The transition from persistence to nonsmooth-fold scenarios in relay control system

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Abstract: In many applications in different fields of science and engineering only two extreme values of the control variable can be easily applied, and a threshold on a suitable output variable is used to discriminate among the two control actions. This relay control introduces a discontinuity, so the closed-loop system is discontinuous piecewise smooth (called Filippov system). When a standard equilibrium attains the threshold, as a system or control parameter is varied, two generic scenarios are possible: the standard equilibrium turns into a pseudo-equilibrium on the discontinuity boundary (persistence), so a stationary solution persists through the bifurcation; the collision and disappearance of the standard equilibrium and a coexisting pseudo-equilibrium (nonsmooth-fold). In this paper we analyze the degenerate situation separating these two scenarios, and we apply our results to a four-dimensional SISO system describing the ecological dynamics of a protected natural resource (a resource that cannot be harvested when below threshold). We show that while profitable exploitation is guaranteed (though often at the threshold) in the persistence scenario, the food chain collapses after a nonsmooth-fold.

Keywords: Discontinuous control, Nonlinear systems, Discontinuities, Sliding, Catastrophe, Ecology.

1. INTRODUCTION

Filippov systems are discontinuous, piecewise smooth, autonomous systems of ordinary differential equations [Filippov, 1988]. They have frequently been used in the last decades as models in various fields of science and engineering (e.g., mechanical systems with dry friction, electronic devices with diodes and transistors, on-off and threshold mechanisms in economics, medicine, and ecology, see di Bernardo et al. [2008a] and refs. therein). In particular, any relay-type control system, where two (or more, but in finite number) control actions are taken according to suitable thresholds set on the output variables, can be formalized as a Filippov system [Dercole et al., 2003]. This is particularly interesting in the nonlinear context, where the study of discontinuity-induced bifurcations [di Bernardo et al., 2008a], involving the interaction of the system’s invariant sets (equilibria, limit cycles, etc.) with the discontinuity boundaries, has been intensively carried out in the last decade, and efficient numerical tools are starting to appear [Dercole and Kuznetsov, 2005, Thota and Dankowicz, 2008]. Paradoxically, most of the available results are concerned with cycles grazing (touching tangentially) one of the discontinuity boundaries [di Bernardo et al., 2001, 2002, Kowalczyk and di Bernardo, 2005], while the structural stability of boundary equilibria has been systematically investigated only for two-dimensional Filippov systems [Kuznetsov et al., 2003, Dercole et al., submitted]. Although this situation has been recently remedied [di Bernardo et al., 2008b,c], general results for n-dimensional Filippov systems are still mostly limited to codimension-one cases [Kowalczyk and di Bernardo, 2005, Kowalczyk et al., 2006, Colombo and Dercole, 2010].

Codim 1 discontinuity-induced bifurcations can be classified into two generic scenarios, respectively characterized by the ‘persistence’ and the ‘nonsmooth-fold’ of the bifurcating solution. Specifically for boundary equilibria in Filippov systems, persistence is the case in which the standard equilibrium hits the discontinuity boundary and turns into a pseudo-equilibrium (a point on the boundary where the two adjacent vector fields are anticollinear), so the stationary solution persists through the bifurcation, while the collision and disappearance of the standard equilibrium with a coexisting pseudo-equilibrium occurs in the nonsmooth-fold case.

In this paper we unfold the codim 2 bifurcation (called ‘generalized’ boundary equilibrium) at which the boundary equilibrium changes scenario, and show that this generically occurs together with a fold bifurcation between pseudo-equilibria. Our analysis provides a normal form for the bifurcation and the asymptotic of the fold curve locally to the codim 2 point.

