Design of Uncertain Nonlinear Feedback Systems with Inputs and Outputs Satisfying Bounding Conditions ⋆

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Abstract: This paper develops a computational method for designing a feedback control system where the plant is an uncertain linear time-invariant convolution subsystem with a static memoryless input nonlinearity. The main design objective is to determine a controller ensuring that the error and the controller output are always within respective bounds for all uncertainties and for all inputs having bounded magnitude and slope. First, using Schauder fixed point theorem, we show that (if exists) a design solution for an uncertain linear system obtained by replacing the nonlinearity with a gain and a bounded disturbance is also a solution for the original problem. Then, by extending a known theory and applying it to the so-obtained linear problem, we derive design inequalities that can readily be solved in practice. The usefulness of the developed method is illustrated by a design example of an uncertain heat-conduction process.

Keywords: Nonlinear control systems, computer-aided control system design, robust control, uncertain dynamical systems, distributed-parameter systems, Popov criterion.

1. INTRODUCTION

Many practical systems possess uncertainties which cannot be neglected in the design. In this case, if one requires to arrive at an accurate and realistic formulation so that the design problem can be solved effectively, then the robustness issue has to be taken into account explicitly.

Usually, one of the principal aims in control systems design is to ensure that a response $v$ is always within a prescribed bound for all possible inputs (i.e., inputs that can happen or are likely to happen in practice). Accordingly, the design criterion is expressed as

$$|v(f, t)| \leq \varepsilon, \; \forall f \; \forall t \in \mathbb{R}$$

where $v(f, t)$ is the value of $v$ at time $t$ in response to a possible input $f$, and $\varepsilon$ is the largest value of $|v(f, t)|$ that can be accepted. Criterion (1) is frequently employed in practice by engineers to monitor the performances of control systems and has been investigated, for various models of possible inputs, by many researchers (see, e.g., Birch and Jackson, 1959; Zakian, 1979, 1996, 2005; Lane, 1995; Silpsrikul and Arunsawatwong, 2010, and the references therein). Moreover, the criterion is useful in the design of critical systems, where any violation of the bound $\varepsilon$ may result in an unacceptable operation (see Zakian, 1989, 2005, and the references therein).

This paper considers a feedback control system shown in Fig. 1, where $\psi$ is a continuous, static and memoryless nonlinear function such that $\psi(0) = 0$, $G_p(s)$ is the plant transfer function, and $G_c(s, p)$ is the controller transfer function with a design parameter $p \in \mathbb{R}^n$. Suppose that $G_p(s)$ has uncertainties and belongs to a set $\mathcal{G}_p$. The system is assumed to be at rest for $t \leq 0$. The input $f$ is known only to the extent that it belongs to a possible set $\mathcal{P}$ (i.e., a set containing all possible inputs) given by

$$\mathcal{P} \triangleq \{ f \in L_\infty \mid \|f\|_\infty \leq M, \|f\|_\infty \leq D \}.$$ (2)

where $L_\infty$ denotes the set of all functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\|f\|_\infty < \infty$, and the bounds $M$ and $D$ are given. As usual, $\|f\|_\infty \equiv \sup\{|f(t)| : t \in \mathbb{R}_+\}$.

![Fig. 1. An uncertain nonlinear feedback system of interest.](image_url)

The set $\mathcal{P}$ is appropriate for characterizing persistent inputs (i.e., inputs varying persistently for all time). Note that $\mathcal{P}$ has long been investigated by many researchers (see, e.g., Birch and Jackson, 1959; Zakian, 1979; Lane, 1995; Silpsrikul and Arunsawatwong, 2010). Evidently, when all the possible inputs have no stepwise discontinuities, using $\mathcal{P}$ as the possible set makes the design formulation more realistic and more appropriate than using $L_\infty$.

For more discussion, see Zakian (1996, 2005).

