 A & -h^T \\
F & -h & A_t - C + F^T T
\end{bmatrix} \geq 0,
\]

\[
\begin{align*}
\|A_t z_{t+1} - b\|_o & \leq U_{\text{max}}, \\
\|A_t z_{t+1} + \mathcal{R}_t(A_0, B_0) (\tilde{B}^{-1} Y_{t+1} - \tilde{B}^{-1} b)_{\text{max}}\| & \leq \gamma \|z_{t+1}\|,
\end{align*}
\]

whenever \(\|z_{t+1}\| > \sqrt{\kappa} \mathcal{R}_t(A_0, B_0) C_1 + \varepsilon\).

solves the optimal control problem (7) subject to (11), and for every initial condition \(x_0 \in \mathbb{R}^n\) successively successive \(k\)-step of the controls \(u_t\) renders the closed-loop system mean-square bounded, i.e., there exists a constant \(\gamma = \gamma(x_0, \varepsilon, C_4, U_{\text{max}})\) such that \(\sup_{t \in \mathbb{N}_0} \mathbb{E}[\|x_t\|^2] < \gamma\).

The assertion of Theorem 4 consists of feasibility and tractability of (15) and the fact that successive \(k\)-step application of the controls \(u_t\) ensures that the closed-loop system is mean-square bounded; the latter is due to the last constraint in (15). We skip the proof of Theorem 4 for reasons of space; it can be established by following the steps of the proof of Theorem 9 below, which is given in detail.

### 3.2 Affine noise and saturated noise feedback

We first state our result concerning mean-square boundedness of the closed-loop system under RH control with the affine noise feedback technique. We require: 4

**Hypothesis 5.** In addition to Hypothesis 1 we assume that the noise sequence \((w_t)_{t \in \mathbb{N}_0}\) takes values from the bounded set \(W = \{\xi \in \mathbb{R}^n \ | \ \|\xi\|_o \leq \gamma\}\) for some \(\gamma > 0\), and that \(U_{\text{max}} = U_{\text{max}}^{*} := \min_{t \in \mathbb{N}_0} \mathbb{E}[\|\bar{w}_t\|^2] / \sigma_{\text{min}}(\mathcal{R}_t(A_0, B_0)) C_1 + \varepsilon\) for some \(\varepsilon > 0\).

**Theorem 6.** Consider the system (12), suppose that the Hypothesis 5 holds, and let \(N_t = k\). Then a solution to the following feasible and tractable optimization problem:

minimize \(\Theta_{k,t}^T \Theta_{k,t} + \Theta_{k,t}^T M \Theta_{k,t}\)

subject to

\[
\begin{align*}
\Theta_{k,t} & \text{ having the structure given in (A.2),} \\
\|\Theta_{k,t}\| & \leq \gamma \|\bar{w}_t\|, \\
\|A_t x_{t+1} + \mathcal{R}_t(A_0, B_0) (\tilde{B}^{-1} Y_{t+1} - \tilde{B}^{-1} b)_{\text{max}}\| & \leq \gamma \|x_{t+1}\|,
\end{align*}
\]

whenever \(\|x_{t+1}\| > \sqrt{\kappa} \mathcal{R}_t(A_0, B_0) C_1 + \varepsilon\).

3 The various vectors and matrices appearing in Theorem 4 below are listed in Appendix §A.2 for convenience.

4 The various vectors and matrices appearing in Theorem 6 below are listed in Appendix §A.3 for convenience.
yields controls defined by (A.1) that
○ solve the optimal control problem (8) subject to (11), and
○ for every initial condition $x_0 \in \mathbb{R}^n$ successive $\kappa$-step applications of the controls $U_{\kappa}$ defined by (A.1) renders the closed-loop system mean-square bounded, i.e., there exists a constant $\bar{\gamma} = \gamma(x_0, \kappa, C_4, U_{\max})$ such that $\sup_{t \in [0,T]} \mathbb{E}_0[|x(t)|^2] \leq \bar{\gamma}$.

The proof is similar to that of (Hokayem et al., 2010a, Theorem 3), and is omitted.