We then demonstrate our results on the relay control of a SISO nonlinear system (see Fig. 1). The system describes the ecological dynamics of a food chain (resource-consumer-predator), where the predator is commercially exploited at adult stage and stocked at juvenile stage (e.g., recreational fishing and hunting). The control strat-
Fig. 1. The structure of the control system.

ergy fixes the total harvesting effort $E$ (including, e.g., quotas, costs of licences, subsidies, limitations on the fishing/hunting gears and seasons), and forbids the exploitation when the population density drops below a prescribed threshold $s$. We show that a stable exploited equilibrium exists for intermediate efforts and attains the threshold for both severe and mild efforts. Surprisingly, while persistence occurs by increasing the effort, the nonsmooth-fold with a pseudo-saddle takes place for sufficiently low effort leaving the resource only harvested by predator juvenile and cuts the food chain: the consumer goes extinct by increasing the effort, the nonsmooth-fold and exploitation inefficiently based on stocking. Let now $\alpha \in \mathbb{R}$. Without loss of generality we assume that when $\alpha = 0$ vector field $f^{(2)}$ has an hyperbolic equilibrium at $x = 0$ lying on the discontinuity boundary $\Sigma$ (boundary equilibrium, BE). Let $A = (f^{(1)})^T$, $b = (f^{(2)})^T$, $c^\top = H_x^0$ (the 0-superscript stands for evaluation at $(x, \alpha) = (0, 0)$) and $q = A^{-1}b$. Assume the genericity of the BE, i.e. that vector field $f^{(2)}$ is locally transversal to $\Sigma$ and the transversality of the bifurcation
\[
c^\top b \neq 0, \quad H_x^0 \neq 0. \quad \text{(G.1,2)}
\]
As shown in di Bernardo et al. [2008b], there are two possible scenarios:

\textbf{Persistence:} at $\alpha = 0$, an equilibrium admissible in region $D_1$ for $\alpha < 0$ becomes a BE and turns virtual in region $D_2$ for $\alpha > 0$. At the same time, a virtual pseudo-equilibrium collides with the BE and becomes admissible (see Fig. 1). There is one admissible (pseudo-)equilibrium on either side of the bifurcation.

\textbf{Nonsmooth-fold:} at $\alpha = 0$, an equilibrium and a pseudo-equilibrium both admissible in region $D_1$ for $\alpha < 0$ collide on a BE and turn virtual in region $D_2$ for $\alpha > 0$. Two/none admissible equilibria are present for $\alpha \leq 0$.

The discriminating condition is:
\[
\text{persistence/nonsmooth-fold if } c^\top q \neq 0. \quad \text{(4)}
\]
Actually, this analysis holds true even for nonhyperbolic BEs, provided that
\[
\text{det}(A) \neq 0 \quad \text{(G.3)}
\]
(e.g., boundary Hopf, see Kuznetsov et al. [2003], Sect. 6, Dercole et al. [2003], and di Bernardo et al. [2008c]).

3. GENERALIZED BOUNDARY EQUILIBRIA

We now consider the codim 2 generalized boundary equilibrium (GBE) bifurcation, where
\[
c^\top q = 0. \quad \text{(5)}
\]
Here, $\alpha \in \mathbb{R}^2$ and, locally to $\alpha = 0$, $H_0(0, \alpha) = 0$ defines the codim 1 BE curve in the $\alpha$-parameter plane. We keep assumptions (G.1–3) (where $H_0^0$ is now the two-dimensional gradient to the BE curve at $\alpha = 0$).

3.1 Transversality

Let us first impose that $\alpha = 0$ separates a persistence BE branch from a nonsmooth-fold one. At a point $\alpha$ on the BE curve, persistence/nonsmooth-fold corresponds to
\[
H_x(0, \alpha)(f^{(1)}_x(0, \alpha))^{-1}f^{(2)}(0, \alpha) \leq 0 \quad \text{(6)}
\]
(see (4) and recall that \( x = 0 \) at the BE), so we need to impose that (6) changes sign (with ‘nonzero velocity’) at \( \alpha = 0 \) along the BE curve, i.e.,
\[
\begin{bmatrix}
H_{010}^1 Q - c^T A^{-1}(f_1^{(0)})^1 q + c^T A^{-1}(f_2^{(0)})^0, \\
H_{001}^1 Q - c^T A^{-1}(f_1^{(0)})^0 q + c^T A^{-1}(f_2^{(0)})^0
\end{bmatrix}
\]
\[
(H_{001}^1)^0 \neq 0,
\]
where \( H_{001}^1 = [H_{001}, -H_{011}]^T \) is tangent to the BE curve (\( \pi/2 \)-clockwise-rotated w.r.t. \( H_{011} \)).