In connection with the system in Fig. 1, the design problem considered in this paper is to determine a controller $G_c(s, p)$ so that the following design criteria are satisfied.
\[
|e(f, t)| \leq E_{\text{max}} \quad |u(f, t)| \leq U_{\text{max}} \\
\sup_{f \in \mathcal{P}} \sup_{t \geq 0} |e(f, t)|, \quad \sup_{f \in \mathcal{P}} \sup_{t \geq 0} |u(f, t)|
\]
where the bounds \( E_{\text{max}} \) and \( U_{\text{max}} \) are given. It is easy to see that the criteria (3) are equivalent to
\[
\sup_{G_p \in \mathcal{G}_p} \hat{e} \leq E_{\text{max}}, \quad \hat{e} \triangleq \sup_{f \in \mathcal{P}} \sup_{t \geq 0} |e(f, t)|, \\
\sup_{G_p \in \mathcal{G}_p} \hat{u} \leq U_{\text{max}}, \quad \hat{u} \triangleq \sup_{f \in \mathcal{P}} \sup_{t \geq 0} |u(f, t)|
\]
where, for each \( G_p(s) \in \mathcal{G}_p \), \( \hat{e} \) and \( \hat{u} \) are sometimes called the peak values of \( e \) and \( u \), respectively.

Evidently, the criteria (4) become useful design inequalities if the numbers \( \sup_{G_p \in \mathcal{G}_p} \hat{e} \) and \( \sup_{G_p \in \mathcal{G}_p} \hat{u} \) can be computed in practice. However, computing such numbers is difficult, since the optimization problems defined in (4) are in general non-convex and infinite-dimensional.

The main objective of this paper is to develop a computational method for determining a controller \( G_2(s) \) satisfying the criteria (4). Essentially, we derive a sufficient condition of the form
\[
\phi_i \leq e_{\text{max}} \quad \text{and} \quad \phi_i \leq u_{\text{max}}
\]
for ensuring the satisfaction of (4), where the associated performance measures \( \phi_i \) and \( \phi_i \) are readily computable. Hence, the condition (5) provides the design inequalities that are suitable for solution by numerical methods.

The key ideas are as follows. First, by extending the results in Mai et al. (2010b) to the case of \( G_p(s) \) with uncertainties, we show that (if exists) a design solution for an uncertain linear system (see Fig. 4) obtained by replacing the nonlinearity with a constant gain and a bounded disturbance also satisfies the original design criteria (4). Second, the design inequalities (5) are derived by using the results obtained from extending Zakian (1983, 1984, 1996, 2005)’s theory of majorants to the case of the system with two inputs. Finally, a useful condition for ensuring the robust stability of the nonlinear system is developed so that a numerical solution of the criteria (4) can be obtained by searching in the space of design parameters.

The organization of this paper is as follows. Section 2 presents the extension of Zakian’s theory of majorants. Section 3 includes the main contribution of the paper, where the inequalities (5) are derived. In Section 4, the stability condition of the uncertain nonlinear system is given. A design of an uncertain heat-conduction process is carried out in Section 5 to illustrate the usefulness of the method. Finally, conclusions are given in Section 6.

2. EXTENSION OF THEORY OF MAJORANTS

Zakian (1983, 1984, 1996, 2005)’s theory of majorants, which was developed for linear time-invariant systems with one input, has been applied successfully to the design of robust control systems (Taiwo, 1986, 2005; Bada, 1987). An extension of the theory to the case of uncertain linear feedback systems with two inputs is presented in this section. (A majorant is an upper bound with certain properties. See Zakian (1984) for the details.)

Consider the linear feedback system shown in Fig. 2. Let \( G_2(s) \) have uncertainties such that \( G_2(s) \in \mathcal{G}_2 \). Let \( f \in \mathcal{P} \) and \( d \in \mathcal{D} \) where
\[
\mathcal{D} \triangleq \{ d \in L_\infty \mid \|d\|_\infty \leq N \}.
\]

For the purpose of the paper, attention is focused only to the case where \( f \in \mathcal{P} \) and \( d \in \mathcal{D} \). However, it is important to note that the results presented in this section are valid for any model of the possible sets.

Fig. 2. A linear feedback system with two inputs.

