Rendezvous of Unicycles with Continuous and Time-invariant Local Feedback

Ronghao Zheng ∗ Zhiyun Lin ∗∗ Ming Cao ∗∗∗

* College of Electrical Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027 P. R. China (e-mail: zhronghao@gmail.com)
** College of Electrical Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027 P. R. China (e-mail: linz@zju.edu.cn)
*** Faculty of Mathematics and Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG, Groningen, The Netherlands (e-mail: m.cao@rug.nl)

Abstract: In this paper, a local feedback control strategy is devised to drive a group of nonholonomic unicycle-type vehicles to a rendezvous point. The proposed control law is continuous and time-invariant. Using tools from graph theory, convergence and ultimate boundedness properties of the overall system are examined. Similar to the results for the rendezvous problem for vehicles governed by linear dynamics, the convergence of the studied overall multi-agent system relies on the connectivity of the interaction graph. The validity of this control law is verified by computer simulations.

Keywords: Cooperative control, Multivehicle systems, Rendezvous, Unicycles

1. INTRODUCTION

In recent years, the problem of coordinating a group of autonomous wheeled vehicles has attracted more and more attention in the fields of both control and robotics. This growing research interest has led to the development of a number of distributed control algorithms capable of causing large groups of mobile autonomous agents to perform group tasks, such as collaborative exploration and environment monitoring (Kumar et al., 2005; Leonard et al., 2007). When coordinating mobile agents in motion, various control strategies have been proposed for mobile agents with different dynamics, especially the challenging case when the agents are subject to nonholonomic constraints. For example, Yamaguchi and Burdick (1998) proposed a time-varying control law for a group of Hilare-type mobile robots to form team formations.

It is of particular interest to study how to gather a group of nonholonomic mobile agents together, and this is usually referred to as the rendezvous problem. When discussing the feasibility problem for multi-unicycle formations, Lin et al. used a time-varying, periodic and smooth strategy to achieve rendezvous (Lin et al., 2005). It has been shown that the control of the angular velocity of a unicycle-type vehicle does not necessarily depend on the information about other unicycles. Dimarogonas and Kyriakopoulos (2007) presented a discontinuous and time-invariant control law for the multi-unicycle rendezvous problem. Under their control law, a group of unicycles come to a common point and converge to the same orientation eventually.

However, global information about each robot’s orientation is needed in their control law, and because of this the control law is not fully distributed.

It is worth pointing out that the rendezvous problem has been studied extensively when neglecting the nonholonomic constraint and focusing on the linear agent dynamics. Some of the early works include Yamaguchi and Burdick (1998) and Ando et al. (1999). Lin et al. (2007a,b) discussed the “stop-and-go” strategies in both synchronous and asynchronous cases. Lin et al. (2007c) proved rendezvous in continuous time. Cortés et al. (2006) used a generalized invariance principle and proximity graphs to design and analyze control laws for gathering mobile autonomous agents.

In this paper, we present a local feedback control strategy that drives a group of multiple kinematic unicycles to rendezvous. The proposed feedback law is continuous and time invariant, which is interesting considering the fact that nonholonomic systems do not satisfy Brockett’s necessary smooth feedback stabilization condition (Brockett, 1983). Note that there are existing works comparing the performances of time-invariant and time-varying controllers for mobile agents. An experimental comparison between these two types of controllers has been reported in Kim and Tsiotras (2002) which shows the better performances of time-invariant strategies. Some drawbacks of time-varying controllers in practical settings include slow convergence and oscillatory behaviors. The controller proposed in this paper is simple in its structure and thus easy to be implemented. It is distributed and only relative measurements of the neighboring agents are needed. Consequently, we do not require all-to-all communication and common reference coordinate frames. For the fixed topology case, convergence to a rendezvous point under
our proposed control law is analyzed, while for the dynamic topology case we prove the ultimate boundedness of the agents around a stationary point that depends on their initial positions.

The rest of the paper is organized as follows. In section 2, we present the equations of the overall multi-unicycle system and the studied local feedback control law. In section 3 we carry out analysis for both the fixed topology case and the dynamic topology case. Simulation results are demonstrated in section 4 and in the end we make concluding remarks in section 5.

