Time Domain Analysis of Steady State Response in Linear Periodic Systems and Its Application to Switching Converters
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Abstract: This note discusses the steady state response of a linear periodic system with a periodic input, especially for linear systems with periodic and piecewise continuous coefficients. The first order approximation of the steady state response with respect to the period of time is obtained for general linear periodic systems. This result is then applied to analyze the buck-boost converter operating in continuous conduction mode. This mathematically clarifies conventional results of switching converters which have been deduced under physical inspections or assumptions.

Keywords: Periodic Structures, Periodic Systems, Periodic Solution, Linear Systems, Time-varying Systems, Converters, Switching Converters, Buck-boost Converters, Power Electronics, Steady State Analysis

1. INTRODUCTION

Switching converters are highly efficient electric power conversion devices and widely used for stabilized power sources. The steady state analyses of the switching converters such as the time averages or the peak-to-peak values of the load voltage or the inductance current are one of the important research issues in the field of power electronics, see e.g., Erickson (2001); Kassakian (1991); Harada (1991); Krein (1990). Several indexes are often obtained from circuit theoretic observations under physical inspections or assumptions. In the author’s opinion, the existences of periodic voltage and current variations in the converters are implicitly assumed but are not mathematically well defined; moreover, the indexes are approximated from those periodic variations.

In order to develop an alternative systematic procedure for analyzing the steady state properties of switching converters, we introduce the concept of steady state response in linear periodic systems. A switching converter operating in continuous conduction mode (CCM) can be modeled by a linear periodic system, whose homogeneous system is asymptotically stable and is stimulated by a periodic signal. In spite of any initial capacitor charge and any initial inductance current, the load voltage and the inductance current converge to the same periodic variations; therefore, the asymptotic stability is crucial in addition to the periodicity. We firstly show that there is a unique asymptotically stable periodic solution, which is called the steady state response, in the linear periodic system. The elements of this steady state response correspond to periodic voltage and current variations in the switching converter. Then, we develop a first order approximation of the steady state response when the period is sufficiently small. Several indexes such as the time averages or the peak-to-peak values of the load voltage or the inductance current can be recalculated with the error bound bounded by the square order of the period.

This note is organized as follows: In Section 2, we illustrate key features of steady state response for a linear time-invariant system. In Section 3, the concept of steady state response is defined by an asymptotically stable periodic solution in a linear periodic system. Then, we prove the unique existence of steady state response when the homogeneous system is asymptotically stable. In Section 4, we develop a first order approximation of the steady state response when the period is sufficiently small. In Section 5, the effectiveness of the proposed method is demonstrated by an illustrative example for the buck-boost converters. The conclusions are given in Section 6.

We use the following notations: Z, R, R^n, R^{n×m} denote the set of integers, real numbers, real vectors, real matrices, respectively. ∥x∥ denotes the Euclidean norm for a vector x ∈ R^n. ∥X∥ denotes the maximum singular value for a matrix X ∈ R^{n×m}. ρ(X) denotes the spectral radius, i.e., the maximum among the absolute values of the eigenvalues, of a matrix X ∈ R^{n×n}. If the function f(t) is periodic with a period T > 0, i.e., f(t + T) = f(t) for all t ∈ R, it is called T-periodic.

2. BACKGROUND

Let us consider a response of a linear time-invariant system

\[ \dot{x} = Ax + Bu, \quad \dot{x} := \frac{dx}{dt} \quad (1) \]

where x denotes a state vector, u denotes a T-periodic input vector, A and B are constant matrices with ap-
propriate dimensions. Suppose that all real parts of the
eigenvalues of $A$ are negative, i.e., the homogeneoussystem
$\dot{x} = Ax$ is asymptotically stable. By taking the Fourier
transformation of (1), the frequency response from the input
$U(j\omega)$ to the state $X(j\omega)$ is described by

$$X(j\omega) = (j\omega I - A)^{-1}BU(j\omega)$$

(2)

where the initial state is 0, i.e., $x(0) = 0$. This frequency
domain analysis has been extended to linear periodic sys-
tems, e.g., several approaches are summarized in Sandberg
(2005); however, it is not straightforward to analyze the
time domain response from those frequency domain analy-