Let \( g_1 \) and \( g_2 \) be the impulse responses corresponding to \( G_1(s) \) and \( G_2(s) \), respectively. The system is described by
\[
v_2 = v_1 * g_1, \quad v_1 = f - g_2 * (d + v_2), \quad G_2(s) \in \mathcal{G}_2.
\]

Suppose that the design problem for the system (7) is to determine \( G_1(s) \) satisfying
\[
\sup_{G_2 \in \mathcal{G}_2} \hat{v}_i \leq V_i, \quad \hat{v}_i \triangleq \sup_{f \in \mathcal{P}, d \in \mathcal{D}} \|v_i\|_\infty, \quad i = 1, 2
\]
where the bounds \( V_1 \) and \( V_2 \) are given. In case that it is desirable to replace \( G_2(s) \) by a fixed transfer function \( G_2'(s) \), the actual system (7) becomes the nominal one shown in Fig. 3 and described by
\[
v_2 = v_1 * g_1, \quad v_1 = f - g_2 * (d + v_2'). \quad (9)
\]

Fig. 3. The nominal system for the system (7).

In the following, we will develop sufficient conditions expressed in terms of the nominal system (9) for ensuring the satisfaction of the design criteria (8). To this end, define
\[
\mu_i \triangleq \int_0^\infty |w_i(\tau)| d\tau, \quad i = 1, 2
\]
where \( w_i \) is the inverse Laplace transform of \( W_i(s) \) given by
\[
W_1(s) \triangleq \frac{G_2(s) - G_2'(s)}{1 + G_1(s)G_2'(s)}, \quad W_2(s) \triangleq G_1(s)W_1(s).
\]

Let \( \hat{v}_i^* \) denote the peak value of \( v_i^* \) and be described by
\[
\hat{v}_i^* \triangleq \sup_{f \in \mathcal{P}, d \in \mathcal{D}} \|v_i^*\|_\infty \quad i = 1, 2.
\]

For each \( G_2(s) \in \mathcal{G}_2 \), upper bounds of \( \hat{v}_i \) are obtained as follows.

**Lemma 1.** Suppose that the system (9) is BIBO stable and let \( \mu_1 < \infty \) and \( \mu_2 < \infty \). For each \( G_2(s) \in \mathcal{G}_2 \), if \( \mu_2 < 1 \), then
\[
\hat{v}_i \leq \phi_i, \quad \phi_i \triangleq \frac{\hat{v}_i^* + \mu_i N}{1 - \mu_2} \quad i = 1, 2.
\]

**Proof.** It is an extension of the result in Zakian (1983). See Mai (2010)’s thesis for the details of the proof. \( \square \)

By noting that \( \mu_i \) and hence \( \phi_i \) depend on \( G_2(s) \in \mathcal{G}_2 \), the following result is readily obtained from Lemma 1.
Proposition 2. Suppose that the system (9) is BIBO stable and let \( \mu_1 < \infty \) and \( \mu_2 < 1 \) for any \( G_2(s) \in G_2 \). For the system (7), the criteria (8) are satisfied if the following hold:

\[
\hat{\phi}_i + N \sup_{G_2 \in G_2} \mu_i \leq V_{i,\max}, \quad i = 1, 2.
\]

(14)

Evidently, inequalities (14) cannot readily be used for determining \( G_1(s) \) by numerical methods, because, for each value of the design parameter \( p \), the supremal operations over \( G_2 \) are required in the evaluation of \( \sup_{G_2 \in G_2} \mu_i \), which is not computationally economical. For this reason, Zakian (1984, 1996) proposes to replace \( \sup_{G_2 \in G_2} \mu_i \) by its majorant \( \mu_i \) and thus arrives at

\[
\hat{\phi}_i \geq \phi_i, \quad i = 1, 2,
\]

(15)

where \( \hat{\phi}_i \) is a majorant of \( \phi_i \) and defined by

\[
\hat{\phi}_i \triangleq \frac{\hat{v}_i + \mu_i N}{1 - \mu_2}, \quad \mu_2 < 1, \quad \mu_i = A[\sigma_i + B|\hat{v}_i(1)| - \sigma_i], \quad \sigma_i = \lim_{t \to \infty} v_i(t, 1),
\]

(16)

\[
A = \sup_{G_2 \in G_2} \{G_2(t, 1), \ z = g_2 - g_2^*, \ b = \sup_{G_2 \in G_2} \{|z(0)| + |\hat{\phi}_i| : G_2 \in G_2 \},
\]

and \( \hat{v}_i(t, 1) \) is the value of \( v_i^* \) at time \( t \) in response to the input \( f(t) = 1(t) \) and \( d(t) = 0 \). As usual, \( 1 \) denotes the unit step function. Note that the values \( A \) and \( B \) (which depend on \( \phi_i^* \) need to be computed only once; the numbers \( \sigma_i \) and \( v_i^* \) (which do not depend on \( \phi_i \) can be obtained easily for each value of \( \mu_i \)). Hence, \( \hat{\phi}_i \) can readily be computed in practice.

