Partial LTR Design of Optimal Output Disturbance Cancellation Controllers for Non-Minimum Phase Plants

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Abstract: For non-minimum phase plants, a partial LTR design of optimal output disturbance cancellation controllers is discussed. A target recoverable by the partial LTR procedure is identified as an output injection with a frequency-shaped gain matrix. The partial LTR result is used to clarify system-theoretic meaning of enforcing the minimum phase LTR procedure. It is shown that the partial LTR procedure provides more design freedom in shaping target feedback property than the enforced minimum phase LTR procedure.

Keywords: Linear systems, Non-minimum phase system, Disturbance cancellation, Loop transfer recovery, Disturbance observer

1. INTRODUCTION

Recently, the design of disturbance cancellation controllers has been formulated as an LQG problem with a performance index explicitly including disturbances. The classical loop transfer recovery (LTR) technique (e.g., Stein and Athans, 1987, Anderson and Moore, 1990, Zheng and Freudenberg, 1990) has been used for systematic design. It should be noted that the standard LTR procedure cannot directly be used due to the lack of the stabilizability of the extended plant consisting of a plant and a disturbance model. For step disturbances entering the plant input side, Guo et al. (1996) have shown that the difficulty can be overcome by a simple modification of the standard LTR procedure. Ishihara et al. (2005, 2008) have extended the modified LTR procedure to the non-minimum phase case using the partial LTR method originally proposed for the standard LQG problems (Xia and Moore, 1987, Ishihara 1995).

In recent conference papers, Ishihara and Guo (2009, 2010) have discussed the LTR design of the optimal disturbance cancellation controllers for step disturbances entering the plant output side. It has been shown that, for minimum phase plants, a procedure similar to the standard LTR procedure can be used for recovering a target identified as estimation error dynamics with integral action. For the non-minimum phase case, they have obtained an explicit representation of the sensitivity matrix obtained by enforcing the recovery procedure found for the minimum phase case. However, the obtained representation does not provide clear system-theoretic meaning on the enforcement of the minimum phase LTR procedure to the non-minimum phase case.

In this paper, we discuss the partial LTR design of output disturbance cancellation controllers for non-minimum phase plants. It is shown that, unlike the standard partial LTR method for LQG controllers, the target feedback property recoverable by the partial LTR method is not a simple frequency-shaped version of a target for the minimum phase case. In addition, we show that the feedback property obtained by enforcing the minimum phase LTR procedure (Ishihara and Guo, 2010) is also obtained by the partial LTR method.

This paper is organized as follows. In section 2, an optimal output disturbance cancellation controller using a partial state estimate feedback is constructed. The target for the partial LTR is given in Section 3. Relation to the enforcement of the minimum phase LTR is discussed in Section 4. Concluding remarks are given in Section 5.

2. DISTURBANCE CANCELLATION CONTROLLER

2.1 Plant and disturbance

Consider a plant with a disturbance given by

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + d(t), \]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input, \( y(t) \in \mathbb{R}^p \) is the output vector and \( d(t) \in \mathbb{R}^q \) is the step disturbance vector satisfying

\[ \dot{d}(t) = 0. \]

It is assumed that \( (A,B,C) \) is a minimal realization with no zero and pole at the origin and non-minimum phase.
Let \( G(s) \) denote the transfer function matrix of the realization \((A,B,C)\). Using the well-known result (e.g., Anderson and Moore, 1990), we can decompose \( G(s) \) as
\[
G(s) = C(sI - A)^{-1} B = G_s(s)C_n(sI - A)^{-1} B,
\]
where \( G_s(s) \) is an all-pass factor satisfying \( G_s(s)G_s(-s) = I \).

Let \((A_s,B_s,C_s,D_s)\) denote a minimal realization of \( G_s(s) \).