Instead, let us consider a periodic response of (1) by
directly integrating (1) as follows:

$$x(t) = e^{AT}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau,$$

(3)

where $x(0)$ will be appropriately selected so that $x(t)$
becomes $T$-periodic. To this end, by substituting $t = T$
into (3), we have

$$x(0) = x(T) = e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau.$$  

(4)

By assumption, $\dot{x} = Ax$ is asymptotically stable; therefore,
we have $\rho(e^{AT}) < 1$. It follows that (1) has a unique
$T$-periodic solution by selecting the initial state by

$$x(0) = x_{ss0} := (I - e^{AT})^{-1}e^{AT} \int_0^T e^{-AT} Bu(\tau)d\tau.$$  

(5)

If $x(0) \neq x_{ss0}$, (3) is decomposed as

$$x(t) = e^{AT}\delta x(0) + x_{ss}(t),$$

(6)

where

$$x_{ss}(t) := e^{AT}x_{ss0} + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

(7)

is shown to be $T$-periodic and

$$\delta x(0) := x(0) - x_{ss0}$$
denotes the initial difference from the $T$-periodic solution
$x_{ss}(t)$. It follows from (6) that $x_{ss}(t)$ is an asymptotically
stable $T$-periodic solution. Namely, $x_{ss}(t)$ is a solution of
(1) with a period $T$, i.e.,

$$x_{ss}(t+T) = x_{ss}(t), \quad \forall t,$$

and any other solution asymptotically converges to $x_{ss}(t)$, i.e.,

$$\lim_{t \to \infty} \|x(t) - x_{ss}(t)\| = 0, \quad \forall x(0).$$

Compared with the frequency response $X(j\omega)$ in (2), the
unique $T$-periodic solution $x_{ss}(t)$ in (7) directly charac-
terizes the asymptotic properties of $x(t)$. Hence, we will
extend the above time domain analysis into linear periodic
systems.

3. STEADY STATE RESPONSE OF LINEAR
PERIODIC SYSTEMS

Consider a linear $T$-periodic system

$$\dot{x} = A(t)x + b(t)$$

(8)

where $x \in \mathbb{R}^n$ denotes a state vector, $A(t) \in \mathbb{R}^{n \times n}$ and
$b(t) \in \mathbb{R}^n$ are supposed to be $T$-periodic and piecewise
continuous with respect to a time $t$. We note that, since we
will analyze the periodic solution for a fixed periodic con-

control input, we introduce (8) instead of $\dot{x} = A(t)x + B(t)u$
by substituting $b(t)$ for $B(t)u$ for notational simplicity. We
also note that, in order to discuss a switching converter,
we consider piecewise continuous $A(t)$ and $b(t)$ rather than
continuous ones.

At the discontinuous points of $A(t)$ or $b(t)$, the time
derivative $\dot{x}$ is not well defined; therefore, it is not
possible to extend the solution $x(t)$ along (8). Instead, we
introduce the integral equation

$$x(t) = x(0) + \int_0^t (A(p)x(p) + b(p))dp$$

(9)

which corresponds to (8). Then, (9) has a unique abso-
olutely continuous solution which is defined over $\mathbb{R}$ for
any choice of $x(0)$; see Coddington (1955). Hereafter, we will
identify the differential equation in (8) and the correspond-
ing integral equation in (9).

Let us prepare necessary notations to represent the so-
lution of (8). Let $\Phi(t)$ denotes a solution of the integral
equation

$$\Phi(t) = I + \int_0^t A(p)\Phi(p)dp,$$

(10)

$$\Phi(0) = I.$$  

(11)

Then, (10) and (11) have a unique absolutely continuous
solution; see Coddington (1955). $\Phi(t)$ becomes a funda-
mental solution of the homogeneous equation

$$\dot{x} = A(t)x,$$

(12)

or equivalently,

$$x(t) = I + \int_0^t A(p)x(p)dp.$$  

(13)

A matrix

$$Q := \Phi(T)$$

(14)

is called the monodromy matrix of (12). Any solution $x(t)$
of (8) is represented by

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t)^{-1}\Phi(p)b(p)dp.$$  

(15)

Let us introduce the concept of steady state response.

Definition 1. Let $x_p(t)$ denotes a solution of (8). $x_p(t)$ is
said to be $T$-periodic if

$$x_p(t+T) = x_p(t), \quad \forall t.$$  

□

Definition 2. Let $x_a(t)$ denotes a solution of (8). $x_a(t)$ is
said to be asymptotically stable if

$$\lim_{t \to \infty} \|x(t) - x_a(t)\| = 0,$$

for any other solution $x(t)$ of (8).