2. MODEL OF MOBILE ROBOTS AND PROJECTION BASED CONTROL

In this section, we introduce the mobile robot model to be studied and then propose a simple distributed control law for the rendezvous problem.

2.1 Model of mobile robots

The unicycle model is considered in the paper, which is common for mobile robots. Suppose there are $n$ mobile robots in the plane. The kinematic model of robot $i$ is thus described by the following equations

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i \\
\dot{\theta}_i \\
\end{bmatrix} = \begin{bmatrix}
\cos \theta_i & 0 & -y_i \\
\sin \theta_i & 0 & x_i \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
v_i \\
\omega_i \\
\end{bmatrix} = G(\theta_i)u_i, \quad (1)
\]

where $(x_i, y_i)$ denotes the position of the representing point of robot $i$ in an inertia frame $\Sigma_g$, $\theta_i$ is its orientation with respect to the $x$-axis of $\Sigma_g$, and $u_i = [v_i, \omega_i]^T \in \mathbb{R}^2$ are the control input of linear velocity $v_i$ and angular velocity $\omega_i$ (see Fig. 1).

![Mathematical model of a mobile robot](image)

Fig. 1. Mathematical model of a mobile robot.

For simplicity of notation, we write the model of $n$ mobile robots in a compact form. Let $x = [x_1 \ldots x_n]^T$, $y = [y_1 \ldots y_n]^T$, $\theta = [\theta_1 \ldots \theta_n]^T$, $v = [v_1 \ldots v_n]^T$, and $\omega = [\omega_1 \ldots \omega_n]^T$, respectively. Then the following equations describe the kinematic model for the overall system of the $n$ mobile robots in the plane,

\[
\begin{cases}
\dot{x} = \text{diag}(\cos \theta)v \\
\dot{y} = \text{diag}(\sin \theta)v \\
\dot{\theta} = \omega
\end{cases} \quad (2)
\]

where $\cos \theta$ and $\sin \theta$ are the notations for the $n$-dimensional vectors $[\cos \theta_1 \ldots \cos \theta_n]^T$ and $[\sin \theta_1 \ldots \sin \theta_n]^T$, respectively, and $\text{diag}(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ returns the diagonal matrix with its diagonal elements being the corresponding elements of its argument.

2.2 Projection based control

Denote $z_i = [x_i y_i]^T$, the position vector of robot $i$. Let $r_i = [\cos \theta_i \sin \theta_i]^T$ be the normalized velocity vector and let $s_i = [-\sin \theta_i \cos \theta_i]^T$ be the normalized lateral vector that is obtained by rotating $r_i$ counterclockwise by $\pi/2$. Suppose a local frame is attached to each robot, denoted by $\Sigma_i$ (Fig. 1), whose $x$-axis is the direction of $r_i$ and whose $y$-axis is the direction of $s_i$.

We consider the following simple control law for each robot,

\[
\begin{aligned}
v_i &= \sum_{j \in N_i} r_i^T (z_j - z_i) \\
\omega_i &= \sum_{j \in N_i} s_i^T (z_j - z_i)
\end{aligned} \quad (3)
\]

or equivalently,

\[
\begin{aligned}
v_i &= \sum_{j \in N_i} [\cos \theta_i(x_j - x_i) + \sin \theta_i(y_j - y_i)] \\
\omega_i &= \sum_{j \in N_i} [-\sin \theta_i(x_j - x_i) + \cos \theta_i(y_j - y_i)]
\end{aligned} \quad (4)
\]

where $N_i$ is the set of robot $i$’s neighbors, which are the robots that robot $i$ is able to sense and interacts with. In other words, the linear velocity of robot $i$ is proportional to the sum of $x$-components of its neighbors’ positions and the angular velocity is proportional to the sum of $y$-components of its neighbors’ positions in its local frame $\Sigma_i$.

Such relative position information can be locally measured by an onboard sensor. An illustration is given in Fig. 2 where only one neighbor is considered.

![Illustration for control law](image)

Fig. 2. An illustration for the control law (3).

