Noisy Network Localization via Optimal Measurement Refinement Part 2: Distance-Only Network Localization

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Abstract: In this paper we review some results on the problem of noisy localization and review the characteristics of networks labeled as easily localizable networks. We then present a more general result on such networks before moving to the main problem of interest in this paper. That is, proposing computationally efficient algorithms to refine noisy distance measurements in easily localizable networks.

1. INTRODUCTION

Localization problems have been well-studied in the context of wireless sensor networks, and it is typically assumed that a small fraction of sensors, called anchors, have a priori information about their global coordinates. Exploiting the fact that the position of these anchor nodes are known in a global coordinate system and that a number of inter-node distances are known, all the other nodes in the network can be localized under a condition which will be discussed later in this paper, i.e. global rigidity of the underlying graph of the network [Eren et al., 2004, Asnæs et al., 2006]. However, many theoretical results assumed a noiseless scenario while in all engineering applications the assumption of having noiseless measurements is not realistic. Despite the recent efforts to formally address the problem of sensor network localization in the presence of measurement noise, [Anderson et al., 2010], this field is still in its infancy. Nonetheless, many computational algorithms are proposed to solve the localization problem, e.g. convex optimization based algorithms [Doherty et al., 2001, Carter et al., 2006, Biswas et al., 2006, Ding et al., 2008], the algorithms using sum of squares relaxation [Shames et al., 2009], graph connectivity based algorithms [Shang et al., 2003], the methods that use multidimensional scaling [Costa et al., 2006], or other methods e.g. described in [Moore et al., 2004, Bachrach and Taylor, 2005].

This paper is another step in formally proposing computationally efficient algorithms in the context of easily localizable networks where the distance measurements are noisy. We first review the previous results on easily localizable networks, e.g. in [Anderson et al., 2009], and extend the existing results to different networks with important engineering implications. Later, we consider the problem of localization in the presence of noisy distance measurements in such easily localizable networks. This paper is part 2 in a series. Part 1 deals with bearing-only network/target localization; see [Bishop and Shames, 2011].

The remainder of this paper is organized as follows. In the next section, we introduce the notation used throughout the paper along with some definitions regarding widely-used graph and network terminology. In Section 3 we outline the conditions under which the sensor network can be localized given only distance measurements between certain sensor pairs. A number of a high-level results are obtained. Then in Section 4 we review the problem of localization when the distance measurements are noisy. In Section 5, we introduce the idea of distance measurements refinement in networks and particularly introduce two algorithms to achieve measurement refinement in easily localizable networks. A conclusion is given in Section 6.

2. NETWORK AND GRAPH NOTATION

Consider $n_s$ sensors with indices in a set $V = \{1, 2, \ldots, n_s\}$ and with positions in $\mathbb{R}^2$ denoted by $p_i$. The sensor $i$ can measure the distance $d_{ij} = d_{ji} = ||p_i - p_j||$ to a different sensor $j$ if $||p_i - p_j|| \leq r$ for some $r > 0$. This relationship can be used to define a graph $G(V, E)$ where $V$ is the set of sensors and $E$ is the set of distance sensing links $(i, j)$ where $(i, j)$ exists iff $||p_i - p_j|| \leq r$.

Definition 1 (Formal Sensor Network). A sensor network $F_p$ is then formally defined by a graph $G(V, E)$ and a map $\Pi : V \rightarrow \mathbb{R}^2$ which takes sensor $i$ in $V$ to its respective position $p_i$ in $\mathbb{R}^2$.

It is common to use the term network to refer to both $G(V, E)$ and $F_p$ when there is no chance of confusion.

Definition 2 (Clique). A clique in an undirected graph $G$ is a subset of its vertices such that every two vertices in the subset are connected by an edge. In other words, a clique is a complete subgraph of a graph. Furthermore, we denote a $k$-clique to be a clique of graph $G$ induced by $k$ nodes.

It is known that finding all of the cliques of a graph is NP-complete [Bomze et al., 1999].