It is now ready to state the main result of this section.

Theorem 3. Suppose that the nominal system (9) is BIBO stable and let \( \mu_1 < \infty \) and \( \mu_2 < 1 \). For the system (7), the criteria design (8) are satisfied if the following hold:

\[
\hat{\phi}_i \leq V_{i,\max}, \quad i = 1, 2.
\]

(17)


From the above, it follows that \( \hat{\phi}_i \) can readily be computed in practice. Consequently, conditions (17) provides useful design inequalities for determining \( G_1(s) \) that satisfies the original criteria (8) by numerical methods.

3. DESIGN OF UNCERTAIN NONLINEAR SYSTEMS

This section focuses on the design problem of the uncertain nonlinear system shown in Fig. 1. By a straightforward extension of the method used in Mai et al. (2010b), it can be shown that a design solution of a linear system, which is obtained by replacing the nonlinearity \( \psi \) with a gain \( K \) and a bounded disturbance \( d \), is also that of the uncertain nonlinear system. Since the linear system is uncertain and subject to two inputs, the results developed in Section 2 can readily be applied and thus yield sufficient conditions for ensuring (4). The conditions provide surrogate design criteria that are in keeping with the method of inequalities (Zakian and Al-Naib, 1973).

Assumption 1. For every input \( f \in P \) and every \( G_p(s) \in G_p \), there are unique \( c : \mathbb{R}_+ \to \mathbb{R} \) and \( u : \mathbb{R}_+ \to \mathbb{R} \) that satisfy the following equations

\[
u = g_c * e, \quad e = f - u * g_p, \quad u = \psi(u).
\]

(18)

where \( g_c \) and \( g_p \) are the inverse Laplace transforms of \( G_c(s) \) and \( G_p(s) \), respectively.

Now, by using the decomposition technique developed in Mai et al. (2010b), the auxiliary linear system shown in Fig. 4 is obtained and described by

\[
u' = g_c * e', \quad e' = f - g_2 * (Ku' + d), \quad K \in \mathbb{R}, \ G_2(s) \in G_2, \ f \in P, \ \text{and} \ d \in D \text{ given by}
\]

\[
D \triangleq \{d : \|d\|_\infty \leq N\}, \quad N \triangleq \sup_{|z| \leq U_{\max}} |\psi(x) - Kx|.
\]

(20)

Fig. 4. The auxiliary linear system.

Oldak et al. (1994) used the similar decomposition technique in connection with the design by quantitative feedback theory (see, e.g., Yaniv, 1999, and the references therein) for feedback systems containing hard nonlinearities. It is important to note, however, that the design problem formulation considered here is different from that in Oldak et al. (1994). In fact, the problem setting and the design objectives are completely different.

For each \( G_p(s) \in G_p \), let \( e' \) and \( u' \) denote the peak values of \( e' \) and \( u' \), respectively, given by

\[
e' \leq \sup_{f \in P} \|e'\|_\infty, \quad u' \leq \sup_{f \in P, d \in D} \|u'\|_\infty.
\]

(21)

Next, let \( A \) denote the convolution algebra whose elements take the form

\[
g(t) = \left\{ \begin{array}{ll} g_c(t) + \sum_{i=0}^{\infty} g_d(t - t_i), & t \geq 0, \\
0, & t < 0, \end{array} \right.\]

(22)

where \( \delta(\cdot) \) is the Dirac delta function, \( 0 = t_0 < t_1 < t_2 \ldots \) are constants, \( \sum_{i=0}^{\infty} |g_d(t)| < \infty \) and \( \int_0^{\infty} |g_d(t)| dt < \infty \) (see, e.g., Desoer and Vidyasagar, 1975). Also, let \( h \) be the impulse response corresponding to the transfer function

\[
H(s) \triangleq \frac{G_p(s)G_c(s)}{1 + KG_p(s)G_c(s)}.
\]

(23)

Assumption 2. The function \( h \) satisfies conditions that \( h \in A \) and \( h \in A \) for all \( G_p(s) \in G_p \).