The control law was originally proposed in Zheng et al. (2009) to study collective behaviors resulted from cyclic pursuit, where it shows that this control law does not lead to rendezvous but ordered circular motions for unidirectional cyclic topologies. However, as we will see in the paper, the interaction topology plays a very important role in the emergent collective behaviors. For bidirectional interaction topologies, we will show that under the same control law, a group of $n$ robots rendezvous, that is, the position states $z_1, z_2, \ldots, z_n$ converge to a common point. To describe a bidirectional interaction topology, we consider an (undirected) graph $G = (V, E)$ where $V$ is the node set with each node corresponding to each robot and $E$ is the edge set such that $(i, j) \in E$ implies $j \in N_i$ and also $i \in N_j$. In the paper, we consider both fixed and dynamic cases. For the dynamic case that the connectivity of the graph changes over time, we will clearly add the explicit dependence on $t$ in the notations, e.g., $G(t), N_i(t)$, etc. Next we introduce some notions from graph theory, which
will be used in the paper. A graph $G$ is said to be connected if for any two nodes $i$ and $j$, there is a path connecting them. A dynamic graph $G(t)$ is said to be uniformly jointly connected if there exists $T > 0$ such that for all $t$, the union graph $G(t, t + T) := (V, \bigcup_{\tau \in [t, t+T]} E(\tau))$ is connected.

For a graph $G$, we use $L$ to denote the Laplacian of the graph, for which the $i,j$th off-diagonal entry is $-1$ if $(i,j) \in E$ and $0$ otherwise, and the $i$th diagonal entry is the negative sum of the off-diagonal entries in the $i$th row. Correspondingly, the notation $L(t)$ is used to represent the Laplacian of the dynamic graph $G(t)$. Notice that for a graph with a finite number of nodes, the patterns of connectivity of the graph are finite, too. Thus, the graph $G$ is connected.

In this section, we present rigorous analysis for the rendezvous of a group of $n$ mobile robots under the control law (3).

3. CONVERGENCE AND BOUNDEDNESS ANALYSIS

In this section, we present rigorous analysis for the rendezvous behavior of a group of $n$ mobile robots under the control law (3).

3.1 Fixed topology

First, we consider the case of fixed topology, that is, the graph $G$ is time-invariant. For the distributed control law (3), we can re-write it in a compact form as follows:

$$
\begin{align*}
\dot{y}_i &= -\delta i \theta (y_i - y_j), \\
\dot{\theta}_i &= -\delta_i \theta (\theta_i - \theta_j).
\end{align*}
$$

(5)

Next we present the rendezvous result for the case of fixed topology.

Theorem 1. Suppose the graph $G$ is connected. Then a group of $n$ mobile robots rendezvous under the control law (3).

Proof: Consider a continuously differentiable function

$$
V(x, y) = \frac{1}{2} x^T L x + \frac{1}{2} y^T L y.
$$

It is well known that for an undirected graph, the Laplacian $L$ is symmetric and positive semi-definite (Godsil and Royle, 2001). Moreover, since the graph is connected, which is assumed in the theorem, it is also known that 0 is a simple eigenvalue of $L$ and its associated eigenvector is $\mathbf{1}$, a vector of all 1 components with suitable dimension (Godsil and Royle, 2001). Thus, the function $V(x, y)$ is positive definite with respect to the set $\{(x, y) | x_1 = \cdots = x_n, y_1 = \cdots = y_n\}$.

Consider the closed-loop system under the control law (5) and take the derivative of $V(x, y)$ along its solution. Then it is obtained that

$$
\dot{V}(x, y) = (x^T L x + y^T L y)
= (x^T L \delta i \theta (y_i - y_j) + y^T L \delta_i \theta (\theta_i - \theta_j))
= (x^T L \delta i \theta (y_i - y_j) + y^T L \delta_i \theta (\theta_i - \theta_j))
= -v^T v = -||v||^2 \leq 0.
$$