**Remark 7.** A glance at the policies in (8) and (9) reveals at once that they are structurally similar. Nevertheless, observe that if the noise is not bounded, contrary to Hypothesis 5, then the policy in (8) is not guaranteed to stay bounded in general irrespective of the constraints on the $\eta's$ and the $\theta's$. In contrast, the one in (9) is bounded provided the function $\varphi$ is. The only exception to this occurs if the $\theta's$ in (8) are all set to zero (the control is thus open-loop); in this case observe that the corresponding optimization problem in Theorem 6 does not require bounded noise in Hypothesis 5.

### 3.3 Affine state feedback

The various vectors and matrices appearing in Theorem 6 below are listed in Appendix §A.4 for convenience.

**Hypothesis 8.** In addition to Hypothesis 1, we stipulate that:
1. $\mathbb{W} = \{\xi \in \mathbb{R}^n \mid ||\xi||_\infty \leq \gamma\}$ for some $\gamma > 0$,
2. for the control policy (10) we fix $K_\epsilon = K$ for all $\epsilon > 0$,
3. we require that for some $\epsilon > 0$,

$$U_{\max} > U_{\max}^* := \sigma_{\epsilon n}(\mathcal{R}(A_{\kappa}, B_{\kappa}))^{-1} \left( \begin{array}{c} \sqrt{\rho_{\kappa m}}(\mathcal{R}(A_{\kappa}, B_{\kappa})) \sigma_1(\mathcal{K}\mathcal{D}) \\
\quad + \mathcal{M}_{\epsilon} \sigma_1(\mathcal{R}(A_{\kappa}, B_{\kappa})) \end{array} \right) C_1 + \frac{\epsilon}{2} + \left\| \mathcal{K}\mathcal{D} \right\|_\infty \gamma,$$

where the matrices $\mathcal{D}$ and $K$ are given in (A.4) and (A.5), respectively.

**Theorem 9.** Consider the system (12), suppose that Hypothesis 8 holds, and let $\eta_t = \kappa$. Then a solution to the following tractable optimization problem:

$$\min_{\eta_t} \mathbb{E}_0 \left( (Q + K^T R K) X_{\kappa} + (K^T R K) \eta_t \right)$$

subject to

- $K$ as in (A.5) and $\eta_t \in \mathbb{R}^{N_t}$,
- $\left\| (\mathcal{K}\mathcal{A})_{\kappa} \right\|_\infty \left( (\kappa \mathcal{B})_{\kappa} + (I_t)_{\eta_t} \right) + \left\| (\mathcal{K}\mathcal{D}) \right\|_\infty \gamma \leq U_{\max}$
- for all $i = 1, 2, \ldots, N_t$,
- $\left\| \bar{A}_{\kappa i} x_{\kappa i} + \mathcal{R}(A_{\kappa}, B_{\kappa}) (\mathcal{K}\mathcal{A})_{\kappa} x_{\kappa i} + (\mathcal{K}\mathcal{B} + I) \eta_{\kappa i} \right\|_{1:x_m} \leq \left\| x_{\kappa i} \right\|_2 - \left( \sqrt{\rho_{\kappa m}}(\mathcal{R}(A_{\kappa}, B_{\kappa})) \sigma_1(\mathcal{K}\mathcal{D}) \right) + \sigma_1(\mathcal{R}(A_{\kappa}, B_{\kappa})) C_1 \sqrt{\kappa} - \frac{\epsilon}{2}$ whenever $\left\| x_{\kappa i} \right\|_2 \geq \left( \sqrt{\rho_{\kappa m}}(\mathcal{R}(A_{\kappa}, B_{\kappa})) \sigma_1(\mathcal{K}\mathcal{D}) \right) + \sigma_1(\mathcal{R}(A_{\kappa}, B_{\kappa})) C_1 \sqrt{\kappa} + \epsilon$,

which verifies the third constraint of (17). Hence $\tilde{\eta}_t$ makes the optimization problem (17) feasible and completes the proof of the first assertion. Regarding the second claim, we prove below that the sequence $(\xi_t)_{t \in \mathbb{N}}$, defined by $\xi_t = \left\| x^*_{\kappa i} \right\|_2$ satisfies both (13) and (14), and from Proposition 2 we will conclude that the orthogonal sub-system in (12) has bounded variance. Consider the sub-sampled process $(x^0_{\kappa i})_{t \in \mathbb{N}}$, given by