Define an extended state vector for a realization of the factorization (3) as
\[
\tilde{\xi}_f(t) = \Phi_f\tilde{\xi}_f(t) + \Gamma_f u(t), \quad y(t) = H_f\tilde{\xi}_f(t),
\]
where
\[
\Phi_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & B_sC_n & A_s \end{bmatrix}, \quad \Gamma_f = \begin{bmatrix} 0 \\ B \end{bmatrix},
\]
\[
H_f = \begin{bmatrix} I & D_sC_n & C_n \end{bmatrix}.
\]

2.2 Disturbance and minimum phase state observer

It can easily be checked that the pair \((H_f, \Phi_f)\) is observable but \((\Phi_f, \Gamma_f)\) is unobservable. By the observability of \((H_f, \Phi_f)\), the estimator for the extended state (4) can be constructed as
\[
\dot{\hat{\xi}}_f(t) = \Phi_f\hat{\xi}_f(t) + \Gamma_f u(t) + K(y(t) - H_f\hat{\xi}_f(t)),
\]
where
\[
\hat{\xi}_f(t) \doteq [d'(t) \ x'_n(t) \ x'_f(t)]',
\]
is an estimate of the extended state vector \(\tilde{\xi}(t)\) and
\[
K = \begin{bmatrix} K_d & K_m' & K_n' \end{bmatrix}
\]
is an optimal observer gain matrix.

To determine the observer gain matrix \(K\), we introduce a stochastic model of the extended plant (5) as
\[
\dot{\xi}_f(t) = \Phi_f\xi_f(t) + \Gamma_f u(t) + \hat{\Gamma}_f v(t), \quad y(t) = H_f\xi_f(t) + v(t),
\]
where \(v(t)\) and \(w(t)\) are mutually independent zero-mean white noise processes with covariance matrices \(\Gamma_f\) and \(W\), respectively, and \(\hat{\Gamma}_f\) is chosen such that \((\Phi_f, \hat{\Gamma}_f)\) is controllable.

The optimal observer gain matrix \(K\) for the stochastic model (10) is given by
\[
K = \Pi_f H_f'V^{-1}
\]
where \(\Pi_f\) is a positive definite solution of the Riccati equation
\[
\Phi_f\Pi_f + \Pi_f\Phi_f' - \Pi_f H_f'V^{-1}H_f\Pi_f + \hat{\Gamma}_f W \hat{\Gamma}_f' = 0.
\]

To construct an output disturbance cancellation controller based on the partial estimate feedback, the estimate of the all-pass state \(x_n(t)\) is not required. Eliminating the all-pass state estimate \(\hat{x}_n(t)\) from (7), we can easily obtain the estimates of the disturbance and the minimum phase state as follows.

**Lemma 1:** Define
\[
\hat{\hat{x}}_m(t) \doteq [d'(t) \ \hat{x}_n(t)]',
\]
where \(\hat{x}_n(t)\) and \(d'(t)\) are estimates given by the observer (7). In addition, define matrices
\[
\Phi_m \doteq \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad \Gamma_m \doteq \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad H_m(s) \doteq \begin{bmatrix} I & G_s(s)C_n \end{bmatrix}.
\]

Then, the estimate \(\hat{\hat{x}}_m(t)\) can be obtained from the observer
\[
\dot{\hat{\hat{x}}}_m(t) = \Phi_m\hat{\hat{x}}_m(t) + \Gamma_m u(t) + K(s)(y(t) - H_m(s)\hat{\hat{x}}_m(t)),
\]
where \(K(s)\) is a frequency-shaped observer gain matrix
\[
K(s) \doteq \begin{bmatrix} K_d(s) \\ K_n(s) \end{bmatrix}
\]
with
\[
K_d(s) = K_d[I + C_s(sI - A_s)^{-1}K_d]^{-1},
K_n(s) = K_n[I + C_s(sI - A_n)^{-1}K_n]^{-1}.
\]
Proof: Omitted. □

For the observer (15), we can easily show the following result on estimation error dynamics.

**Lemma 2**: Define the estimation error vector for the observer (15) as

\[ \hat{\xi}_n(t) \triangleq \xi_n(t) - \hat{\xi}_n(t). \]  

(18)

Then the estimation error \( \hat{\xi}_n(t) \) is described by

\[ \hat{\xi}_n(t) = \Phi_n \hat{\xi}_n(t) - K(s)H_n(s)\hat{\xi}_n(t), \]  

(19)

which can be regarded as a feedback system consisting of open loop error dynamics \( \hat{\xi}_n(t) = \Phi_n \hat{\xi}_n(t) \) and output injection with the frequency-shaped gain matrix \( K(s) \). In addition, the sensitivity matrix at the output side of the estimation error feedback system is given by

\[ S_n(s) \triangleq [I + H(s)(sI - \Phi_n)^{-1}K(s)]^{-1}, \]  