Definition 3 (Minor). A minor of a graph $G$ is a graph obtained from $G$ by a sequence of (i) edge deletions and (ii) edge contractions, where an edge contraction is an operation which identifies the two vertices belonging to an edge, removes the edge and merges together the two vertices.
3. NETWORK LOCALIZATION WITH DISTANCE MEASUREMENTS

Suppose that the sensor positions are not known a priori but must be estimated. The following definitions are standard in graph theory and have been used in the study of network localizability [Aspnes et al., 2006].

Definition 4 (Congruent Networks). A network $F_p$ and a network $F_q$ are said to be congruent if there is an isometry $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(q_i) = p_i$.

Definition 5 (Equivalent Networks). Now $F_q$ and $F_p$ are said to be equivalent if their underlying graphs are identical, i.e. $G_q = G_p = G$, and $\|p_i - p_j\| = \|q_i - q_j\|$ for all $(i, j) \in E$.

Definition 6 (Rigidity). A network $F_p$ is rigid if there exists a sufficiently small positive $\epsilon$ such that if $F_q$ is equivalent to $F_p$ and $\|p_i - q_i\| \leq \epsilon$ for all $i \in V$ then $F_q$ is congruent to $F_p$. Intuitively, a rigid network is one that cannot flex.

There exist rigid networks $F_p$ and $F_q$ which are equivalent but not congruent.

Definition 7 (Global Rigidity). A network $F_p$ at $p = \{p_1, \ldots, p_n\}^\top$ is globally rigid if every network $F_q$ which is equivalent to $F_p$ is also congruent to $F_p$.

Generally, rigidity and global rigidity are generic properties of networks. This means that the rigidity (global rigidity) of a generic realization of a graph $G$ depends only on the graph $G$ and not the particular realization.

Before continuing we propose the following proposition.

Proposition 1 (Special Case of Kuratowski Reduction Theorem). The (underlying) globally rigid graph $\gamma(G, E)$ of a network in $\mathbb{R}^2$ has a $K_4$ minor, where $K_n$ denotes the complete graph on $n$ vertices.

Definition 8 (Noiseless Distance Network Localization Problem). The network localization problem involves determining $p_i, \forall i \in V$, given the graph topology $G$, the positions of some of the sensors $p_j, j \in V', V' \subset V$, and the set of distances $d_{ij}$. The network is said to be localizable if $p_i, \forall i \in V$ can be determined uniquely given the positions of some of the sensors $p_j, j \in V', V' \subset V$, and the set of distances $d_{ij}$.

In particular, we make the following assumption.

Assumption 1. The sensors are in general position such that the localizability of a network can be characterized by the topology of $G$.

In other work [Aspnes et al., 2006], it has been shown that the (noiseless version of the) sensor network localization problem in $\mathbb{R}^2$ is in principle solvable if and only if the graph $G$ has the property of generic global rigidity [Jackson and Jordan, 2005] and there are at least three noncollinear anchor nodes (the nodes with known positions). The qualifying words ‘in principle’ imply that no consideration is being given at this point to the nature of a particular algorithmic procedure for carrying out the localization calculation. The word ‘generic’ also deserves comment. Others sometimes remove the word generic from the term generic global rigidity with mild abuse of nomenclature, as for example in [Jackson and Jordan, 2005], where the term ‘global rigidity’ is used instead of ‘generic global rigidity’ as a descriptor of certain graphs. The term ‘uniquely realizable’ has also been used for generic global rigidity, see [Connelly, 2005, Jackson and Jordan, 2005]. It turns out that there can be special networks which have at least three noncollinear anchors but which are not localizable though the associated graphs are globally rigid. They are exceptional, and in them, special relationships exist among the sensor positions, e.g. groups of sensors may be collinear. Thus the reason for using the term generic is to highlight the need to exclude the problems arising from such networks. For further discussion on the use of the word generic one may refer to [Aspnes et al., 2006, Tay and Whiteley, 1985].

The network localization problem can be written in the form of the following set of equations

$$ \|p_i - p_j\|^2 = d_{ij}^2, \quad p_i = p_i, \forall i \in V'. $$

(1)

where $p_i$ is a variable in the equation set and where $p_i, i \in V'$, is held at the known position of anchor $i$. More generally, $p_i, i \in V$, is used to denote the sensor positions when they are held constant at the true values.