To find $S = \{(x, y, \theta) | \dot{V} = 0\}$, since $\dot{V} = 0$ implies $v = 0$, we can infer that $x = 0$ and $y = 0$. So $Lx$ and $Ly$ remain constant. Next we show that $Lx$ and $Ly$ must be 0. To see this, suppose by contradiction that $Lx \neq 0$ or $Ly \neq 0$. From the formula (5) and the fact $v = 0$ it follows that $[Lx, Ly]$ is in the kernel of $[\delta i \theta, \delta_i \theta]$. On the other hand, due to linear independence of the row vectors of $[\delta i \theta, \delta_i \theta]$ and $[\delta (-\sin \theta), \delta \cos \theta]$, it can then be concluded that $[Lx, Ly]$ does not lie in the kernel of $[\delta (-\sin \theta), \delta \cos \theta]$, which in turn implies $\omega \neq 0$ and $\theta$ does not remain constant. Therefore, the nonzero constant vector $[Lx, Ly]$ can not be in the kernel of $[\delta i \theta, \delta_i \theta]$ anymore and $v$ becomes nonzero, a contradiction. Hence, the only solution that can stay identically in $S$ is the solution satisfying $Lx = 0$ and $Ly = 0$, which means $x_1 = \cdots = x_n$ and $y_1 = \cdots = y_n$. Thus, by LaSalle’s invariance principle (Theorem 4.1, Khalil (1996)), all the trajectories converge to the set satisfying $x_1 = \cdots = x_n$ and $y_1 = \cdots = y_n$. That is, all the mobile robots rendezvous.

Finally, we generalize the control law (3) and show that rendezvous still occurs whenever the control takes correct signs of feedback information. It means that imperfect sensing does not affect the collective behavior to a certain extent.

Consider the following alternative

$$
\begin{align*}
\dot{v} &= -f(\delta i \theta (Lx + \delta \cos \theta) \delta \sin \theta (Ly) \\
\dot{\theta} &= -h(\delta (-\sin \theta) Lx + \delta \cos \theta (Ly))
\end{align*}
$$

(6)

where for any vector $\zeta \in \mathbb{R}^n$,

$$
\bar{f}(\zeta) = [f_1(\zeta_1), \ldots, f_n(\zeta_n)]^T
$$

and

$$
\bar{h}(\zeta) = [h_1(\zeta_1), \ldots, h_n(\zeta_n)]^T
$$

satisfying

- $\zeta_i f_i(\zeta_i) \geq 0$,
- $f_i(\zeta_i) = 0$ if and only if $\zeta_i = 0$,
- $\zeta_i h_i(\zeta_i) \geq 0$,
- $h_i(\zeta_i) = 0$ if and only if $\zeta_i = 0$.

The functions $\bar{f}()$ and $\bar{h}()$ represent any general function satisfying the properties above. One typical example is the sign function. Now comes the second rendezvous result.

Theorem 2. Suppose the graph $G$ is connected. Then a group of $n$ mobile robots rendezvous under the control law (6).

Proof: Again, we consider the continuously differentiable function

$$
\dot{V}(x, y) = \frac{1}{2} x^T L x + \frac{1}{2} y^T L y.
$$

Consider the closed-loop system under the control law (6) and take the derivative of $V(x, y)$ along its solution. Then it is obtained that
\[
\begin{align*}
\dot{V}(x, y) &= (x^T L \dot{x} + y^T L \dot{y}) \\
&= (x^T L \text{diag}(\cos \theta)v + y^T L \text{diag}(\sin \theta)v) \\
&= (x^T L \text{diag}(\cos \theta) + y^T L \text{diag}(\sin \theta))v.
\end{align*}
\]

From the properties of \(f\) and \(h\), we thus obtain

\[
\dot{V}(x, y) \leq 0
\]

and \(\dot{V}(x, y) = 0\) if and only if \(v = 0\). The rest of the proof is identical to the proof for Theorem 1 using LaSalle’s invariance principle.

### 3.2 Dynamic topology

Second, we consider the case of dynamic topology, that is, the graph is time-varying and is denoted by \(\mathcal{G}(t)\). For this case, it is hard to prove the convergence to a common point rigorously though sufficient simulations showed this is true. As the first step, we show in this section that for the dynamic topology case, a group of \(n\) mobile robots are ultimately bounded under the control law (3) and moreover a point related to the initial positions of the group of mobile robots is invariant under the control law.