(20)

which can be rewritten as

\[ S_n(s) = \left[ I + \frac{1}{s}K_d(s) + G_n(s)C_n(sI - A)^{-1}K_n(s) \right]^{-1}. \]  

(21)

Proof: Omitted. □

**Remark 1**: Note that the sensitivity matrix of the estimation error dynamics for the disturbance and the state observer for non-factored plant \( (A, B, C) \) is given by

\[ S(s) = \left[ I + \frac{1}{s}K_d + C(sI - A)^{-1}K_s \right]^{-1}, \]  

(22)

where \( K_d \) and \( K_s \) are observer gain matrices for the disturbance and the state, respectively. The sensitivity matrix (21) can be viewed as a frequency-shaped version of (22) in the sense that (21) is obtained from (22) by replacing \( C, K_d, K_s \) with \( G_n(s)C_n, K_d(s) \) and \( K_n(s) \), respectively.

2.3 Disturbance cancellation control law

Introduce a quadratic performance index defined as

\[ J_\rho \triangleq \int_0^\infty \rho[y'(t)y(t) + u'(t)[u(t) - \overline{u}]]dt \]  

(23)

where \( \rho \) is a positive weighting coefficient and \( \overline{u} \) is the steady state vector of \( u(t) \).

On the assumption that all the extended states are perfectly measurable, the optimal control law for the performance index (23) is obtained as follows.

**Lemma 3**: Assume that the extended state (4) is measurable. Consider the state feedback control law

\[ u(t) = -F_n x_n(t) - [G_n(0)C_n(-A + BF_n)^{-1}B] \hat{d}(t), \]  

(24)

where \( F_n \) is a feedback gain matrix such that the matrix \( A - BF_n \) is stable. The feedback gain matrix \( F_n \) minimizing (23) is given by the gain matrix of the standard optimal regulator with the performance index

\[ J_\rho \triangleq \int_0^\infty \rho[y'(t)y(t) + u'(t)[u(t) - \overline{u}]]dt \]  

(25)

for the plant \( (A, B, C_n) \).

Proof: Omitted. □

Applying the separation principle to the above lemma, we can obtain the optimal output feedback control law as follows.

**Proposition 1**: Consider the plant (1) and the disturbance (2). Assume that the estimates \( \hat{x}_n(t) \) and \( \hat{d}(t) \) are obtained from the observer (15). Then, the optimal output feedback control law for the performance index (23) is given by a partial state estimate feedback form as

\[ u(t) = -F_n \hat{x}_n(t) - F_d \hat{d}(t), \]  

(26)

where \( F_n \) is the optimal feedback gain matrix minimizing the performance index (25) for the minimum phase plant \( (A, B, C_n) \) and

\[ F_d = [G_n(0)C_n(-A + BF_n)^{-1}B]^{-1}. \]  

(27)

Proof: Omitted. □

The structure of the output feedback control system is shown in Fig. 1.

3. PARTIAL LTR

3.1 Output sensitivity matrix

The optimal output feedback control (26) can be written as

\[ u(t) = -\Psi \hat{\xi}_n(t), \]  

(28)

where

\[ \Psi \triangleq [F_n, F_d]. \]  

(29)

and \( \hat{\xi}_n(t) \) is defined in (13). The transfer function matrix of the output feedback controller from \( y(t) \) to \( -u(t) \) can be expressed in the right factorization form as

\[ C(s) \triangleq \Psi(sI - \Phi_n + \Gamma_n \Psi)^{-1}K(s) \times [I + H_n(s)(sI - \Phi_n + \Gamma_n \Psi)^{-1}K(s)]^{-1}, \]  

(30)

where \( \Phi_n, \Gamma_n, H_n(s) \) are defined in (14) and \( K(s) \) is given by (16) and (17).

Using the expression (30), we can easily obtained a representation of the output sensitivity matrix.

**Lemma 4**: Consider the control system consisting of the plant (1) and the output feedback controller (30). The sensitivity matrix at the plant output side can be expressed as
\[ \Sigma_n(s) \triangleq [I + H_n(s)(sI - \Phi_n + \Gamma_n \Psi)^{-1}K(s)]S_n(s), \]  

(31)

where \( S_n(s) \) is the sensitivity matrix defined in (21) related to estimation error dynamics of the observer (15) with the frequency-shaped gain matrix (16).