Solving this set of equations for $p_i, i = 1, \ldots, n_S$, is known to be NP-hard. However, there are certain networks for which an algorithm exists that solves the localization problem in polynomial time. We review these networks in the next section.

3.1 Easily Localizable Networks

Firstly, we will consider a special form of a graph known as a trilateration graph.

Definition 9 (Trilateration Graph). A graph $G(V, E)$ is called a trilateration graph if the vertices can be ordered as $\{1, \ldots, |V|\}$ such that $(1, 2), (1, 3), (2, 3) \in E$ and at least $(i, j) \in E$, $(i, k) \in E$, and $(i, l) \in E$ for each $i \in \{4, \ldots, |V|\}$ and at least three $j, k, l \in \{1, \ldots, l - 1\}$. Furthermore, we label nodes 1, 2, and 3 as seeds of the graph.

The ordering of the vertices in a trilateration graph is called the trilaterative ordering. For an arbitrary fixed graph, known to be a trilateration graph, the seeds 1, 2, and 3 may not be unique nor the ordering.

We have the following result for the localization of trilateration graphs; see [Anderson et al., 2009].

Theorem 1. Suppose that the positions of three sensors $i, j, k$ in $V$ are known absolutely in $\mathbb{R}^2$. Suppose further that $G$ is a trilateration graph. Then the network is distance localizable in $\mathbb{R}^2$.

The algorithm requires one to search for the seeds. There are $\binom{n}{3}$ naive choices for the seeds. At best, the algorithm runs in linear time. At worst, the run time is less than cubic in the number of sensors.

Corollary 1. If $G$ is an arbitrary graph containing a trilateration subgraph on the entire set of sensors, and three sensors $i, j, k$ in $V$ are known absolutely in $\mathbb{R}^2$, then the network is absolutely distance localizable in $\mathbb{R}^2$.

Next we extend this idea to $\nu$-trilateration graphs, which are also easily localizable. First, we define the concept of power graphs.

Definition 10 (Graph Powers). Given $G(V, E)$ then a graph $G^k$ is the $k$th power graph of $G$ and has the same vertex set as $G$ and an edge set defined by $E = \{(i, j) :$ a path of length $k$ or less exists between $i$ and $j \in G\}$. 

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A graph raised to the power of its diameter is complete. Next, the idea of an n-lateration graph is introduced.

**Definition 11** (n-lateration Graph). An n-lateration graph \( G(V,E) \) with \( |V| > n \) has a subset of \( n \) vertices which form a complete subgraph. Label these vertices \( \{1, \ldots, n\} \). Further, there exists an ordering of the remaining \( |V| - n \) vertices \( \{n+1, \ldots, |V|\} \) such that an edge exists between each \( i \in \{n+1, \ldots, |V|\} \) and \( n \) distinct vertices in \( \{1, \ldots, i-1\} \). The ordering of the vertices is called the n-lateration ordering while the initial \( n \) vertices that form a complete subgraph is the seed.

Common examples of n-lateration graphs found in applications are bilateration, trilateration and quadrilateration graphs. Throughout the paper, graphs that have an n-lateration subgraph are called n-lateration graphs themselves (or are said to have the n-lateration property). It is obvious that neither the seed nor the ordering of an n-lateration graph is necessarily unique. An n-lateration graph in which the seed and ordering is unique is said to be a pure n-lateration graph.

### 3.2 Properties of n-Lateration Graphs

First, we define another important concept in graph theory with applications in sensor networks localization.

**Definition 12** (Planar Graph). A graph is planar if and only if it does not contain a subgraph that is homeomorphic to the complete graph on five vertices or the complete bipartite graph on six vertices.

Conceptually, a planar graph is a graph that can be drawn on the plane such that its edges intersect only at their endpoints. Now, we present a number of results concerning n-lateration graphs are now covered.