$$x^0_{t+1} = A^0_{\kappa i} x^0_{\kappa i} + \mathcal{R}(A_{\kappa}, B_{\kappa}) u_{\kappa i} + \mathcal{R}(A_{\kappa}, B_{\kappa}) u_{\kappa i},$$

where $u_{\kappa i} := \left[ \begin{array}{c} u_{\kappa i} \\
\vdots \\
u_{\kappa i} \end{array} \right]$ and $w^0_{\kappa i} := \left[ \begin{array}{c} w^0_{\kappa i} \\
\vdots \\
w^0_{\kappa i} \end{array} \right]$. We define the filtration $(\mathcal{F}_t)_{t \in \mathbb{N}}$ as $\mathcal{F}_t := (x_{\kappa i}, \ldots, x_{\kappa i})$, and observe that $(\xi_t)_{t \in \mathbb{N}}$ is $\mathcal{F}_t$-adapted. Now,

$$u_{\kappa i} := \left[ \begin{array}{c} \mathcal{K}\mathcal{A} x_{\kappa i} + (\mathcal{K}\mathcal{B} + I) \eta_{\kappa i} + \mathcal{K}\mathcal{D} W \right]_{1:x_m}.$$

Consequently,
Employing the third constraint in (17) we arrive at
\[
\mathbb{E}^\lambda \left[ \|x_{t+1}^o\| - \|\hat{x}_{t+1}^o\| \right] \\
\leq - \left( \sqrt{\text{max}} \left( \mathbb{E} \left[ (\mathbb{R}_A(A, B)(\mathcal{K}D) \right] + \frac{\lambda}{2} \right) \right) \\
+ \mathbb{E}^\lambda \left[ \mathbb{R}_A(A, B)(\mathcal{K}D) \right]_{t+1:n} \\
+ \mathbb{R}_A(A, B)w_{t+1:n} \right] \\
\leq - \frac{\epsilon}{2} \quad \text{on } \{ \|x_t^o\| > J := \mathbb{R}_A(A, B)C_1 \}
\]
This verifies (13). It can be shown, as in (Ramponi et al., 2010), that there exists a constant $M = M(C_1, U_{\text{max}}) > 0$ such that
\[
\mathbb{E} \left[ \|x_{t+1}^o\| - \|\hat{x}_{t+1}^o\| \right] \leq M.
\]
Therefore, (14) is verified. Letting $J := \text{max}(J', \|x_0\|)$, we conclude from Proposition 2 that the sub-sampled process ($x_t^o$) is mean-square bounded, i.e., there exists a constant $\bar{y}_0 = \bar{y}_0(x_0, \epsilon, C_4, U_{\text{max}}) > 0$ such that $\mathbb{E} \|x_t^o\| \leq \bar{y}_0$. Application of triangle inequalities and the system dynamics (12) show that there exists $\gamma_0 = \gamma_0(x_0, \epsilon, C_4, U_{\text{max}}) \geq \bar{y}_0$ such that $\mathbb{E}_\lambda \left[ \|x_t^o\| \right] \leq \gamma_0$. Moreover, since $A_\kappa$ is Schur stable, the control is bounded, and the noise is bounded in fourth moment, there exists (Ramponi et al., 2010) a constant $\gamma_s > 0$ such that $\mathbb{E} \|x_t^o\| \leq \gamma_s$ for all $t$. Finally, setting $\bar{y} = \gamma_0 + \gamma_s$ completes the proof.

4. NUMERICAL EXAMPLE

We consider a neutrally stable system for the simulation and compare the performance for all the RH techniques considered here. The neutrally stable system is given as:
\[
x_{t+1} = \begin{bmatrix} \cos(\varphi) - \sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \cos(\theta) - \sin(\theta) \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t + w_t, \quad (20)
\]
with $\varphi = 3\pi/4$ and $\theta = \pi/4$. To ensure a fair ground for comparison among the RH techniques, we require that the noise sequence $(w_t)_{t\in\mathbb{N}}$ is i.i.d. uniformly distributed over the set $\mathcal{W} = \left[-\frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}\right]$; thus, $\|w_t\|_2, \|w_t\|_\infty \leq 10$. Notice that all the four eigenvalues of $A$ lie on the unit circle and are simple, also the pair $(A, B)$ is reachable with reachability index $\kappa = 4$. The state and control weight matrices were chosen to be $Q_i = I_{4\times4}$ and $R_i = 1$ respectively, and the initial condition was $x_0 = \begin{bmatrix} 20 & 50 & 25 & 100 \end{bmatrix}^T$. The control horizon, $N_c$, was fixed at 4, equal to the reachability index $\kappa$ and the optimization horizon was chosen to be 6. We simulated the system for 50 different extractions of the noise sequence with the above data and the policy proposed in (Hokayem et al., 2010a) clearly outperforms the others.