For the dynamic case, we clearly re-write the control law (3) explicitly depending on \(t\), that is,

\[
\begin{align}
v_i &= \sum_{j \in \mathcal{N}(t)} r^T_j (z_j - z_i), \\
\omega_i &= \sum_{j \in \mathcal{N}(t)} s^T_j (z_j - z_i).
\end{align}
\]

To show the ultimate boundedness of the \(n\) mobile robots, we define a new state \(p_i = z_i + r_i\) for each robot \(i\), which is a point one unit ahead of the representing point in the direction of its heading. Differentiating \(p_i = z_i + r_i\) results in the identity

\[
p_i = \dot{z}_i + \dot{r}_i = v_i r_i^T + \omega_i s_i.
\]

Thus,

\[
\begin{align}
\dot{p}_i &= \sum_{j \in \mathcal{N}(t)} r^T_j (z_j - z_i) + \sum_{j \in \mathcal{N}(t)} s^T_j (z_j - z_i) \\
&= (r_i r^T_i + s_i s^T_i) \sum_{j \in \mathcal{N}(t)} (z_j - z_i) \\
&= \sum_{j \in \mathcal{N}(t)} (z_j - z_i).
\end{align}
\]

Writing the system in the matrix form, we have

\[
\dot{p} = -(L(t) \otimes I_2)z,
\]

where \(I_2\) is the 2-by-2 identity matrix

\[
p = [p_1^T p_2^T \ldots p_n^T]^T \quad \text{and} \quad z = [z_1^T z_2^T \ldots z_n^T]^T.
\]

Now we present our first result for the dynamic topology case.

**Theorem 3.** For a group of \(n\) mobile robots under the control law (7), the centroid of \(p_1, p_2, \ldots, p_n\) is stationary.

**Proof:** Define the centroid of \(p_1, p_2, \ldots, p_n\) as

\[
c := \frac{1}{n} (1^T \otimes I_2)p.
\]

From (8), it is calculated that

\[
\dot{c} = -\frac{1}{n} n^T \otimes I_2 (L(t) \otimes I_2)z.
\]

Applying the associativity property of the Kronecker product, we have

\[
\dot{c} = -\frac{1}{n} (1^T L(t) \otimes I_2)z.
\]

Note that at any \(t\), the Laplacian \(L(t)\) is symmetric and satisfies \(1^T L(t) = 0\). It then follows that \(\dot{c} = 0\) and so the centroid \(c\) is stationary.

Next, we show that the new representing points \(p_1, p_2, \ldots, p_n\) are ultimately bounded around the centroid \(c\). In other words, it implies that the \(n\) mobile robots are ultimately bounded. The result is given below.

**Theorem 4.** Suppose the graph \(\mathcal{G}(t)\) is uniformly jointly connected. Then a group of \(n\) mobile robots are ultimately bounded under the control law (7).

The proof requires the following lemma. Its proof is given in appendix.

**Lemma 5.** Consider the system

\[
\dot{x} = -B(t)Q^T(t)x + B(t)Q^T(t)w
\]

where the external input \(w(t)\) satisfies \(\|w(t)\| \leq \bar{w}\) for a constant \(\bar{w}\) and \(B(t)\) satisfies the following assumptions:

\(\text{A1.}\) There exists a constant \(\delta > 0\) such that for almost all \(t \geq 0, \|B(t)\| \geq \delta;\)

\(\text{A2.}\) There exist constants \(T, \mu > 0\) such that for all \(t \geq 0,\)

\[
\mu I \leq \int_0^t B(\tau)B^T(\tau)d\tau.
\]

Then the system (9) is ultimately bounded.