**Proposition 2.** A pure n-lateration graph on a finite set of vertices is planar if and only if \( n \leq 3 \).

Now we present a result relating n-lateration graphs to power graphs.

**Theorem 2.** Given a connected graph \( G(V,E) \) with \( |V| > k \) then \( G^k \) has a k-lateration subgraph.

The proof is given in the Appendix. A version of the preceding theorem was proven in [Fang et al., 2009] for trilateration and quadrilateration graphs. An important physical consequence of the previous result exists. Imagine an invertible map \( \Pi^{-1} : \mathbb{R}^d \rightarrow V \) taking points in some Euclidean d-space to the vertices of a graph. Suppose a link exists between points \( p_i \) and \( p_j \) if and only if \( ||i - j|| \leq r \) for some \( r > 0 \). The existence of a link between \( i \) and \( j \) induces an edge between \( \Pi^{-1}(p_i) \) and \( \Pi^{-1}(p_j) \) in \( G(V,E) \). Then Theorem 2 implies that substituting \( k \times r \) for \( r \) induces the graph \( G^k \). Then \( G^k \) has the k-lateration property. The importance of this result is highlighted by the problem of localizing a point configuration in d-space using the Euclidean distance between adjacent points in the graph topology. A sufficient condition for localizability is as follows.

**Theorem 3.** A network of \( |V| \) points in \( \mathbb{R}^{d-1} \) with the underlying graph \( G(V,E) \) and known positions for at least \( d \) points is localizable in polynomial time given values for \( ||p_i - p_j|| \) for all pairs \( (i,j) \) such that \( (i,j) \in E \) if \( G(V,E) \) has the d-lateration property and all points are in general position.

The aforementioned theorem has immediate application in electrical engineering, communication systems, chemistry, and physics. Theorem 2 implies that given a connected point configuration in \( \mathbb{R}^d \) it is sufficient to increase the radius of interaction between points \( d \)-fold to guarantee localizability. Of course, this is not necessary in general. Moreover, as we will see later, higher-order lateration graphs are easier to localize and thus increasing the radius of interaction beyond \( d \)-times the original radius has additional benefits (including redundancy).

### 4. NOISY DISTANCE LOCALIZATION

We assume that the measured distance \( \bar{d}_{ij} \) satisfies

\[
d_{ij}^2 = d_{ij}^2 + e_{ij},
\]

where the quantity \( e_{ij} \) being a (typically small) error in the squared distance (rather than in the distance itself); thus \( d_{ij} \) remains the actual distance and \( e_{ij} \) constitutes the measurement noise effect. Now, suppose that each squared distance \( d_{ij}^2 \) in (1) is replaced by \( d_{ij}^2 + e_{ij} \). Then in the absence of any knowledge of the noise, it is natural to consider the following set

\[
||p_i - p_j||^2 = d_{ij}^2 + e_{ij}
\]

This equation set is overdetermined and, in general, will have no solution due to the presence of inconsistent (independent) errors in each of the equations.

The simplest example of this problem involves localizing a single sensor given noisy measurements of its distance from three anchor sensors, as treated in [Cao et al., 2006]. There are two unknowns, the coordinates of the single sensor to be localized. There are three scalar equations perturbed by noise, and there is generally no solution. The obvious remedy is to try for an approximate solution, and that is what is done in general. This work extends [Cao et al., 2006] in a general way by accounting for arbitrary networks configurations. We also go further and consider cases, i.e. network configurations, where the localization problem is realistically solvable.

Despite the inability to solve the noisy equation set (3), the notion of localization, albeit approximate localization, still makes sense: clearly, it would be appropriate to seek those coordinate values of \( p_i \), for \( i \in V' \) such that \( V' \subseteq V \setminus V' \) solving the following minimization problem:

\[
\min_{p_i \in V'} \sum_{i,j \in V} ||p_i - p_j||^2 - d_{ij}^2 + e_{ij}
\]

subject to

\[
p_i = p_i \forall i \in V'
\]

(Of course, other measures for the error between \( ||p_i - p_j||^2 \) and \( (d_{ij}^2 + e_{ij}) \) could be used.)