□

Definition 3. Let $x_{ss}(t)$ denotes a solution of (8). $x_{ss}(t)$ is
said to be a steady state response of (8) if it is $T$-periodic
and asymptotically stable.

□

Now, we explicitly prove the existence of steady state
response for asymptotically stable systems.

Theorem 1. Consider the linear $T$-periodic system in (8)
with piecewise continuous $A(t)$ and $b(t)$. Suppose that
(12) is asymptotically stable. Then, there exists a unique
steady state response of (8) given by

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\[ x_{ss}(t) := \Phi(t)(I - Q)^{-1} \int_0^T \Phi(t - p)^{-1} b(t - p) dp. \] (16)

The outline of the proof is as follows: Firstly, we prove that (13) has a unique \( T \)-periodic solution if \( \det(I - Q) \neq 0 \). This fact has been proved in Halanay (1966) for continuous \( A(t) \) and \( B(t) \); then, we extend this result to piecewise continuous \( A(t) \) and \( B(t) \) in the same way. It can be shown that \( x_{ss}(t) \) in (16) is \( T \)-periodic indeed. Next, by decomposing the solution \( x(t) \) in the similar way to (6), it can be shown that \( x_{ss}(t) \) is asymptotically stable. Hence, \( x_{ss}(t) \) is a unique steady state response of (8).

We note that \( x_{ss}(t) \) in (16) depends on the choice of \( b(t) \) which includes the effects of control input. However, its existence is guaranteed regardless of the choice of \( b(t) \) when (12) is asymptotically stable.

We also note that the \( T \)-periodic solution in Halanay (1966) is given by

\[ x_{ss}(t) = \int_0^T K(t, p)b(p) dp \]

\[ K(t, p) := \begin{cases} \Phi(t)(I - Q)^{-1}\Phi(p)^{-1} & (p \leq t) \\ \Phi(t + T)(I - Q)^{-1}\Phi(p)^{-1} & (t \leq p) \end{cases}, \]

which appears to be different from \( x_{ss}(t) \) in (16). However, by using the relation \( \Phi(t + T) = \Phi(t)Q \) and \( b(t + T) = b(t) \), it can be shown that they are equivalent indeed. Hereafter, we continue our discussion based on the simpler form of \( x_{ss}(t) \) in (16).

4. FIRST ORDER APPROXIMATION OF STEADY STATE RESPONSE

In practice, it is often requested to compute the average or the envelop of the steady state response rather than the rigorous one. Hence, we consider the first order approximation of the steady state response when the period \( T \) is sufficiently small.

As discussed in the previous section, we consider the linear \( T \)-periodic system in (8). We suppose that \( A(t) \) and \( b(t) \) are piecewise continuous and that (12) is asymptotically stable. Moreover, we suppose that \( A(t) \) and \( b(t) \) are represented by time-scale transformations of 1-periodic matrix-valued functions.

Assumption 1. Suppose that \( A(t) \) and \( b(t) \) are generated from

\[ A(t) = A_m \left( \frac{t}{T} \right), \quad b(t) = b_m \left( \frac{t}{T} \right), \quad \forall t, \]

(17)

where \( A_m(t) \) and \( b_m(t) \) are 1-periodic and piecewise continuous.

We note that this assumption is always satisfied for a fixed \( T \). We impose this assumption in order to admit the time-scale contraction as a function of \( T \). This operation is motivated by analyzing switching converters, which will be demonstrated in the next section.

Under Assumption 1, we introduce the following notations:

\[ \|A\| := \sup_{t \in [0, T]} \|A(t)\| = \sup_{t \in [0, 1]} \|A_m(t)\| \]

(18)

\[ \|b\| := \sup_{t \in [0, T]} \|b(t)\| = \sup_{t \in [0, 1]} \|b_m(t)\| \]

(19)

\[ \bar{A} := \frac{1}{T} \int_0^T A(t) dt \]

(20)

\[ \bar{b} := \frac{1}{T} \int_0^T b(t) dt \]

(21)

\[ \bar{A}_m := \frac{1}{T^2} \int_0^T A(p) dp \]

(22)

Since \( A_m(t) \) and \( b_m(t) \) are 1-periodic, \( \|A\|, \|b\|, \bar{A}, \bar{b}, \bar{A}_m \) does not depend on the period \( T \).