**Proof of Theorem 4:** We define a matrix \(Q \in \mathbb{R}^{n \times (n-1)}\) with the properties \(Q^T I_n = 0\) and \(Q^T Q = I_{n-1}\). Then \(QQ^T = I_n - \sum_{i=1}^n I_i^T I_i\) and thus

\[
L(t)QQ^T = L(t).
\]

Let \(q = (Q^T \otimes I_2)p\). Then

\[
p = I_n \otimes c + (Q \otimes I_2)q
\]

where \(c \in \mathbb{R}^2\) is the centroid of \(p_1, \ldots, p_n\). From (8), it can be obtained that

\[
\dot{q} = -(Q^T \otimes I_2)(L(t) \otimes I_2)z
\]

where \(r = [r_1^T, \ldots, r_n^T]Q^T\). Letting \(w = (Q^T \otimes I_2)r\) and using (10) and (11), we then obtain that

\[
\dot{q} = -[Q^T L(t)Q \otimes I_2]g + [Q^T L(t)Q \otimes I_2]w.
\]

Since \(\|r\| = \sqrt{n}\), it is then known that \(\|w\|\) is upper-bounded. Note that the Laplacian matrix \(L(t)\) is symmetric for any \(t\), so \([Q^T L(t)Q \otimes I_2]\) can be written in the form of \((B(t)B^T(t)\) with a suitable \(B(t)\). Thus the system (12) is of form (9) as in Lemma 5. Moreover, assumption A1 of Lemma 5 holds. Next, we show assumption A2 of Lemma 5 also holds.

We know that the graph is uniformly jointly connected by assumption. That is, there exists \(T\) such that for any \(t\), the union graph \(\mathcal{G}(t, t+T)\) is connected. For every period \([t, t+T]\), suppose the graph switches at \(t_1, t_2, \ldots, t_k\) with \(t < t_1 < t_2 < \cdots < t_k < t+T\). Denote the corresponding Laplacian matrices at intervals \([t, t_1), [t_1, t_2), \ldots, [t_k, t+T)\) by \(L_0, L_1, \ldots, L_k\), respectively. Then

\[
\int_t^{t+T} Q^T L(\tau)Qd\tau = Q^T LQ.
\]
where $\bar{L} = (t_1 - t)L_0 + (t_2 - t_1)L_1 + \cdots + (t + T - tk)L_k$.
That the union graph $\bar{G}(t, t + T)$ is connected implies, $L_0 + L_1 + \cdots + L_k$ has a simple zero eigenvalue, and so does $\bar{L}$. Since $Q^T LQ$ inherits all eigenvalues of $\bar{L}$ except the one at zero and moreover the switching of the graph is subject to a dwell time, there exists a constant $\mu > 0$ such that the smallest eigenvalue $\lambda_{\min}(Q^T LQ) \geq \mu$, which is equivalent to say
$$\int_{t}^{t+T} Q^T L(\tau) Q d\tau \geq \mu I_n$$
for any $t$. Furthermore, for Kronecker product we know that $\int_{t}^{t+T} Q^T L(\tau) Q d\tau \geq \mu I_n$ is equivalent to
$$\int_{t}^{t+T} [(Q^T L(\tau) Q) \otimes I_2] d\tau \geq \mu I_{2n}.$$
Hence, assumption A2 holds.

Thus, applying Lemma 5, it is concluded that $q(t)$ is ultimately bounded. Moreover, from Theorem 3 we know that the centroid of the group of mobile robots is stationary. Therefore, the trajectories of the $n$ mobile robots are ultimately bounded under the control law (7).

4. SIMULATIONS

The results we obtained are demonstrated through two simulations presented in this section. In the simulation screenshots below, the initial posture of each mobile robot is drawn in blue and its final posture is drawn in red.

The first screenshot (Fig. 3) presents the simulated trajectories of 4 mobile robots under our control law for the fixed topology case where the graph is shown in Fig. 4.

![Fig. 3. Screenshot: the fixed topology case](image)

Fig. 3. Screenshot: the fixed topology case

The second screenshot (Fig. 5) presents the simulated trajectories of 3 mobile robots under our control law for the dynamic topology case. The dynamic graph $\bar{G}(t)$ is defined as follows:
$$\bar{G}(t) = \begin{cases} G_a, & \text{if } t \mod 3 \in [0, 1) \\ G_b, & \text{if } t \mod 3 \in [1, 2) \\ G_c, & \text{if } t \mod 3 \in [2, 3) \end{cases}$$
which periodically switches among three graphs shown in Fig. 6. From the simulation, we can see that the three mobile robots not only are ultimately bounded but also rendezvous.