Now we know that if all \( e_{ij} \) are zero, there is generally a unique solution to the minimization problem, namely the solution of the usual localization problem, which yields a zero value for the cost function, and \( p_i \) is known. Let \( ||e|| \) denote the Euclidean norm, so that \( ||e||^2 = \sum_{(i,j) \in E} c_{ij}^2 \), and \( e \) is obtained by stacking all \( e_{ij} \). Note that \( e_{ij} = 0 \) for \( i,j \in V' \). The questions of interest to us here are: Is it guaranteed that, at least for suitably small \( ||e|| \), there will still be a unique solution to the minimization problem, and that its distance from the solution of the localization problem with zero measurement noise will go continuously to zero as the value of \( ||e|| \) goes to zero?

These questions are answered in the following result from [Anderson et al., 2010]:
Theorem 4. Consider a globally rigid and generic network \( F_p \). Let \( V' \subset V \) denote vertices of \( G \) corresponding to anchor nodes, of which there are at least three and for which the value of \( p_i \) is known. Let \( d_{ij} \) denote the measured distance between nodes \( i \) and \( j \) when \( (i, j) \in E \). Then there exists a suitably small positive \( \Delta \) and an associated positive constant \( c \) such that if the measurement errors in the squares of the distances obey \( ||e|| < \Delta \), the solution to the noisy localization problem is unique and there holds \( ||p - p|| \leq ||e|| \).

This establishes that a network can be approximately localized when the inter-node distance measurements are contaminated with a sufficiently small noise. In the next section we consider the problem of improving the accuracy of the localization via refining the distance measurements.

5. DISTANCE MEASUREMENT REFINEMENT IN SENSOR NETWORKS

Consider a globally rigid graph \( G(V, E) \) and a set of inter-node distance measurements. The problem of distance measurements refinement is to find a set of distances \( d^*_i \) for all \( (i, j) \in E \) such that the following set of equations is consistent.

\[
||p_i - p_j||^2 = d^*_{ij}^2 \quad \forall i \in V
\]

We first address this problem for a network with a \( K_4 \) underlying graph before considering other more general networks.

5.1 Distance Measurement Refinement in a \( K_4 \) Network

Let us define the Cayley-Menger matrix of a single \( n \)-tuple of points \( p_0, \ldots, p_{n-1} \) in \( d \)-dimensional space as,

\[
M(p_0, \ldots, p_{n-1}) \triangleq \begin{bmatrix}
0 & d_{01}^2 & \cdots & d_{0,n-1}^2 \\
d_{10}^2 & 0 & \cdots & d_{1,n-1}^2 \\
\vdots & \vdots & \ddots & \vdots \\
d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\]

Let \( V(p_0, \ldots, p_{n-1}) \) denote the volume of a Euclidean simplex \( \{p_0, p_1, \ldots, p_{n-1}\} \), then the following relation can be given in the determinant of the Cayley-Menger matrix

\[
V^2(p_0, p_1, \ldots, p_{n-1}) = \frac{(-1)^n}{2^{n-1}((n-1)!)^2} \det |M|
\]

where \( \det |M| = \det [M(p_0, p_1, \ldots, p_{n-1})] \) is the determinant of the Cayley-Menger matrix. The following theorem is introduced in a number of places and stems from the above definition of the volume of a simplex.

Theorem 5. Consider an \( n \)-tuple of points \( p_0, \ldots, p_{n-1} \) in \( d \)-dimensional space. If \( n \geq d+2 \), then the Cayley-Menger matrix is singular, namely \( \det |M(p_0, \ldots, p_{n-1})| = 0 \).

That is, considering an \( n \)-tuple of points in \( d \)-dimensional space with \( n \geq d+2 \), a geometrical constraint exists between the inter-point distances that must be satisfied.