Let us introduce the Landau notation for matrix-valued functions.

Definition 4. Let \( H(T) \in \mathbb{R}^{n \times m} \) denotes a function of \( T > 0 \) and \( k \) denotes a nonnegative integer. Then, \( H(T) \) is denoted by

\[ H(T) = O(T^k) \]

as \( T \) goes to 0 if and only if there exists positive constants \( T_1 \) and \( M \) such that the norm of \( H(t) \) is bounded by

\[ \|H(T)\| \leq MT^k, \quad 0 < \forall T \leq T_1. \]

(23)

Now, by applying the successive approximation method in Coddington (1955), we derive the first order approximation of \( x_{ss}(t) \) in (16) when the period \( T \) is sufficiently small.

Theorem 2. Consider the linear \( T \)-periodic system in (8) with piecewise continuous \( A(t) \) and \( B(t) \). Suppose that (12) is asymptotically stable and that Assumption 1 is satisfied. Then, the steady state response \( x_{ss}(t) \) in (16) is approximated as follows:

\[ \sup_{t \in \mathbb{R}} \left\| x_{ss}(t) - \left( \xi_0 + T \left\{ \xi_{10} + \xi_{11} \left( \frac{t}{T} \right) \right\} \right\| = O(T^2), \]

(24)

where \( \xi_0 \) and \( \xi_{10} \) are constant vectors defined by

\[ \xi_0 := -\bar{A}^{-1}\bar{b} \]

(25)

\[ \xi_{10} := \bar{A}^{-1}\bar{A}_m \bar{A}^{-1}\bar{b} - \bar{A}^{-1} \int_0^1 A_m(p) \int_0^p b_m(q) dq dp, \]

and \( \xi_{11} \) is a 1-periodic and absolutely continuous vector-valued function defined by

\[ \xi_{11}(\alpha) := \int_0^\alpha b_m(p) dp - \int_0^\alpha A_m(p) dp \bar{A}^{-1}\bar{b}. \]

(26)

Due to the page limitation, we omit the proof, which is based on the successive approximation method for an integral equation on a bounded interval in Coddington (1955).

In (23), \( \xi_0 + T \left\{ \xi_{10} + \xi_{11} \left( \frac{t}{T} \right) \right\} \) denotes the first order approximation of \( x_{ss}(t) \). We note that \( \xi_0 \) and \( \xi_{10} \) are constant vectors which depends neither \( t \) or \( T \) and that \( \xi_{11}(\alpha) \) is a vector-valued function which does not depend
on $T$; in other words, the function form of $\xi_{11}(\alpha)$ is fixed regardless of the choice of $T$. Hence, once $\xi_0$, $\xi_{10}$, and $\xi_{11}(\alpha)$ are computed, the above first order approximation can be obtained by time scale transformation for any $T$.

Let us compare the first order approximation with the conventional approximation obtained from the state averaged model

$$\dot{x}_{\text{ave}} = \bar{A}x_{\text{ave}} + \bar{b}. \quad (27)$$

The equilibrium point of (27) gives the zeroth order component $\xi_0$; however, it is not possible to obtain the first order component. Hence, the more accurate approximation is obtained in Theorem 2.

In general, it is difficult to calculate the fundamental matrix $\Phi(t)$ analytically; only its existence and a few algebraic properties are known. In contrast, as shown in (18)–(22) and (24)–(26), the first order approximation can be directly calculated from $A(t)$ and $b(t)$. Hence, the above first order approximation may relax a computational problem to calculate the accurate function in (16).

Moreover, if $A(t)$ and $b(t)$ are piecewise constant, it follows from (26) that the first order approximation becomes piecewise linear with respect to time $t$. Then, approximate values of the average and the peak-to-peak value can be easily computed. These approximate calculations are important in analyzing switching converters; in fact, the authors are motivated by this problem. In the next section, we illustrate the effectiveness of the proposed method for the buck-boost converter.

5. APPLICATION TO SWITCHING CONVERTERS

The proposed method can be applicable to analyze the static properties of any switching converters operating in continuous conduction mode (CCM). In this section, we analyze the buck-boost converters for illustrative purposes.