3.2 GBE normal form
Consider the auxiliary ODE system
\[
\dot{x} = (1 - \lambda)f^{(1)}(x, \alpha) + \lambda f^{(2)}(x, \alpha),
\]
\[
\dot{\lambda} = H(x, \alpha).
\]
whose equilibria are pseudo-equilibria of system (1) (admissible if \( 0 < \lambda < 1 \), virtual if \( \lambda < 0 \) or \( \lambda > 1 \)).

Along the BE curve, \((x, \lambda) = (0, 0)\) is an equilibrium of (7) with Jacobian
\[
J(\alpha) = \begin{bmatrix}
f_1^{(0)}(0, \alpha) & f_2^{(0)}(0, \alpha) \\
0 & H_2(0, \alpha)
\end{bmatrix}.
\]
Since
\[
\frac{\det(J(\alpha))}{\det(f_2^{(0)}(0, \alpha))} = H_2(0, \alpha)(f_1^{(0)}(0, \alpha))^{-1} f_2^{(0)}(0, \alpha)
\]
and \(\text{sign}(\det(J_2^{(0)}(0, \alpha))) = \text{sign}(\det(A))\) for small \(\|\alpha\|\), (G.3) and (G.4) imply that \((0, 0)\) is hyperbolic for \(\alpha \neq 0\).

By contrast, (G.3) and (5) (and (4)) suggest that \(J^0\) has a simple zero eigenvalue with associated right and left eigenvectors \([q^*, l^*]^T\) and \([p^*, 1]^T\), where \(p^* = -c^T A^{-1}\) and
\[
p^* q + 1 = c^T A^{-2} b + 1 \neq 0.
\]
Thus, moving along the BE curve, \((0, 0)\) generically undergoes a transcritical bifurcation [Kuznetsov, 2004] at \(\alpha = 0\).

We now introduce a new pair of parameter 
\(\beta \in \mathbb{R}^2, \beta = (\beta_1, \beta_2)\), \(\beta_0 = 0\), and a new coordinate \(u \in \mathbb{R}\) along the one-dimensional parameterized center manifold associated with the equilibrium \((0, 0)\) close to \(\alpha = 0\), i.e.,
\[
x = h(u, \beta), \lambda = l(u, \beta),
\]
with \(h^0 = 0\) and \(l^0 = 0\) (the 0-superscript also stands for evaluation at \((u, \beta) = (0, 0)\)).

Let
\[
\beta_1 = -H(0, \alpha),
\]
so that \(\beta_1\) plays the role of the \(\alpha\) parameter in the codim 1 analysis of Section 2 and the BE bifurcation has equation \(\beta_1 = 0\). For the moment, we keep \(\beta_2\) as an unspecified function of \(\alpha\) such that \(\beta_2^0 = 0\) and the \(2 \times 2\) Jacobian \(\beta_2\) is nonsingular (this will be imposed later).

We apply the standard (parameter-dependent) homological approach [Beyn et al., 2002, Sect. 11]. It allows to compute (up to any finite order) the free component \(\beta_2 \in \beta_2(\alpha)\) of the parameter change, the center manifold (10) and the restriction of system (7) on that manifold. What must be a priori assumed is the structure of this restriction, i.e. which coefficient we expect to be generically nonzero. In our case the structure is
\[
\dot{u} = g(u, \beta) = g_{010} \beta_1 (1 + O(\|\beta\|))
\]
\[
+ u (g_{011} \beta_1 + \frac{1}{2} g_{200} u + O(\|u, \beta\|^2)),
\]
the most generic structure that when \(\beta_1 = 0\), i.e., on the BE curve, is nothing but the transcritical normal form (with parameter \(\beta_2\)) plus higher-order terms. Coefficients \(g_{010}, g_{011},\) and \(g_{200}\) are generically nonzero. When there is freedom on both components of the parameter change, they can be normalized at \(\pm 1\). Here, however, only \(\beta_2\) is free, so that the \(g\)-coefficients are treated as unknowns.