REFERENCES


Appendix A. DEFINITIONS OF VARIOUS MATRICES

A.1 Augmented system vectors and matrices

\[
X_t := \begin{bmatrix} x_t \\ \vdots \\ x_{Nt} \end{bmatrix}, \quad U_t := \begin{bmatrix} u_{t} \\ \vdots \\ u_{Nt-1} \end{bmatrix}, \quad W_t := \begin{bmatrix} w_{t} \\ \vdots \\ w_{Nt-1} \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} I_{m} \\ \vdots \\ A^{Nt} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ \vdots \\ A^{Nt-1}B \\ \vdots \\ A^{Nt} \end{bmatrix}, \quad \mathcal{D} := \begin{bmatrix} 0_{m} \\ \vdots \\ A_{m} \end{bmatrix}, \quad \mathcal{Q} := \text{diag}(Q_0, Q_1, \ldots, Q_N), \quad \text{and} \quad \mathcal{R} := \text{diag}(R_0, R_1, \ldots, R_{N-1}).
\]

A.2 The vectors and matrices appearing in Theorem 4

\[
F = \hat{B}^{-1/2} \hat{D}^{-1} h = c - \hat{D}^{-1} \hat{B}^{-1} b,
\]

\[
c = \sum_{k=1}^{N} \hat{C}_{k-1}^{T} Q \hat{A}_{k-1} x_{kt},
\]

\[
C = \sum_{k=1}^{N} \hat{C}_{k-1}^{T} Q \hat{C}_{k-1}, \quad b = \sum_{k=1}^{N} \hat{B}_{k-1}^{T} Q \hat{C}_{k-1}, \quad \hat{B} = \hat{R} + \sum_{k=1}^{N} \hat{B}_{k-1} Q \hat{B}_{k-1}, \quad \hat{D} = \sum_{k=1}^{N} \hat{D}_{k-1} Q \hat{C}_{k-1}, \quad \hat{R} = \text{diag}(R_0, \ldots, R_{N-1}), \quad \hat{A}_{k-1} = \prod_{i=1}^{k-1} A, \quad \hat{B}_{k-1} = \begin{bmatrix} (\prod_{j=1}^{k-1} A) B \\ \vdots \\ 0_{m0(N-k)m} \end{bmatrix}, \quad \text{and} \quad \hat{C}_{k-1} = \begin{bmatrix} (\prod_{j=1}^{k-1} A) \cdots I_{m} 0_{m0(N-k)n} \end{bmatrix}.
\]

A.3 The vectors and matrices appearing in Theorem 6

Over an entire horizon the controls can be written in a compact form as

\[
U_t = \Theta_t W_t + \eta_t, \quad \text{(A.1)}
\]

where

\[
\eta_t := \begin{bmatrix} 0_t \\ \vdots \\ 0_{Nt-1} \end{bmatrix} \quad \text{and} \quad \Theta_t := \begin{bmatrix} 0_{m} \\ \vdots \\ 0_{m0} \end{bmatrix}, \quad \text{(A.2)}
\]

A.4 The vectors and matrices appearing in Theorem 9

Over an entire horizon the controls can be written in a compact form as

\[
U_t = \mathcal{K}_t X_t + \eta_t, \quad \text{(A.3)}
\]

where

\[
\mathcal{A} := \begin{bmatrix} I_{m} \\ \vdots \\ A^{Nt} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ \vdots \\ A^{Nt}B \end{bmatrix}, \quad \mathcal{D} := \begin{bmatrix} 0_{m} \\ \vdots \\ A \end{bmatrix}, \quad \mathcal{Q} := \text{diag}(Q_0, Q_1, \ldots, Q_N), \quad \text{and} \quad \mathcal{R} := \text{diag}(R_0, R_1, \ldots, R_{N-1}).
\]