5.1 Modeling Buck-Boost Converters

Consider a buck-boost converter consisting of a voltage source $V_i$, an ideal switch $S$, an inductor $L$, a capacitor $C$, and a resistive load $R$ in Fig. 1. Dynamics of the converter can be represented by $i_L$, the current through the inductor $L$, and $v_o$, the voltage across the load $R$. We assume that the state of switching devices $S$ and $D$ can be either ‘$S$=ON and $D$=OFF’, or ‘$S$=OFF and $D$=ON’.

That is, the converter always behaves in CCM.

We will analyze the voltage conversion from $V_i$ to $v_o$ when $S$ repeats ON and OFF periodically with a period $T$. The duty ratio $d$ is defined by the ratio of duration ON divided by the whole duration ON and OFF. $d$ satisfies $0 < d < 1$ by definition. In practice, the buck-boost converter does not work ideally; for example, the load may be fluctuated, there may be effects of parasitic circuit elements, and $d$ is feedback controlled in order to keep $v_o$ almost constant. However, the effect of feedback control can be regard as quasi stationary compared with the switching period; accordingly, the duty ratio $d$ is supposed to be constant.

In this case, the buck-boost converter can be modeled by a nonlinear periodic system by using the Kirchhoff’s current law and Kirchhoff’s voltage law. Here, we suppose that the buck-boost converter operates in CCM. Then, the buck-boost converter can be modeled by a linear periodic system:

$$\dot{x} = A(t)x + b(t), \quad x := \begin{bmatrix} v_o \\ i_L \end{bmatrix} \quad (28)$$

where

$$A(t) := A_m \left( \frac{t}{T} \right) \quad (29)$$

$$b(t) := b_m \left( \frac{t}{T} \right) \quad (30)$$

$$A_m(t) := \begin{cases} A_1 & (k \leq t < k + d, \ k \in \mathbb{Z}) \\ A_2 & (k + d \leq t < k + 1, \ k \in \mathbb{Z}) \end{cases} \quad (31)$$

$$b_m(t) := \begin{cases} b_1 & (k \leq t < k + d, \ k \in \mathbb{Z}) \\ b_2 & (k + d \leq t < k + 1, \ k \in \mathbb{Z}) \end{cases} \quad (32)$$

$$A_1 := \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix} \quad (33)$$

$$A_2 := \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \quad (34)$$

$$b_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (35)$$

$$b_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (36)$$

It is obvious that $A(t)$ and $b(t)$ satisfies Assumption 1.

5.2 Asymptotic Stability and Existence of Steady State Response in Buck-boost Converters

In this subsection, we study the asymptotic stability of $\dot{x} = A(t)x$ and the existence of the steady state response in the buck-boost converter. To this end, we study the eigenvalues of the monodromy matrix $Q$, where

$$Q = \Phi(T) = e^{A_2T(1-d)} e^{A_1Td}.$$  

The characteristic polynomial of $Q$ satisfies the following identity:

$$\det(Q - \lambda I) = \lambda^2 - (\tr Q)\lambda + \det Q,$$

where $\det Q$ is the determinant of $Q$ and $\tr Q$ is the trace of $Q$. By a direct calculation, we have

$$\tr Q = 2e^{\sigma g}, \quad \det Q = e^{2\sigma},$$

where

$$\sigma := -\frac{T}{2RC} \quad (< 0)$$

$$g := f_1(1 - d) \cosh \sigma d + \sigma f_2(1 - d) \sinh \sigma d.$$
Let $\lambda_1$ and $\lambda_2$ denote the eigenvalues of $Q$. Firstly, we consider the case where $\lambda_1$ and $\lambda_2$ are complex numbers. Since $Q$ is a real matrix, $\lambda_1$ and $\lambda_2$ are complex conjugate. It follows from $\lambda_1 \lambda_2 = \det Q = e^{2\sigma} < 1$ that $|\lambda_1|^2 = |\lambda_2|^2 < 1$. Hence, we have $\rho(Q) < 1$. Next, we consider the case where $\lambda_1$ and $\lambda_2$ are real numbers. By a direct calculation, we have

$$\frac{\partial g}{\partial d} = \omega^2 f_2(1 - d) \cos \sigma d.$$

It then follows that $|g| < \cos \sigma$ for $0 < d < 1$. By a direct calculation, we have

$$\det(\pm I - Q) = 1 \mp 2e^{\sigma} g + e^{2\sigma} > 1 - 2e^{\sigma} \cos \sigma + e^{2\sigma} = 0.$$ 

By using these inequalities, it can be shown that $\rho(Q) < 1$.