![Fig. 4. Fixed topology.](image)

Fig. 4. Fixed topology.

![Fig. 5. Screenshot: the dynamic topology case.](image)

Fig. 5. Screenshot: the dynamic topology case.

![Fig. 6. Dynamic topology.](image)

Fig. 6. Dynamic topology.

5. CONCLUSION

In this paper, a local feedback control strategy that drives a system of multiple nonholonomic unicycles to a rendezvous point in terms of position is introduced. The proposed control law is continuous and time-invariant using only local information. Adopting tools from graph theory, convergence and ultimate boundedness of the overall system are examined. Similar to the linear case, the convergence relies on the connectivity of the interaction topology.

APPENDIX

The proof of Lemma 5 follows the lines of proof for uniformly globally exponentially stability of gradient algorithms (Panteley and Loria, 2000).

**Proof of Lemma 5:** Let $V(x) = \|x\|^2$. Then along the solution of (9), it is obtained that
$$\frac{d}{dt} V(x(t)) = -2x^T(t)B(t)B^T(t)x(t)+2x^T(t)B(t)B^T(t)w(t).$$
Integrating both sides from $t$ to $t + T$ and using $2a^T b \leq 2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$, we obtain
\[ V(x(t + T)) − V(x(t)) ≤ −∫_t^{t+T} \|BT(\tau)x(\tau)\|^2 d\tau + ∫_t^{t+T} \|BT(\tau)w(\tau)\|^2 d\tau, \]

where the solution
\[ x(\tau) = x(t) − ∫_t^\tau B(s)BT(s)[x(s) − w(s)]ds. \]

Denote
\[ H_1 = ∫_t^{t+T} \|BT(\tau)x(\tau)\|^2 d\tau \]

and
\[ H_2 = ∫_t^{t+T} \|BT(\tau)w(\tau)\|^2 d\tau. \]

Then it can be easily obtained that
\[ H_2 ≤ \delta^2\bar{w}^2T. \tag{13} \]

Substituting the solution \( x(\tau) \) into \( H_1 \) we obtain, using \((a − b)^2 ≥ \frac{1}{4a^2}a^2 − \rho \bar{a}^2 \) for \( \rho > −1 \), the triangle inequality, Cauchy-Schwartz inequality, and assumptions A1, A2 of Lemma 5
\[ −H_1 ≤ \frac{\rho}{1 + \rho} ∫_t^{t+T} \|BT(\tau)x(\tau)\|d\tau + \rho ∫_t^{t+T} \|B(\tau)\|^2 x(t) + \|B(s)\|^2 \|BT(s)[x(s) − w(s)]\|^2 ds d\tau ≤ \frac{\rho}{1 + \rho} \|x(t)\|^2 \]
\[ + \rho \lambda ∫_t^{t+T} ∫_t^\tau \|BT(s)[x(s) − w(s)]\|^2 ds d\tau \]

where
\[ ∫_t^{t+T} ∫_s^\tau \|B(s)\|^2 \|BT(s)[x(s) − w(s)]\|^2 ds d\tau ≤ T ∫_t^{t+T} \|BT(s)[x(s) − w(s)]\|^2 ds \]
\[ ≤ T(\lambda H_1 + H_2). \]

Choose \( \rho = \frac{\lambda}{\delta^2T + \lambda} \) where \( \lambda > 0 \). Then it can be obtained that
\[ −H_1 ≤ −\frac{\mu\lambda}{(\delta^2T + \lambda)(1 + \lambda)} \|x(t)\|^2 + \frac{\lambda}{1 + \lambda} H_2. \tag{14} \]

Hence, using (14) and (13) we obtain
\[ V(x(t + T)) − V(x(t)) ≤ −\frac{\mu\lambda}{(\delta^2T + \lambda)(1 + \lambda)} \|x(t)\|^2 + \frac{\delta^2\bar{w}^2T}{1 + \lambda}, \]

which implies the solutions will eventually satisfy
\[ \|x(t)\|^2 ≤ \frac{\delta^2\bar{w}^2T(\delta^4T + \lambda)}{\mu\lambda} \]

and remains true thereafter. Thus, the conclusion follows.

\section*{REFERENCES}