Proof: Omitted. \( \square \)

### 3. 2 Asymptotic expression of output sensitivity matrix

Using Lemma 4, we can obtain an asymptotic expression of the output sensitivity matrix (31) as follows.

**Proposition 2:** Assume that the frequency-shaped observer gain matrix \( K(s) \) is fixed. Consider the optimal feedback gain matrix \( F_n \) for the performance index (23). Then, as the weighting coefficient \( \rho \) in the performance index (25) approaches infinity, the sensitivity matrix \( \Sigma_n(s) \) given by (31) for the output feedback controller approach the matrix

\[ \Sigma_n(s) \triangleq \Theta(s)S_n(s), \]  

(32)

where \( S_n(s) \) is the sensitivity matrix defined in (21) and

\[ \Theta(s) \triangleq I + \frac{1}{s}[I - G_n(s)G_n^{-1}(0)]K_s(s). \]  

(33)

Proof: Note that the optimal feedback gain matrix \( F_n \) for the performance index (23) is obtained as an optimal gain matrix for the standard regulator problem with the performance index (25). It follows from the well known result for the asymptotic behaviour of the optimal gain matrix for the standard regulator problem (e.g., Anderson and Moore), that the optimal gain matrix \( F_n \) for the performance index (20) with sufficiently large \( \rho \) can be written as

\[ F_n = \rho C_n, \]  

(34)

where \( C_n \) is the output matrix of the minimum phase image of \( G(s) \). Using the submatrices in (14) and (16), we can write

\[ H(s)(sI - \Phi_n + \Gamma_n \Psi)^{-1}K(s) = G_n(s)C_n(sI - A + BF_n)^{-1}K_n(s) + \frac{1}{s}K_s(s), \]  

(35)

\[ -\frac{1}{s}G_n(s)C_n(sI - A + BF_n)^{-1}B \times [G_n(0)C(-A + BF_n)^{-1}B]^{-1}K_s(s). \]

Replacing \( F_n \) in (35) with (34), we can write the first and third matrices in (35) as

\[ G_n(s)C_n(sI - A + BF_n)^{-1} \]  

(36)

\[ = G_n(s)C_n(sI - A)^{-1}G_n(s)C_n(sI - A)^{-1}B \]  

\[ = [I + \rho C_n(sI - A)^{-1}B]^{-1}\rho C_n(sI - A)^{-1} \]  

\[ \to 0 \quad (\rho \to \infty), \]

where we have used the matrix inversion lemma and the assumption that the matrix \( A \) is non-singular. Using (35)-(37) in (31), we can easily show that the sensitivity matrix \( \Sigma_n(s) \) approaches \( \Sigma_n(s) \) defined in (32) as \( \rho \to \infty \). It is worth noting that the matrix (33) has no pole at \( s = 0 \). \( \square \)

The above result includes the following results as special cases.

**Remark 2:** For minimum phase plants, the sensitivity matrix (32) is reduced to \( S(s) \) defined in (22) since it is obvious that \( \Theta(s) = I \) and \( S_n(s) = S(s) \). Therefore, the above result includes the result of Ishihara and Guo (2009) as a special case.

**Remark 3:** Assume that the observer gain matrix \( K_s(s) \) for the disturbance estimation is set to zero, then it readily follows from (33) that \( \Theta(s) = I \). The recoverable sensitivity matrix (32) reduces to

\[ \Sigma_n(s) = S_n(s) = \left[ I + G_n(s)C_n(sI - A)^{-1}K_n(s) \right]^{-1}, \]  

(38)

which is a sensitivity matrix recovered by the partial LTR method for the standard LQG problem. Note that (38) is a frequency-shaped version of the target for the minimum phase case.

### 3. 3 Target for the partial LTR

Note that the system-theoretic meaning of the sensitivity matrix \( S_n(s) \) in (32) is clarified in Lemma 2 in terms of estimation error dynamics of the disturbance and minimum phase state observer with frequency-shaped observer gain matrix. It is tempting to consider that \( S_n(s) \) is a recoverable target sensitivity matrix. Unfortunately, Proposition 2 shows that this is not the case.

The sensitivity matrix (32) can be used for the target sensitivity matrix recoverable by the partial LTR procedure using the weighting coefficient \( \rho \) in the performance index (25). However, the expression (32) does not readily provide its system-theoretic meaning.

The following result clarifies the meaning of (32).