The homological equation is composed of
\[
h_u(u, \beta) g(u, \beta) = \frac{1}{2} - l(u, \beta) f_1^{(1)}(h(u, \beta), \alpha(\beta))
\]
\[
+ l(u, \beta) f_2^{(2)}(h(u, \beta), \alpha(\beta)),
\]
\[
l_u(u, \beta) g(u, \beta) = H(h(u, \beta), \alpha(\beta)),
\]
that simply impose the invariance of the center manifold under (7). We substitute into (13) the expansions
\[
f_1^{(1)}(x, \alpha) = Ax + \frac{1}{2} B(x, \alpha),
\]
\[
f_2^{(2)}(x, \alpha) = b + A^2 x + A^2 \beta + \cdots,
\]
\[
H(x, \alpha) = H^0 x + H_0^0 \alpha + \frac{1}{2} H^0_{xx} x^2
\]
\[
+ H_0^0 \beta + \cdots,
\]
\[
l(u, \beta) = l_{100} u + l_{101} \beta + \frac{1}{2} l_{200} u^2 + l_{111} u \beta + \cdots,
\]
\[
\alpha(\beta) = a_{101} \beta_1 + a_{012} \beta_2 + \cdots,
\]
where \(h_{10}, b_{20}\) (n-dimensional columns), \(l_{10}, b_2,\) and \(h_{01}, h_{00}\) in (\(R\)), \(b_{11}, h_{10}, h_{11}\) in \((n \times 2)\) matrices, \(h_{11}, b_{00}\) in \((2 \times 2)\) matrices, and \(a_{10}, a_{01}\) (2-dimensional column) are further unknowns to be determined, while \(A^0, A^1, B, B_1, H^0, H^0_{xx}, H^0_{\beta_2}\) are known matrices and multilinear operators (recall that \(A_1 = f_0^0 = 0\)).

Note that due to choice (11), the Jacobian
\[
a_0 = [a_{10}, a_{01}] = (\beta_0^0)^{-1}\begin{bmatrix}
\beta_0^0_{202} & H_0^0_{101} - H_0^0_{201} - H_0^0_{111} - H_0^0_{021}
\end{bmatrix}
\]
depends nonlinearly on the unknowns \(\beta_{2,1}^0\), \(i = 1, 2, 3\). Since parameter scaling is arbitrary and the parameter change must be invertible, we impose
\[
det(\beta_{0}^{0}) = H_{200} H_{011} - H_{001} H_{101} = 1,
\]
and therefore fix
\[
a_{10} = [\beta_{202}^0, -\beta_{201}^0]^T, \text{ and } a_{01} = (H_0^0)^0.
\]
By balancing the powers \(u, \beta, u^2, u\beta\) in (13), we find the results in Table 1, where \([h_{10}, l_0]^T\) is in the nullspace of \(J_0^0\) (see (18a)), while all other vectors \([h_{11}, l_1]^T\) in the center manifold expansion (14d) \(e\) are taken in the range of \(J_0^0\) by imposing
\[
p^T h_{1i} + l_{1i} = 0, \quad i = 010, 001, 20, 110, 101.
\]
This can be seen in the fourth equation of (18b) for \(i = 010\), as a result in (18c), and in the fact that \(h_{10}^T l_0^T = (J_0^0)^{\text{INV}} (h_{10}^T, l_0^T)^T\) represents the unique solution of \(J_0^0 h_{1i}^T l_{1i} = (h_{10}^T, l_{10}^T)^T\) (solvable since \(p^T h_{1i} + l_{1i} = 0\) in (18d, f, h)) in the range of \(J_0^0\), that can be computed by solving the nonsingular bordered system
\[
\begin{bmatrix}
A & b & q & c^T & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
h \\
l \\
0
\end{bmatrix}
\begin{bmatrix}
h' \\
l' \\
0
\end{bmatrix}
\]
\[ h_{10} = q, \ l_{10} = 1 \] (18a)

