Differential Game Model and Its Solutions
for Force Resource Complementary via
Lanchester Square Law Equation *

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Abstract: This paper investigates the warfare command game problem on both sides with force 
resource complementary in the fixed time. Under some moderate assumptions, force resource 
complementary differential game is proposed, which ensures that the optimal strategies are 
obtained based on Lanchester square law equation, and the optimum conditions are given by the 
quantitative analysis. In analyzing and solving the optimal strategies of an numerical example, 
feasibility and effectiveness of the model and the approach is demonstrated. The research results 
may provide a theoretical reference for combat command decision making and games.

1. INTRODUCTION
Lanchester theory of combat (Taylor, 1983; Ekchian, 1982; 
Helmbold, 1975) is a typical scientific method to predict 
the outcome of military battles by the quantitative anal-
ysis. Lanchester equation theory has been widely used 
to analyze real wars and determine tactics in war game 
simulation. In the last few years, Some warfare command 
decision making and game problems (Li, 2000; Sha, 2003) 
based on Lanchester equation have been an active area of 
considerable research. On these problems, there are some 
related results recently. The optimal resource allocation 
and redistribution models in military conflicts based on 
Lanchester square law equation is proposed in Sheeba and 
Ghose (2006; 2008) and Ghose(2002) etal. A mathemat-
ical model to solve the maximum remaining force 
problem when the total force is not superior to the enemy 
is proposed in Chen and Jing, etal. (2009). Mathematic 
tactics to research the military problems and the firepower-
assignment differential game models are researched in Sha, 
(2003). Differential game model to analyze the firepower-
assignment problem of backhanded sight gunshot is es-
established and its solutions are solved according to Lanch-
ester’s second linear law in Huang and Xu (2006). Lanches-
ter square law equation model extended to a (2, 2) 
conflict is proposed in Colegrave and Hyde (1993). With 
the complexity of the warfare command systems, the warfare 
command decision making and game problems with the 
additional battle factors have been the focus of major 
research interests in recent years. Firepower-assignment 
optimization model in information war using Lanchester 
theory and differential games theory is investigated in Li 
and Sun (2008). A class of troops support differential game 
models and gave the optimal strategies of troops support 
is proposed in Li and Chen (2004). Two-player zero-sum 
differential game in which the players have the asymmetric 
information on the random terminal payoff is researched in 

However, most of research results are obtained by consid-
ering the outcome of the battle in the termination time 
of the battle, that is, the operational side is eliminated 
or the enemy side is eliminated. For the research on the 
change of the force strengths on both sides in the fixed 
time, there are a few related results. Thus, our purposed 
in this paper, then, is to investigate the warfare command 
decision making problem based on the Lanchester square 
law equation with force resource complementary in the 
fixed time. We establish the differential game model and 
its solution for the force resource complementary and give 
the optimum conditions, and the corresponding optimal 
strategies are analyzed quantitatively. Numerical example 
is also proposed to show the feasibility and effectiveness of 
the proposed model and the approach.

This paper is organized as follows. In Section 2, we 
give the differential game problem formulation with force 
resource complementary in military conflict. In Section 
3, we propose the optimum conditions and the solving 
approach of the optimal strategies. After that, in section 
4, The optimal strategies is obtained by an application 
example presented to show the effectiveness of the model 
and the approach. Finally, in Section 5, we conclude the
paper, summarize the results obtained, and lay out some possible directions for future research.

2. DIFFERENTIAL GAME PROBLEM STATEMENT

Consider the following Lanchester square law equation with force resource complementary is described as

\[
\begin{align*}
\dot{x}(t) &= -\alpha y(t) + u(t), \\
\dot{y}(t) &= -\beta x(t) + v(t),
\end{align*}
\]

(1)

where \(x(t)\) and \(y(t)\) are the strengths of two opposing forces surviving at the time \(t\). \(\alpha > 0\) and \(\beta > 0\) are the constant attrition coefficients that reflect the effectiveness of forces in the unit time. \(x(t_0) = x_0\) and \(y(t_0) = y_0\) are considered as the initial force strengths of two military forces. \(u(t) > 0\) and \(v(t) > 0\) are the control input variables.

**Remark 1.** \(u(t)\) and \(v(t)\) are the control inputs in some warfare command decision making problems, which can change the battle situation and the force strengths of two opposing forces. At present, many researchers consider the reinforcement factor and the self-destroying factor as the control inputs.