It should be noted that by virtue of the representation (22) in \( A \), the plant transfer function \( G_p(s) \) in (18) can be lumped- or distributed-parameter systems as long as \( h \) satisfies Assumption 2. For example, the plant can be a system with time-delays or a heat conduction process.

The main result is stated below and can be proved by using the method in Mai et al. (2010b), which is essentially an application of Schauder fixed point theorem (see, e.g., Baños and Horowitz, 2004).

Theorem 4. Let Assumptions 1 and 2 be satisfied. For the system in Fig. 1, the design criteria (4) are satisfied if, for the system in Fig. 4, the following hold:

\[
\sup_{G_c \in G_c} \|e'\|_\infty, \quad \sup_{G_p \in G_p} \|u'\|_\infty.
\]

(24)
Proof. The details of the proof can be found in Mai (2010)’s thesis. □

According to Theorem 4, the design problem of the nonlinear system can be replaced by that of the auxiliary linear system subject to an additional disturbance \( d \). However, computing the performances \( e^* \) and \( u^* \) given by (24), in general, are difficult due to the plant uncertainty. Therefore, it is desirable to replace (24) with sufficient conditions by using the results developed in Section 2.

Consider the nominal system shown in Fig. 5 where \( f \in \mathcal{P} \), \( d \in \mathcal{D} \) and \( G_p^*(s) \) denotes the nominal transfer function to be used for \( G_p(s) \in \mathcal{G}_p \).

\[
\begin{align*}
& e^* \quad G(s) \quad u^* \quad K \quad d \quad u^* \quad G_p^*(s) \quad y^* \\
& f
\end{align*}
\]

Fig. 5. The nominal system for the system in Fig. 4.

Assume that the nominal system is BIBO stable. As a result, the following limits exist

\[
\sigma_1 \triangleq \lim_{t \to \infty} e^*(t, 1), \quad \sigma_2 \triangleq \lim_{t \to \infty} u^*(t, 1),
\]

(25)

where \( e^*(t, 1) \) and \( u^*(t, 1) \) are the values of \( e^* \) and \( u^* \) at time \( t \) in response to the inputs \( f(t) = 1(t) \) and \( d(t) = 0 \). Next, define

\[
\hat{\mu}_1 \triangleq A[\sigma_1] + B[\sigma_2] - \sigma_1, \quad \hat{\mu}_2 \triangleq A[\sigma_2] + B[\sigma_1] - \sigma_2,
\]

where

\[
A = \sup \{||z|| : G_p \in \mathcal{G}_p, \quad z = g_p - g_p^*, \quad B = \sup \{||z|| : G_p \in \mathcal{G}_p\}. \]

(26)

Let \( \hat{e}^* \) and \( \hat{u}^* \) denote the peak values of \( e^* \) and \( u^* \) and be given by

\[
\hat{e}^* \triangleq \sup_{f \in \mathcal{P}, \ d \in \mathcal{D}} ||e^*||_{\infty}, \quad \hat{u}^* \triangleq \sup_{f \in \mathcal{P}, \ d \in \mathcal{D}} ||u^*||_{\infty}.
\]

(27)

It is now necessary to state the sufficient conditions to ensure the stability of (24).