**Proposition 3:** Consider the disturbance and the minimum phase state observer with frequency-shaped observer gain matrices defined by

\[ K_j(s) \triangleq G_n(s)G_n^{-1}(0)K_s(s)\Theta^{-1}(s), \]  

(39)

\[ K_m(s) \triangleq K_n(s)\Theta^{-1}(s), \]

where \( K_j(s) \) and \( K_m(s) \) are frequency-shaped observer gain matrices defined by

\[ G_n(s)C_n(sI - A + BF_n)^{-1}B[\begin{bmatrix} G_n(0)C_n(0) \end{bmatrix}^{-1}B]^s \]  

(37)

\[ = G_n(s)C_n(sI - A)^{-1}B[\begin{bmatrix} G_n(0)C_n(0) \end{bmatrix}^{-1}B]^s \]  

\[ \times [I + \rho C_n(sI - A)^{-1}B][\begin{bmatrix} G_n(0)C_n(0) \end{bmatrix}^{-1}B]^s \]  

\[ \to G_n(s)G_n^{-1}(0) \quad (\rho \to \infty), \]
matrices used in the controller (30) and \( \Theta(s) \) is defined in (33). Then the estimation error is generated by the output injection shown in Fig. 2 and the sensitivity matrix at the output side of the open loop error dynamics is given by (32).

Proof: Let \( S_n^\prime(s) \) denote the sensitivity matrix at the point \( P_o \) in the Fig.2. It readily follows from Fig. 2 that

\[
S_n^\prime(s) = \left[ I + \frac{1}{s} K_\beta(s) + G_n(s)C_n(s) - A \right]^{-1} K_\alpha^\prime(s). \quad (40)
\]

Using the expression (39) in (40), we can easily show that \( S_n^\prime(s) = \Sigma_n^\prime(s) \). □

The frequency-shaped observer gain matrices defined in (39) have somewhat complex structure, which reflects the non-standard structure of the disturbance cancellation controller.

The design procedure based on the partial LTR method is summarized as follows:

Step 1: By solving the Riccati equation (12) with covariance matrices as design parameters, determine the observer gain matrices (9) such the target system shown in Fig. 2 has desired feedback properties.

Step 2: Construct the output feedback disturbance cancellation controller (26) with the observer gain matrices determined in Step 1. Increase the scalar design parameter \( \rho \) until the feedback properties of the output feedback controller are sufficiently close to the target.

4. RELATION TO THE ENFORCED PROCEDURE

For the output disturbance cancellation controllers, Ishihara and Guo (2010) have obtained explicit representation of the sensitivity matrix achieved by enforcing the minimum phase LTR procedure (Ishihara and Guo, 2009). In this section, we discuss the relation between the two LTR procedures.

The result obtained by Ishihara and Guo (2010) is summarized as follows.

Lemma 5: Consider an optimal output disturbance cancellation controller for the non-minimum phase plant \((A, B, C)\). Assume that an optimal disturbance and state observer with the optimal gain matrix

\[
L = \begin{bmatrix} L_o \\ L_s \end{bmatrix}
\]

is used in the controller. In addition, assume that the optimal feedback gain matrix for the performance index (25) is used in the controller. Then, as the weighting coefficient \( \rho \) in the performance index (25) tends to infinity, the sensitivity matrix at the plant output side approaches

\[
\tilde{S}(s) \triangleq \Xi(s)S(s),
\]

where \( S(s) \) is the sensitivity matrix defined in (19) and

\[
\Xi(s) \triangleq I + \left[ C - G_n(s)C_n(s) - A \right]^{-1} L_s + \frac{1}{s} \left[ I - G_n(s)G_n(0) \right] L_o.
\]

Proof: See Ishihara and Guo (2010). □

In this section, we assume that the optimal observer gain matrix (41) is obtained from the stochastic model

\[
\begin{align*}
\dot{\xi}(t) &= \Phi \xi(t) + \Gamma u(t) + \bar{F}w(t), \\
y(t) &= H \xi(t) + v(t),
\end{align*}
\]

where

\[
\xi(t) \triangleq \begin{bmatrix} d'(t) \\ x'(t) \end{bmatrix},
\]

\[
\Phi \triangleq \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad \Gamma \triangleq \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \bar{F} \triangleq \begin{bmatrix} \bar{B} \\ B \end{bmatrix}, \quad H \triangleq \begin{bmatrix} I & C \end{bmatrix},
\]

\( v(t) \) and \( w(t) \) are mutually independent zero-mean white noise processes with the covariance matrices \( W \) and \( V \), respectively, and \( \bar{B} \) in \( \bar{F} \) is chosen such that \((\Phi, \bar{F})\) is controllable.