Consequently, we have $\rho(Q) < 1$ for any $d$ satisfying $0 < d < 1$. It follows that $\dot{x} = A(t)$ is asymptotically stable. The prior condition for Theorem 1 is satisfied; therefore, there is a unique steady state response in the buck-boost converter operating in CCM.

5.3 Explicit Description of Steady State Response in Buck-boost Converters

In this subsection, we obtain the explicit description of the state response in the buck-boost converter. By substituting the fundamental solution

$$\Phi(t) = \begin{cases} 
e^{\lambda_1(t-kT)Q^k}, & (kT \leq t < (k+d)T, k \in \mathbb{Z}) \\ e^{\lambda_2(t-(k+d)T)Q^k}, & ((k+d)T \leq t < (k+1)T, k \in \mathbb{Z}) \end{cases}$$

and $b(t)$ into (16), the steady state response $x_{ss}$ can be explicitly computed as follows:

$$x_{ss}(t) = \begin{cases} v_{oss}(t) i_{Lss}(t) \\ i_{Lss}(t) \end{cases},$$

where $v_{oss}(t)$ and $i_{Lss}(t)$ are $T$-periodic piecewise continuous functions given by

$$v_{oss}(t) := \begin{cases} v_{oss1} \left( \frac{t}{T} - k \right), & (kT \leq t < (k+d)T, k \in \mathbb{Z}) \\ v_{oss2} \left( \frac{t}{T} - k \right), & ((k+d)T \leq t < (k+1)T, k \in \mathbb{Z}) \end{cases}$$

and $v_{oss1}(t)$, $v_{oss2}(t)$, $i_{Lss1}(t)$, $i_{Lss2}(t)$ are continuous functions given by

$$i_{Lss1}(t) := \begin{cases} i_{Lss1} \left( \frac{t}{T} - k \right), & (kT \leq t < (k+d)T, k \in \mathbb{Z}) \\ i_{Lss2} \left( \frac{t}{T} - k \right), & ((k+d)T \leq t < (k+1)T, k \in \mathbb{Z}) \end{cases}$$

and $v_{oss1}(t)$, $v_{oss2}(t)$, $i_{Lss1}(t)$, $i_{Lss2}(t)$ are continuous functions given by

$$v_{oss1}(t) := \frac{V_i \omega^2}{2(\cos \sigma - g)} \times de^{(2\sigma-d)T} f_2d$$

$$v_{oss2}(t) := -\frac{V_i \omega^2}{2(\cos \sigma - g)} \times de^{(\sigma-1-d)T} f_2d$$

$$i_{Lss1}(t) := \frac{V_i T}{L} \frac{\alpha + L}{2(V_i T)} \times de^{\sigma(1+d)T} f_2d$$

$$i_{Lss2}(t) := -\frac{V_i T}{L} \frac{\alpha - L}{2(V_i T)} \times de^{\sigma(1-d)T} f_2d$$

In this example, the sizes of matrix exponentials are two and the symbolic computations can be carried out. However, when the sizes of matrix exponentials are larger than two, it is not easy to carry out the symbolic computations. In such cases, the first order approximation of $x_{ss}(t)$ in (23) may relax the computational problems.

5.4 First Order Approximation of Steady State Response in Buck-boost Converters

In this subsection, we obtain the first order approximation of the steady state response generated in the buck-boost converter. Since $A(t)$ and $b(t)$ satisfies Assumption 1, Theorem 2 is applicable. By substituting $A(t)$ and $b(t)$ into (24)–(26), the first order approximations of $v_{oss1}(t)$, $v_{oss2}(t)$, $i_{Lss1}(t)$, and $i_{Lss2}(t)$ with respect to $T$ can be obtained as follows:

$$v_{oss1}(t) \simeq \tilde{v}_{oss1}(t) := \frac{2RCd + Td^2}{2(1-d)RC} \frac{d}{(1-d)RC} V_i$$

$$v_{oss2}(t) \simeq \tilde{v}_{oss2}(t) := \frac{2RCd + Td^2}{2(1-d)RC} \frac{d^2}{(1-d)^2RC} (t-T) V_i$$

$$i_{Lss1}(t) \simeq \tilde{i}_{Lss1}(t) := \frac{d}{(1-d)^2R} \frac{2L}{T} V_i$$

$$i_{Lss2}(t) \simeq \tilde{i}_{Lss2}(t) := \frac{d}{(1-d)^2R} \frac{2L}{(t-T)d} V_i$$

where $\simeq$ denotes the $T^2$-order approximation uniformly in $t$. 