\[
\begin{bmatrix}
A b 0 & -q \\
\cdot c^\top & 0 & H_0^a \cdot -1 \\
0 & 1 & 0 & \cdot H_0^a \\
0 & 0 & 0 & \cdot 1
\end{bmatrix}
\]

\[ h_{001} = 0, \ l_{001} = 0 \] (18c)

\[ g_{200} = \frac{1}{p \cdot q + 1} \left( p^\top (B(q, q) + 2(A_2 - A)q) + H^a \cdot (q, q) \right) \cdot \left( h_{20} \right)^{\text{INV}} - B(q, q) + 2(A_2 - A)q \\
\]

\[ \begin{bmatrix}
h_{10} \\
l_{10}
\end{bmatrix} = \left( \begin{bmatrix} h_{001} \end{bmatrix} \right)^{\text{INV}} \left( g_{200} \right) \left( \begin{bmatrix} q \end{bmatrix} \right) - \left( B(q, q) + 2(A_2 - A)q \right) \]

\[ \begin{bmatrix}
h_{101} \\
l_{101}
\end{bmatrix} = \left( \begin{bmatrix} h_{010} \end{bmatrix} \right)^{\text{INV}} \left( g_{101} \right) \left( \begin{bmatrix} q \end{bmatrix} \right) - \left( B(q, q) + 2(A_2 - A)q \right) \]

At a generic GBE, the \((n+4) \times (n+4)\) matrix \(M\) in (18b) is shown to be nonsingular, as well as \(g_{010} \neq 0\) and \(g_{101} \neq 0\) (details and proofs will be reported in Della Rossa and Dercole [submitted]). By adding more terms in the expansions in (12) and (14), one can obtain an approximation of the center manifold (10) and of the parameter change \(\alpha(\beta)\) (and of its inverse \(\beta(\alpha)\)) up to any finite order. In particular, function \(\beta_2(\alpha)\) can be further specified, though here we have only computed its linear approximation (see the first of (16)). This is enough to impose (15) and to check for the genericity of the transcritical bifurcation of system (7), namely

\[ g_{200} \neq 0. \] (G.5)

### 3.3 GBE unfolding

We now perform the bifurcation analysis of the normal form (12) in the \(\beta\)-parameter plane close to \(\beta = 0\). When \(\beta_1 = 0\), the equilibrium \(u = 0\) of the normal form (12) corresponds to the BE \(x = 0\) (with \(\lambda = 0\)) of system (1). Since \(\lambda\) increases in the direction of \(q\), crossing/sliding occurs for negative/positive values of \(u\). By contrast, when \(\beta_1 \neq 0\), \(u = 0\) is not equilibrium of (12) and neither it corresponds to the \(f^{(1)}\)-tangent point separating crossing from sliding along the \(x\)-center-manifold.

When \(\beta_1 = 0\), the normal form (12) also has the equilibrium\(\bar{u}(\beta_2) = -2g_{101}/g_{200}\) \(\beta_2 + O(\beta_2^2), \) (19) while two equilibria collide at \(-g_{101}/g_{200} \beta_2\) along a fold bifurcation curve of equation

\[ 2g_{101}/g_{200} \beta_2 + O(\beta_2^2) = \frac{1}{2} \beta_2^2 + O(\beta_2^3). \] (20)

Depending on the signs of \(g_{101}/g_{200}\) and \(g_{010}/g_{200}\) in (19) and (20), there are four cases; the two with \(g_{101}/g_{200} < 0\) \((\bar{u} > 0\) for \(\beta_2 > 0\)) being depicted in Fig. 3. When \(g_{010}/g_{200} < 0\) \((\text{top/bottom panel}), the parabola-like curve \(20\) is on the right/left \((\beta_1 \geq 0\). The other two cases are symmetric with respect to the \(\beta_2\)-axis.