Theorem 5. Suppose that the nominal system in Fig. 5 is BIBO stable and that \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \) defined in (26) are finite. The criteria (24) for the system in Fig. 4, and hence the criteria (4), are satisfied if \( \hat{\mu}_2 < 1 \) and if

\[
\hat{\phi}_e \leq E_{\text{max}}, \quad \hat{\phi}_e \triangleq \frac{\hat{e}^* + \hat{\mu}_1 N}{1 - \hat{\mu}_2}, \quad \hat{\phi}_u \leq U_{\text{max}}, \quad \hat{\phi}_u \triangleq \frac{K\hat{u}^* + \hat{\mu}_2 N}{K(1 - \hat{\mu}_2)}.
\]

(28)

Proof. From Theorems 3 and 4, the proof follows. □

Note that \( \phi_{ef} \) and \( \phi_{uf} \) can be computed by using available methods (see, e.g., Satoh, 2005; Silpsrikul and Arunsawatwong, 2010, and the references therein). In this work, the approach developed in Silpsrikul and Arunsawatwong (2010) is employed because it is simple and efficient. Therefore, the values \( \hat{e}^* \) and \( \hat{u}^* \) can readily be obtained in practice. Hence, it is evident that the inequalities (29) are more suitable and more computationally tractable than the original design criteria (4).

From the above discussion, it is easy to verify that

\[
\hat{\phi}_e = (N||e^*_d(\delta)||_1 + N\hat{\mu}_1)/(1 - \hat{\mu}_2), \quad \hat{\phi}_u = (NK||u^*_d(\delta)||_1 + K\hat{\mu}_1 + N\hat{\mu}_2)/K(1 - \hat{\mu}_2).
\]

(30)

Evidently, \( \hat{\phi}_e \) and \( \hat{\phi}_u \) depend on the value of the gain \( K \). Thus, to achieve a better design, \( K \) can be allowed to be an additional design parameter. To this end, let \( \hat{p} \triangleq [p^T, K]^T \in \mathbb{R}^{n+1} \) and the design problem is now to determine a design parameter \( \hat{p} \) satisfying

\[
\hat{\phi}_e(\hat{p}) \leq E_{\text{max}}, \quad \hat{\phi}_u(\hat{p}) \leq U_{\text{max}}
\]

(32)

with \( \hat{\mu}_1(\hat{p}) < \infty, \hat{\mu}_2(\hat{p}) < 1 \). Thus, the inequalities (33) are appropriately called the surrogate design criteria.

4. STABILITY CONDITIONS

Clearly, a necessary condition for a design parameter \( \hat{p} \) to satisfy the design criteria (4) is that

\[
\hat{\phi}_p(\hat{p}) < \infty \quad \text{and} \quad \hat{\phi}_u(\hat{p}) < \infty \quad \text{for all} \ G_p(s) \in \mathcal{G}_p.
\]

(33)

Such a point \( \hat{p} \) is said to be a stability point of the original nonlinear system (i.e., stability, in the sense that \( \hat{e} \) and \( \hat{u} \) are finite).

From (26), one can see that if \( A \) and \( B \) are finite, then the stability of the nominal system (in Fig. 5) in the sense that \( \hat{e}^* < \infty \) and \( \hat{u}^* < \infty \) implies that (for \( i = 1, 2 \))

\[
\hat{\phi}_e(\hat{p}) < \infty, \quad \hat{\phi}_u(\hat{p}) < \infty \quad \text{and} \quad \hat{\mu}(\hat{p}) < \infty.
\]

(34)

See Zakian (1983, 1984) for more details. In this regard, to search for a numerical solution of (33) in the space \( \mathbb{R}^{n+1} \) (or \( \mathbb{R}^n \) if \( K \) is chosen to be fixed), the algorithm needs to start from a stability point of the nominal system, i.e., a point \( \hat{p} \) that satisfies (35). (Recall that finding a controller satisfying (33) is in general a non-convex problem.) From Theorems 4 and 5, it follows that when the search algorithm cannot locate a solution of (33), no conclusion can be drawn about (34). That is, only the finiteness of \( \hat{e}^* \) and \( \hat{u}^* \) cannot in general guarantee the stability of the original nonlinear system in Fig. 1.

Note further that the points \( \hat{p} \) satisfying either \( \hat{\phi}_p(\hat{p}) = \infty \) or \( \hat{\phi}_u(\hat{p}) = \infty \) can form a connected region, which obviously contains neither a solution of (4) nor that of (33). Hence, in finding a design solution of (4) by solving the surrogate criteria (33), it is required that the algorithm should start from a point \( \hat{p} \) satisfying both (35) and (34). Furthermore, to facilitate the algorithm to move towards a solution (if one exists), it is useful to perform the search only in the region such that both (35) and (34) are satisfied.