Now, consider a military conflict between two opposing forces. Let \(Y\) denote the defending side and \(X\) denote the attacking side, and consider \(u\) and \(v\) as the rate of the reinforcement on both sides. We address the optimization problem for determining the optimal reinforcement strategies \(u^*\) and \(v^*\) to satisfy an object function \(J(u, v)\), which consists of the surviving force strengths of two opposing forces in the fixed time \(T\).

\[
J(u, v) = x(T) - y(T)
\]

(2)

where \(x(T)\) and \(y(T)\) are the residual of force strengths on both sides in the fixed time \(T\).

The objective of this paper is to solve the optimal reinforcement strategies for system (1) and the object function \(J(u, v)\). To continue our solving process, the following assumption is needed for the above decision making problem.

**Assumption 1.** According to the index performance, suppose that \(u > 0\) and \(v > 0\) satisfy

\[
\begin{align*}
u_1 &\leq u \leq u_2 \\
v_1 &\leq v \leq v_2
\end{align*}
\]

(3)

where \(u_1\) is the minimum rate of force strengths complementary of \(x\), \(u_2\) is the maximum rate of force strengths complementary of \(x\); \(v_1\) is the minimum rate of force strengths complementary of \(y\), \(v_2\) is the maximum rate of force strengths complementary of \(y\).

**Assumption 2.** Denote that \(X_0\) and \(Y_0\) are the total of the force strengths complementary of two military forces, the following conditions are satisfied.

\[
\begin{align*}
\int_0^T u(t)dt &\leq X_0; \\
\int_0^T v(t)dt &\leq Y_0,
\end{align*}
\]

(5)

\[
\begin{align*}
\int_0^T u_2 dt &\geq X_0; \\
\int_0^T v_2 dt &\geq Y_0,
\end{align*}
\]

(6)

**Remark 2.** the inequities (5) and (6) represent the limit of the force strengths complementary on both sides, that is, the strategies can not be kept with the maximum complementary rates \(u_2\) and \(v_2\) from 0 to \(T\).

Then, the game problem can be described as follows. the attacking side \(x(t)\) will to get the optimal strategy \(u^*\) to get the maximum value of \(J(u, v)\), the enemy side \(y(t)\) will get the optimal strategy \(v^*\) to get the minimum value of \(J(u, v)\), that is,

\[
J(u^*, v^*) = \max_u \min_v J(u, v).
\]

(7)

3. THE OPTIMUM CONDITION AND THE SOLVING APPROACH

The objective of this paper is to get the optimal strategies of the differential game, that is, we should get the optimal value of \(J(u, v)\). Thus, we can give the optimum conditions and the solving approach for the force strengths complementary strategies in this section.

Firstly, we can get the solutions of (1) as follows

\[
x(t) = \frac{1}{2}(x_0 - \sqrt{\alpha \beta} y_0) e^{\sqrt{\alpha \beta} t} + \frac{1}{2}(x_0 + \sqrt{\alpha \beta} y_0) e^{-\sqrt{\alpha \beta} t} + \frac{1}{2} \int_0^t ((u - \sqrt{\alpha \beta} v) e^{\sqrt{\alpha \beta} (t-\tau)} + (u + \sqrt{\alpha \beta} v) e^{-\sqrt{\alpha \beta} (t-\tau)}) d\tau
\]

\[
y(t) = \frac{1}{2}(y_0 - \sqrt{\alpha \beta} x_0) e^{\sqrt{\alpha \beta} t} + \frac{1}{2}(y_0 + \sqrt{\alpha \beta} x_0) e^{-\sqrt{\alpha \beta} t} + \frac{1}{2} \int_0^t ((v - \sqrt{\alpha \beta} u) e^{\sqrt{\alpha \beta} (t-\tau)} + (v + \sqrt{\alpha \beta} u) e^{-\sqrt{\alpha \beta} (t-\tau)}) d\tau
\]

By using the solution of (1), the index performance \(J(u, v)\) can be presented as

\[
J(u, v) = \Delta_1 + \int_0^T (\Delta_2 u - \Delta_3 v) dt
\]