Note that persistence occurs for \(\beta_2 > 0\) when \(\bar{u}\) in (19) is positive/negative (for \(\beta_2 > 0\)) and the parabola is on the right/left. Vice versa, \(\beta_2 > 0\) is the nonsmooth-fold BE branch. Hence, persistence/nonsmooth-fold occurs for \(\beta_2 > 0\) if \(g_{010}/g_{101} \leq 0\).

Of course, only the admissible pseudo-equilibria of system (1) (i.e., those on the sliding part of the \(x\)-center-manifold) are of interest, so that the branch of the fold curve corresponding to virtual pseudo-equilibria (dashed in Fig. 3) is not a bifurcation of system (1), while the admissible branch is the fold bifurcation between pseudo-equilibria (PLP) that generically emanates from the GBE point.
The asymptotic of the PLP curve in the original parameters \((a_1, a_2)\) can be obtained by substituting the expansion

\[ \beta(a) = b_10a_1 + b_{01}a_2 + \cdots \]

into (20). As we limited our computations to the linear coefficients \(a_10\) and \(a_{01}\) in (14f), we obtain

\[ b_{10} = \begin{bmatrix} -H_{a_11}^0 \\ -H_{a_21}^0 \\ \end{bmatrix}, \quad b_{01} = \begin{bmatrix} -H_{a_11}^0 \\ -H_{a_21}^0 \\ \end{bmatrix}, \]

where \(a_{10} = [a_{101}, a_{102}]^T\) is given by (18b), and simply provide the linear asymptotic

\[ a_1 = H_{a_11}^0 \epsilon, \quad a_2 = -H_{a_21}^0 \epsilon \tag{21} \]

for small \(\epsilon\), i.e., the tangent direction to the BE curve.

Finally, we need to prove that no other bifurcations of system (1) are rooted at \(\beta = 0\). Indeed, the normal form (12) only takes into account bifurcations of equilibria and pseudo-equilibria, but the global dynamics of (1) near \(x = 0\) is not considered. Obviously, bifurcations rooted at a generic GBE cannot involve a standard cycle (the only possibility left open by (G.3) is that of a shrinking cycle undergoing a Hopf bifurcation at the GBE, but this possibility is ruled out if we assume the BE \(x = 0\) to be hyperbolic). Local bifurcations of a sliding cycle that shrinks around \(x = 0\) while approaching the BE bifurcation are also ruled out if we assume (say genericity condition (G.6)) that the sliding cycle, whether it exists, remains hyperbolic while approaching the GBE. What remains to be excluded is a global bifurcation involving the codim1 connection between a tangent point of vector field \(f^{(1)}\) with a pseudo-equilibrium. This can be done analyzing the boundary value problem describing the connection (details reported in Della Rossa and Dercole [submitted]).

4. AN ECOLOGICAL CASE STUDY

We now apply our results to a relay control system (see Fig. 1). The nonlinear system under control is a 4-dimensional demographic model describing the ecological dynamics of a thritrophic food chain composed of resource (biomass density \(x\)), consumer \((y)\), and predator \((z)\).Adult predator individuals \((z_A)\) are commercially exploited (e.g., recreational fishing and hunting) and the ecosystem is constantly refurnished with predator juveniles \((z_J,\) etc., at larvae or other immature stages).