Now consider a \( K_4 \) graph, with vertex set \( \{i, j, k, l\} \) and a set of measured inter-node distances. For this graph to be realizable in \( \mathbb{R}^2 \), the Cayley-Menger determinant corresponding to the inter-node distances should be equal to zero, i.e. the volume of the tetrahedron defined by the four nodes should be zero. So the problem is to find a set of \( e_{ij}^* \) such that the Cayley-Menger determinant over the square of distances \( (d^*_{ij})^2 = d_{ij}^2 - e_{ij}^* \) is equal to zero and \( \sum(e_{ij}^*)^2 = ||e^*||^2 \) is minimum. Hence we have the following optimization problem.

\[
e^* = \arg\min ||e||^2
\]

subject to \( \det |M_i| = 0, \quad i = 1, \ldots, m \) \hspace{1cm} (6)

\[
e_{ij}^* = 0, \quad i, j \in V'
\]

While the cost function is quadratic, this optimization problem generally cannot be solved analytically due to the cubic constraint, and one needs to apply numerical methods to solve it. However, for a special case of the problem introduced in [Cao et al., 2006] a solution is provided.

Definition 13 (Consistent \( K_4 \) Network). We call a \( K_4 \) network with the distances \( d_{ij}^* \) consistent if and only if the Cayley-Menger determinant associated with this network vanishes.

5.2 Distance Measurements Refinement in a General Network

In this section we extend the method introduced in the previous section to refine the distance measurements in a \( K_4 \) network to a more general network. This is a significant extension to [Cao et al., 2006] and is one of the main contributions of this work. First we propose a definition for a consistent network.

Definition 14 (Consistent Network). We call a network with the measured distances \( d_{ij}^* \) consistent if and only if the Cayley-Menger determinant can be realized in a plane. In other words there exists a mapping \( \Pi: V \rightarrow \mathbb{R}^2 \) such that \( ||\Pi(i) - \Pi(j)|| = d_{ij}^* \) for all \( i, j \in V \).

The aim of this section is to find estimates for \( e_{ij} \), such that the following set of equations becomes consistent.

\[
||p_i - p_j||^2 = d_{ij}^2 - e_{ij}^*
\]

First, we have the following trivial result.

Proposition 3. Consider a network with the underlying graph \( G(V, E) \) and noisy measurements satisfying the conditions of Theorem 4. This graph is consistent if and only if it does not have a \( K_4 \) minor.

We have the following corollary.

Proposition 4. A globally rigid graph \( G(V, E) \) is not generically consistent.

This proposition is equivalent to saying that the system of equations (3) is not consistent. Now, we state the following theorem that helps to frame the solution to the problem of distance refinement in distance measuring networks.

Theorem 6. A network with the underlying graph \( G(V, E) \) and noisy measurements satisfying the conditions of Theorem 4 is consistent if and only if every subgraph of \( G \) (excluding itself) that has at least a \( K_4 \) minor is consistent.

So to make the distance measurements consistent we propose the following optimization problem.

\[
e^* = \arg\min ||e||^2
\]

subject to \( G_i \subset G \) are consistent, \( i = 1, \ldots, m' \) \hspace{1cm} (8)

where \( G_i \) are all the subgraphs of \( G \) (excluding itself) containing at least a \( K_4 \) minor. Solving this problem under the above
constraint is not computationally efficient. However, we can relax the constraint to a computable constraint. To do so, define the set \( \Phi(G, 4) = \{ \phi_1(G, 4), \ldots, \phi_m(G, 4) \} \) of all the 4-cliques of \( G \). Now for any given network with underlying graph \( G \) we relax the constraint in (8) and propose the following optimization problem that solving which will result in having consistent distance measurements and refined \( d_{ij} \).

\[
e^* = \arg\min ||e||^2 \quad \text{subject to} \quad \det |M_i| = 0, \quad i = 1, \ldots, m \tag{9}
\]

where \( \det |M_i| \) is the Cayley-Menger determinant corresponding to clique \( \phi_i(G, 4) \), \( e^* \) is the estimated vector of \( e_{ij} \) terms arranged using the same ordering as in \( e \). Solving this optimization problem in general cases is computationally expensive, and as mentioned earlier solving the localization problem for general matrices is not typically computationally efficient either. However, as we discussed in Section 3.1, there are network structures where the localization can be carried out in polynomial time. In the next section we address the problem of distance refinement in such easily localizable networks. But, first we make some comments on the problem of finding cliques of fixed size in a general graph that is a part of solving the optimization problem (9). Note that is is possible to define a \( k \)-clique finding algorithm for an arbitrary graph (specifically \( n \)-latetration graphs, \( k \leq n \)) that has a worst-case polynomial complexity in the number of vertices, see [Chiba and Nishizeki, 1985, Downey and Fellows, 1995a,b] for more details.