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In this example, the first order approximations can be directly computed from their exact values; namely, by computing the Taylor expansions of \( v_{oss1}(t), v_{oss2}(t), i_{Lss1}(t), \) and \( i_{Lss2}(t) \) around \( T = 0 \), we can recalculate \( \tilde{v}_{oss1}(t), \tilde{v}_{oss2}(t), \tilde{i}_{Lss1}(t), \) and \( \tilde{i}_{Lss2}(t) \), respectively.

5.5 Static Properties of Buck-Boost Converters

Based on the proposed concept of steady state response, the time averages of the load voltage and the inductance currency are described by

\[
V_o := \frac{1}{T} \int_0^T v_{oss}(t) dt
\]

and the peak-to-peak values of the load voltage and the inductance currency are described by

\[
\Delta V_o := \left( \max_{0 \leq t < T} v_{oss}(t) \right) - \left( \min_{0 \leq t < T} v_{oss}(t) \right)
\]

\[
\Delta I_L := \left( \max_{0 \leq t < T} i_{Lss}(t) \right) - \left( \min_{0 \leq t < T} i_{Lss}(t) \right)
\]

The first order approximations of \( v_{oss}(t) \) and \( i_{Lss}(t) \) are linear on the intervals \([0, Td)\) and \([Td, T)\), respectively. Hence, the first order approximations of the averages are computed as follows:

\[
V_o \approx d \frac{\tilde{v}_{oss1}(0) + \tilde{v}_{oss1}(Td)}{2} + (1 - d) \frac{\tilde{v}_{oss2}(Td) + \tilde{v}_{oss2}(T)}{2}
\]

\[
I_L \approx d \frac{\tilde{i}_{Lss1}(0) + \tilde{i}_{Lss1}(Td)}{2} + (1 - d) \frac{\tilde{i}_{Lss2}(Td) + \tilde{i}_{Lss2}(T)}{2}
\]

The first order approximations of the peak-to-peak values are computed as follows:

\[
\Delta V_o \approx \tilde{v}_{oss1}(0) - \tilde{v}_{oss1}(Td) = \frac{T d^2}{(1 - d) R C} V_i
\]

\[
\Delta I_L \approx \tilde{i}_{Lss1}(Td) - \tilde{i}_{Lss1}(0) = \frac{T d}{L} I_i
\]

These approximations (53)–(56) are explained in many textbooks of power electronics, see e.g., Erickson (2001); Kassakian (1991); Harada (1991), and are regarded as important indexes of static characteristics of the buck-boost converter. Those values are derived from circuit-theoretic observations. In order to analyze the other types of switching converter or the effects of the parasitic circuit elements, it is necessary to repeat circuit-theoretic observations with empirical insights, see e.g., Aloisi (2005).

The approximations (53)–(54) are zeroth order components and can be derived from the state average model in (27); however, the approximations (55)–(56) are first order components and cannot be derived from the state average model in (27). Hence, the first order analysis in Theorem 2 is more accurate than the conventional analysis based on the state averaged model.

As illustrated in this section, by introducing the concept of steady state response, the steady state behavior of the switching circuit is explicitly characterized as an asymptotically stable periodic solution. By introducing the first order approximations of steady state response, the static properties such as the averages or the peak-to-peak values can be directly computed from the coefficients of the systems. Moreover, the first order approximations are applicable to any switching circuit including the effects of parasitic circuit elements when the switching circuit is modeled by a linear periodic system. In summary, the proposed method gives an alternative systematic procedure for analyzing static properties of switching converters operating in CCM.

6. CONCLUSIONS

This note gives a mathematical framework for analyzing the steady state response of a linear periodic system. We prove the unique existence of the steady state response when the homogeneous system is asymptotically stable. By supposing that the coefficients of the system are generated from a time scale transformation, we also derive the formula to compute the first order approximation, by which the computational problems due to the fundamental solutions can be relaxed. The proposed procedure gives a unified framework to analyze the static properties of switching converters operating in continuous conduction mode.

REFERENCES