The traditional approaches to the control of exploited natural resources modulate the harvesting effort \((E)\) by solving optimal control problems that take into account profit, cost of effort, and extinction risks [Clark, 1990]. However, the optimal control strategy can hardly be implemented in practice for organizational and institutional difficulties, and the management of natural resources is much more often performed by fixing only a few control parameters (e.g., quotas, costs of licenses, subsidies, limitations on the fishing/hunting gears and seasons). Often, to avoid high risks of extinction of the exploited population, harvesting is forbidden when the population density drops below a prescribed threshold \((s)\). Such a rule is supported by conservation ecologists, who associate an infinite cost to the loss of a population, and is economically justified for sufficiently low discount factors.

The system's equations are

\[
\begin{align*}
\dot{x} &= r x \left(1 - \frac{x}{K}\right) - \frac{a x}{b + x} y - \frac{a_J x}{b_J + x} z_J, \\
\dot{y} &= \frac{a x}{b + x} y - \frac{a_A y}{b_A + y} z_A - dy, \\
\dot{z}_J &= c_J \frac{a_J x}{b_J + x} z_J + f \frac{a_A y}{b_A + y} z_A - (r + d_J) z_J + m, \\
\dot{z}_A &= c_A (1 - f) \frac{a_A y}{b_A + y} z_A + \tau z_J - (A + E) z_A
\end{align*}
\]

in \(D_1\), where \(z_A > s\), while the extra mortality term \(-E z_A\) is added to (22d) in \(D_2\).

The parameters in (22) are all positive and have the following interpretation: \(r\), resource intrinsic growth rate; \(K\), resource carrying capacity; \(a_1\), maximum consumption rates; \(b_1\), half saturation constants—the food density at which half of the maximum consumption rate is realized; \(d_J\) / \(c_J\), mortalities/efficiencies; \(m\) stocking rate; \(f\), reproduction/growth energy allocation fraction; \(\tau\), mean maturation time; \(E\) and \(s\), exploitation effort and protection threshold (see, e.g., Thieme [2003] for detailed descriptions of this and other ecological models).

We performed a bifurcation analysis with respect to the control parameters \((s, E)\) (see Fig. 4). All other parameters are fixed at values (specified in the figure) for which system (22) admits a trivial stable pseudo-equilibrium at which the consumer species is extinct. This cuts the food chain and starves adult predators; their density is at the threshold \(s\) though totally based on stocking. The interesting question is whether or not the relay control is able to prevent this ecologically and economically unfavorable situation. Our analysis shows that this requires exploitation to be allowed at sufficiently low densities \((s\) small) with sufficiently high effort \((E)\) (in order to control the density of predator juveniles). In fact, for sufficiently small \(s\) and intermediate \(E\) (region 1 in the figure), there is a stable exploited equilibrium, i.e., above the discontinuity boundary (the linear manifold \(z_A = s\)). This profitable equilibrium is however separated by a pseudo-saddle from the trivial pseudo-equilibrium. For both higher and lower efforts, the stable exploited equilibrium hits the boundary and becomes virtual, since the superpredator is either too exploited or uncontrolled and therefore too aggressive on the prey (the resource at juvenile stage; the consumer at adult stage). The two BE bifurcations are, however, of different nature: in the first case, exploitation is limited after the bifurcation at a stable nontrivial pseudo-equilibrium (transition 1–2, persistence scenario), while the trivial pseudo-equilibrium becomes the global attractor in the second case (transition 1–4, nonsmooth-fold scenario).

The two BE branches \((\text{BE}_1\text{ and BE}_2)\) meet at a generic GBE (codim 2) point (conditions (G.1,2)–(G.6) are satisfied) and the unfolding, locally to the GBE point, corresponds to the top panel of Fig. 3. The normal form coefficients and the linear asymptotic (21) of the PLP curve are \(g_{010} = 0.2057\), \(g_{101} = -0.5005\), \(g_{200} = 0.2551\), \(s = 0.4206\), \(E = 0.51 + \epsilon\).

REFERENCES

Fig. 4. Bifurcation analysis of system (22), locally to the GBE point (bifurcation diagram is obtained with Matcont [Dhooge et al., 2002]; phase portraits obtained with the integrator described in Piironen and Kuznetsov [2008]).