For the nonlinear system considered in the paper, the problem of computing \( \hat{p} \) satisfying (34) can be solved by using the results developed in Mai et al. (2010c) where the nonlinearity is a sector-bounded function.
For each $G_{c}(s, p)$, define the set of composite transfer functions

$G \triangleq \{ G(s) = G_{c}(s, p)G_{p}(s), \forall G_{p}(s) \in G_{p} \}$ . (36)

and let $g$ denote the inverse Laplace transform of $G(s) \in G$.

**Assumption 3.** The functions $g_{p}$, $g_{c}$ and $g$ satisfies conditions that $g_{p}(s)$, $g_{c}(s)$, $g(s)$ are all in $G_{p}$ for all $G_{p}(s) \in G_{p}$ and there exists $\alpha > 0$ such that

$$\int_{0}^{\infty} e^{2\alpha t} g^{2}(t) dt < \infty, \quad \forall G(s) \in G.$$ (37)

The nonlinearity $\psi$ is said to lie in the sector $[k_{1}, k_{2}]$, denoted by $\psi \in [k_{1}, k_{2}]$, if $\psi(0) = 0$ and $k_{1} \leq \psi(\sigma) / \sigma \leq k_{2}$ for all $\sigma \neq 0$.

The boundedness of $e$ and $u$ can be guaranteed by using the following theorem.

**Theorem 6.** Let Assumption 3 be satisfied. The responses $e$ and $u$ are bounded for any $f \in P$ and $g \in G$ for any $\psi \in [k_{1}, k_{2}]$.

Now, it is easy to see that a practical sufficient condition that Theorem 6 is also applicable to the case where the nominal system; hence, (35) is satisfied.

**Proof.** The proof is completed by the direct application of the results in Mai et al. (2010c).

Following Mai et al. (2010c), condition (38) is equivalent to

$$k_{0} < k_{\max},$$ (39)

where $k_{\max}$ is the supremal value of the allowable sector bound. In this work, $k_{\max}$ is computed by using the convex hull of the Popov plots of all $G(s) \in G$ rather than by using all the plots directly, since the obtained convex hull always has a simple shape and can be computed efficiently in practice (see, e.g., Barber et al., 1996). For details on this, see Mai (2010)’s thesis and also Mai et al. (2010c).

In addition, from the above theorem of the Nyquist criterion (Desoer, 1965; Desoer and Wu, 1968) (see also Desoer and Vidyasagar, 1975), it can be shown that the nominal system in Fig. 5 is BIBO stable for any $K \in [0, k_{\max}]$.

Accordingly, the following is readily obtained.

**Corollary 7.** If Assumption 3 and condition (38) are satisfied, then the system in Fig. 5 is BIBO stable for any $K \in [0, k_{\max}]$.

In other words, if $p$ is a stability point of the nonlinear system and if $0 \leq K < k_{\max}$, then $p$ is a stability point of the nominal system; hence, (35) is satisfied.

Furthermore, by making use of appropriate loop transformations, it can be shown (see, e.g., Mai et al., 2010c) that Theorem 6 is also applicable to the case where $G(s)$ contains one pole at the origin.

Now, it is easy to see that a practical sufficient condition for (39), and also (38), is

$$\phi_{0}(p) \leq -\gamma, \quad \phi_{0}(p) \triangleq k_{0} - k_{\max}(p),$$ (40)

where $\gamma$ is a small positive number. The inequality (40) can be used for computing stability points of the original uncertain system. Further, when the number of elements in $G$ is infinite, the Popov plots are to be computed for a sufficiently large number of elements in $G$. In this connection, designers may use the number $\gamma$ as a marginal tolerance for the error caused by this approximation.