Proposition 5 (Complexity of \( k \)-clique Finding Algorithm). Let \( k \leq n \). The complexity of finding the entire set of \( k \)-cliques in a general graph is polynomial in \( O(|V|^k k^2) \) in the worst-case and is linear in \( O(|E|) \) when the graph is planar (e.g. pure \( n \)-latetration with \( n \leq 3 \)).

### 5.3 Noisy Localization in an \( n \)-latetration Network

In this section we propose two algorithms for refining the distance measurements in \( n \)-latetration networks. First, we introduce “batch refinement” algorithm in the next subsection. Later we will introduce the second algorithm called “sequential refinement” algorithm. Moreover, we make the following assumption in this section.

Assumption 2. We assume that \( 3 \leq v \leq n \) of the nodes in the seed of the \( n \)-latetration network are anchors.

**Batch Refinement** First, we have the following theorem.

**Theorem 7.** Consider an \( n \)-latetration network, \( n \geq 3 \), with underlying graph \( G(V, E) \). If \( |V| \leq 2n \) for all \( i \in V \), \( G \) has at least one 4-clique containing \( i \).

In the light of this theorem we propose Algorithm 1 to refine the distance measurements and localize the nodes in an \( n \)-latetration graph in simultaneous batches. Consider Assumption 2 and let the seeds form the complete graph \( G_0 \).

**Sequential Refinement** In this section we propose Algorithm 2 based on the method introduced in [Cao et al., 2006] to refine the distance measurements and localize an \( n \)-latetration graph sequentially. Consider Assumption 2 and let the seeds form the complete graph \( G_0 \).

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**Algorithm 1** Batch Distance Refinement and Localization of an \( n \)-latetration Network

1. \( S = \emptyset; \)
2. \( \ell = n + 1; \)
3. for \( \ell \leq n \) do
4. \( S^* = S \)
5. \( S = S \cup \mathcal{I}(\ell), \forall (i, i) \in \mathcal{E}; \)
6. if \( \exists i \) such that \( \ell \in \phi_i(S, 4) \) then
7. \( S = S^*; \)
8. solve \( e^* = \arg\min ||e||^2 \quad \text{subject to} \quad \det |M_i| = 0, \quad i = 1, \ldots, m \)
9. \( e_{ij}^* = 0, \quad i, j \in \mathcal{V}; \)
10. localize nodes \( 1, \ldots, \ell - 1 \) using \( d_{ij}^* = d_{ij} + e_{ij}; \)
11. \( S = K_{\ell - 1} \) with nodes \( 1, \ldots, \ell - 1; \)
12. end if
13. \( \ell = \ell + 1; \)
14. end for

**Algorithm 2** Sequential Distance Refinement and Localization of an \( n \)-latetration Network

1. \( S = \emptyset; \)
2. Refine the distances in \( S \) using the \( v \) anchors in the seed using Cayley-Menger method introduced in [Cao et al., 2006];
3. Localize the nodes in the \( S \) using the refined distances;
4. \( \ell = n + 1; \)
5. for \( \ell \leq n \) do
6. \( S = S \)
7. \( S = S \cup \mathcal{I}(\ell), \forall (i, i) \in \mathcal{E}; \)
8. Refine the \( n \) distances to \( \ell \) in \( S \) using \( n \) of the previously localized nodes in \( S^* \) using Cayley-Menger method introduced in [Cao et al., 2006];
9. \( \ell = \ell + 1 \)
10. end for

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6. CONCLUSION

The idea of noisy and imprecise distance measurements in network localization is studied. A distance refinement for a general network guarantees that the network is consistent, i.e. realizable in \( \mathbb{R}^2 \), via guaranteeing consistency of all the non-consistent blocks of the network. It is acknowledged that the problem in its general form is not easily solvable and hence the consistency conditions are relaxed and two computationally efficient algorithms are proposed to refine imprecise and noisy distance measurements in easily localizable networks.