The optimal observer gain matrix for the above stochastic model is given by

\[
L = \Pi H' V^{-1}
\]

where \( \Pi \) satisfies the Riccati equation

\[
\Phi \Pi + \Pi \Phi' - \Pi H' V^{-1} H \Pi + \bar{F} W \bar{F}' = 0.
\]

To simplify discussion, we assume that the plant has a single unstable zero. The following result gives an explicit representation of the all-pass factor.

Lemma 6: Assume that the plant has a single real unstable zero at \( s = z \ (z > 0) \). Let \( \xi \in \mathbb{R}^n \) and \( \eta \in \mathbb{R}^m \) denote the vectors satisfying

\[
\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} \begin{bmatrix} zI - A \\ -B \\ -C \\ 0 \end{bmatrix} = 0, \quad \eta' \eta = 1.
\]

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Then \( G(s) = C(sI - A)^{-1}B \) is factored as
\[
G(s) = G_a(s)C_a(sI - A)^{-1}B, \tag{50}
\]
where
\[
G_a(s) = I - \frac{2s\eta^2}{s + a}, \quad C_a = C - 2s\eta^2\xi^T.
\tag{51}
\]
Proof: The factorization is obtained as a special case of the general result. See e.g., Anderson and Moore (1990).

Using the above representation, we can show the relation between the solutions of the Riccati equations (12) and (48).

**Lemma 7:** Assume that the matrix \( \hat{F}_j \) in (12) is chosen as
\[
\hat{F}_j = \begin{bmatrix} \hat{F} \\ 0 \end{bmatrix}, \tag{52}
\]
where \( \hat{F} \) is defined in (46). Then the solution \( \Pi_j \) of the Riccati equation (12) is expressed as
\[
\Pi_j = N\Pi N', \tag{53}
\]
where \( \Pi \) is a solution of the Riccati equation (48) and
\[
N = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & -\xi^T \end{bmatrix}. \tag{54}
\]
Proof: Using the identities
\[
\Phi_j N = N\Phi, \quad N\hat{F} = \hat{F}_j, \quad H_j N = H, \tag{55}
\]
which can easily be checked by use of Lemma 5, we can show that relation (53) holds for the solutions of the Riccati equations (12) and (48).

The following result gives a clear system-theoretic meaning for enforcing the LTR procedure effective for the minimum phase case to the non-minimum phase case.

**Proposition 4:** Assume the stochastic models (10) and (48) with (52) are used to determine the optimal gain matrices \( K \) and \( L \), respectively. Then, the sensitivity matrix defined in (42) coincides with (32), i.e.,
\[
\hat{S}(s) = \Sigma^*_a(s). \tag{56}
\]
Proof: Using Lemma 7, we can show that
\[
K_a = L_a, \quad K_m = L_m, \quad K_s = -\xi^T L_s. \tag{57}
\]
Choosing a realization of the all-pass factor \( G_a(s) \) as
\[
A_a = -z, \quad B_a = \eta^T, \quad C_a = -2s\eta, \quad D_a = I, \tag{58}
\]
we can show that
\[
(C - G_a(s)C_m)(sI - A)^{-1}L_s = C_a(sI - A_a)^{-1}K_s, \tag{59}
\]
Note that the frequency-shaped gain matrices \( K_a(s) \) are given by (17). Using (57) and (59) in (42) and (43), we can show that the identity (56) holds.

The above result shows that the feedback property achieved by enforcing the minimum phase LTR procedure coincides with the target feedback property recoverable by the partial LTR method.

**Remark 4:** The result in Proposition 4 can easily be extended to plants with multiple unstable zeros using the generalized version of the factorization given in Lemma 6.

**Remark 5:** Note that the assumption (52) is essential to obtain the equivalence result (56). However, the assumption is not necessary for the target of the partial LTR method. For example, \( \hat{F}_j \) with the second block row in (52) needs not to be a zero matrix. Therefore, the partial LTR method provides more design freedom than the enforced LTR procedure in shaping target feedback property.

5. CONCLUDING REMARKS

Extensions to more general class of output disturbances are under current investigation.

REFERENCES