**5. NUMERICAL EXAMPLE**

Consider an uncertain heat-conduction process whose transfer function is described by

$$G_{p}(s) = \frac{a}{\sqrt{s} \sinh \sqrt{s} \lambda}, \quad a \in [18, 21], \quad \lambda \in [0.9, 1.1].$$ (41)

It is known (Curtain and Zwart, 1995) that

$$G_{p}(s) = \frac{a}{\lambda s} + 2a \sum_{n=1}^{\infty} \frac{(-1)^{n}}{\lambda s + n^{2} \pi^{2}}.$$ (42)

$$g_{p}(t) = \frac{a}{\lambda} + \frac{2a}{\lambda} \sum_{n=1}^{\infty} (-1)^{n} e^{-n^{2} \pi^{2} t / \lambda}, \quad t > 0.$$ (43)

From (42) and (43), it follows that the theorems stated in Sections 3 and 4 are applicable for any proper $G_{c}(s, p)$.

Now assume that the nonlinearity $\psi$ is described in Fig. 6, where $z_{0} = 0.2$, $k_{1} = 0.2$ and $k_{2} = 1$. Obviously, $\psi \in [k_{1}, k_{2}]$.

![Fig. 6. The input nonlinearity of the heat-conduction process.](image-url)

Assume that the control objective is to keep the error $e$ and the control input $u$ staying within the ranges $\pm E_{\max}$ and $\pm U_{\max}$, respectively, for all time and for all inputs in a possible set $P$ given by (2) where

$$E_{\max} = 6, \quad U_{\max} = 5, \quad M = 50, \quad D = 25.$$ (44)

Accordingly, the design problem is to determine the controller transfer function $G_{c}(s)$ so that the following criteria are fulfilled:

$$\phi_{0}(p) \leq -0.1, \quad \phi_{c}(p) \leq 6, \quad \phi_{u}(p) \leq 5, \quad \mu_{2}(p) < 1.$$ (45)

To this end, assume that the controller transfer function takes the form

$$G_{c}(s, p) = \frac{p_{1}(s^{2} + p_{2}s + p_{3})}{s^{2} + p_{4}s + p_{5}},$$ (46)

where $p = [p_{1}, p_{2}, p_{3}, p_{4}, p_{5}]} \in \mathbb{R}^{5}$. The nominal model $G_{c}(s)$ of the plant is chosen with

$$a = 20, \quad \lambda = 1.$$ (47)

Since $G_{c}(0) \neq 0$ and $G_{c}(0) = \infty$, it follows that $\sigma_{1} = 0$ and $\sigma_{2} = 0$. Thus,

$$\mu_{1} = B \| e^{*}(1) \|_1, \quad \mu_{2} = B \| u^{*}(1) \|_1.$$ (48)

Note that the computation of $B$ involves an extensive numerical search in $\mathbb{R}^{2}$. The search reveals that

$$B = 4.1032 \quad \text{at} \quad a = 21 \quad \text{and} \quad \lambda = 0.9.$$ (49)

In this work, inequalities (45) are solved by using the MBP algorithm (Zakian and Al-Naib, 1973). A design solution

$$\hat{p} = [7.24, 28.79, 200.44, 142.06, 3785.7, 0.97]^{T}$$ (50)

is located and the performance measures are

$$\phi_{0}(\hat{p}) = -2.03, \quad \phi_{c}(\hat{p}) = 5.86, \quad \phi_{u}(\hat{p}) = 4.74.$$ (51)

Also, $\mu_{1}(\hat{p}) = 0.55$ and $\mu_{2}(\hat{p}) = 0.34$. Fig. 7 shows the Popov plots of the systems for various values of $a$ and $\lambda$. 

10974
developed here can also be applied to the case of uncertain systems with an output nonlinearity, as carried out in Mai et al. (2010a).

REFERENCES


6. CONCLUSIONS

In this paper, we have developed a practical method for designing the feedback control system shown in Fig. 1. The control objective is to ensure that $e$ and $u$ stay within the specified ranges $\pm E_{\text{max}}$ and $\pm U_{\text{max}}$, respectively, for all time and for all inputs $f \in P$ in the presence of uncertainties. The results of this paper are based on the design method in Mai et al. (2010b) and from the extension of Zakian’s theory of majorants, which provides a useful tool for dealing with uncertain systems. The effectiveness of the developed method is illustrated by the design example of an uncertain heat-conduction process.

In connection with Zakian’s framework, this work can be seen as an adjunct to the principle of matching (Zakian, 1979, 1996, 2005). It is of interest to note that the method