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**REFERENCES**


Appendix A. PROOF OF THEOREM 2

A.1 Proof when $G$ is a Path Graph

Consider first a path graph $G$ on $|V| > k$ vertices. Label the vertices sequentially such that, apart from the terminals, vertex $i$ is connected to $i + 1$ and $i - 1$. The degree of $i$ is 2 in general and 1 for the two terminal vertices. Now it follows that vertices greater than $k$-hops from the terminal vertices in $G$ will have degree $2k$ in $G^k$. More generally, a vertex $l$-hops from one terminal vertex and $m$-hops from the other terminal vertex will have degree $(\delta(l, k) + \delta(m, k))$ in $G^k$ where $\delta(x, y) = x$ when $x < y$ or $y$ otherwise. If the graph vertices are counted ‘from left to right’ and $l$ is the left-most terminal vertex and $m$ is the right-most terminal vertex, then $\delta(l, k)$ is the number of vertices some vertex $i$ is connected to on the left while $\delta(m, k)$ is the number of vertices the same vertex $i$ is connected to on the right.

In order from the terminal vertex 1 through to vertex $k$, the vertex degrees are thus $k$ through to $2k$ and increasing by 1 from $i$ to $i + 1$ in $\{1, \ldots, k\}$. Moreover, the induced graph on $\{1, \ldots, k\}$ in $G^k$ is complete and will be a seed. Take vertices $\{k + 1, \ldots, |V|\}$. On the left (counting from left to right), vertex $i \in \{k + 1, \ldots, |V|\}$ is connected to $\delta(i, k) = k$ vertices. This completes the proof when $G^k$ is a connected path graph.

A.2 Proof when $G$ is a Traceable Graph

A traceable graph $G$ has a Hamiltonian path. Take this path and apply the preceding argument. The resulting graph $G^k$ has a $k$-lateral subgraph.

A.3 Proof when $G$ is a Tree

Take the maximal path of $G$ and call it $P_G$. Note that so long as $|V(G)| > k$ then $|V(P_G)| > k$. Apply the preceding argument in Section A.1 to $P_G$ such that $P_G^k$ has the $k$-lateral property. In particular, there is a seed in $P_G^k$ of $k$ vertices forming a complete graph. Without detailing where the seeds are, it is easy to claim an order for $V(P_G^k)$ given by $\{1, \ldots, |V(P_G^k)|\}$.

Take any vertex $j$ that is not on $P_G$ but is a one-hop neighbour of a vertex $i$ on $P_G$. It follows that vertex $j$ in $G^k$ is connected to the $k - 1$-hop neighbours of $i$ in $P_G$ and that these vertices (including $i$ itself) are before $j$ in any ordering given that a previously specified order for $V(P_G)$ is $\{1, \ldots, |V(P_G)|\}$.

Take any vertex $l$ that is not on $P_G$, but is a one-hop neighbour of a vertex $i$ on $P_G$ but is a one-hop neighbour of a vertex $j$ which is a one-hop neighbour of vertex $i$ on $P_G$. Succinctly, vertex $l$ is not on $P_G$ but is a two-hop neighbour of a vertex $i$ on $P_G$. Now a trivial argument can be made to show that $l$ is connected to the $k - 2$-hop neighbours of $i$ in $P_G$ and that these vertices (including $i$ itself) are before $l$ in any ordering of the vertices. Ordering $j$ before $l$ on the branch implies immediately that $l$ is connected to $k$ vertices with a lower ordering.

This sequence can be continued to complete the proof.

A.4 Proof when $G$ is an Arbitrary Connected Graph

Consider an arbitrary spanning tree of $G$ denoted by $T_G$. Take this tree and apply the preceding argument. The resulting graph $G^k$ has a $k$-lateral subgraph.