(8)

where

\[
\Delta_1 = \frac{1}{2}(x_0 - y_0 + \sqrt{\alpha \beta} x_0 - \sqrt{\alpha \beta} y_0) e^{\sqrt{\alpha \beta} T} + \frac{1}{2}(x_0 - y_0 - \sqrt{\alpha \beta} x_0 + \sqrt{\alpha \beta} y_0) e^{-\sqrt{\alpha \beta} T}
\]

\[
\Delta_2 = \frac{1}{2}((1 + \sqrt{\alpha \beta}) e^{\sqrt{\alpha \beta} (T-t)} + (1 - \sqrt{\alpha \beta}) e^{-\sqrt{\alpha \beta} (T-t)}
\]

\[
\Delta_3 = \frac{1}{2}((1 + \sqrt{\alpha \beta}) e^{\sqrt{\alpha \beta} (T-t)} + (1 - \sqrt{\alpha \beta}) e^{-\sqrt{\alpha \beta} (T-t)}
\]

With the constant numbers \(x_0\), \(y_0\) and \(T\), we can solve the optimal value of the index performance (8) by researching the optimal value of the index performance (9) as follows,

\[
f(u, v) = \int_0^T (\Delta_2 u - \Delta_3 v) dt
\]

(9)

**Lemma 1** (Li and Chen, 2004). If there exist the constant numbers \(\lambda > 0\) and \(\mu > 0\), and the force strengths...
complementary rates \( u^*(t) \in (u_1, u_2), v^*(t) \in (v_1, v_2) \), such that

\[
f(u, v^*) - \lambda \int_0^T u dt + \mu \int_0^T v^* dt \\
\leq f(u^*, v^*) - \lambda \int_0^T u^* dt + \mu \int_0^T v^* dt \\
\leq f(u^*, v) - \lambda \int_0^T u^* dt + \mu \int_0^T v dt
\]

then, the index performance (9) satisfies

\[
f(u^*, v^*) = \max \min \left\{ f(u, v) \left| \begin{array}{l}
\int_0^T u dt \leq \int_0^T u^* dt; \\
\int_0^T v dt \leq \int_0^T v^* dt
\end{array} \right. \right\}
\]

Theorem 1. If there exist the constant \( \lambda > 0 \) and \( \mu > 0 \), and the force strengths complementary rates \( u^*(t) \in (u_1, u_2), v^*(t) \in (v_1, v_2) \) satisfy

\[
\int_0^T u dt \leq \int_0^T u^* dt; \int_0^T v dt \leq \int_0^T v^* dt
\]

such that,

\[
f(u^*, v^*) = \max \min \left\{ f(u, v) \right\}
\]

then,

\[
J(u^*, v^*) = \max \min J(u, v)
\]

that is, \( u^*(t) \) and \( v^*(t) \) can be considered as the optimal strategies of the differential game.

Proof: Firstly, we can construct the Lagrange function as follows,

\[
L(u, v, \lambda, \mu, t) = \Delta_2 u - \Delta_3 v - \lambda u + \mu v
\]

According to the above equation, we can find that \( L(u, v, \lambda, \mu, t) \) is the separable function, then, we can discuss the warfare strategies \( u \) and \( v \) to research the optimal value of \( J(u, v) \), respectively.

a) when \( \Delta_2 > \lambda \), we can get that

\[
L(u^*, v, \lambda, \mu, t) = \max_u \left( (\Delta_2 - \lambda)u - (\Delta_3 - \mu)v \right)
\]

From the equation (15), for the differential game, it is deduced that when the strategy satisfies \( u^* = u_2 \), we can obtain the optimal value of \( L(u, v, \lambda, \mu, t) \) satisfies

\[
L(u^*, v, \lambda, \mu, t) = \max_u L(u, v, \lambda, \mu, t).
\]

Otherwise, we can assume that there exists the optimal strategy \( u_m \) (\( u_1 < u_m < u_2 \)) such that

\[
\max_u L(u, v, \lambda, \mu, t) = (\Delta_2 - \lambda)u_m - (\Delta_3 - \mu)v
\]

With \( (\Delta_2 - \lambda)u_m \geq (\Delta_2 - \lambda)u \), we can get that

\[
\max_u L(u, v, \lambda, \mu, t) \leq (\Delta_2 - \lambda)u_2 - (\Delta_3 - \mu)v
\]

Then, the assumption is not hold, that is,

\[
L(u^*, v, \lambda, \mu, t) > L(u, v, \lambda, \mu, t)
\]

b) According to a), we can also get that when \( \Delta_2 < \lambda \), the optimal force strengths complementary rates is \( u_1 \). It satisfies

\[
L(u^*, v, \lambda, \mu, t) > L(u, v, \lambda, \mu, t).
\]

c) when \( \Delta_3 > \mu \), we can get

\[
L(u, v^*, \lambda, \mu, t) = \min_v \left( (\Delta_2 u - \Delta_3 v - \lambda u + \mu v) \right) = \min_v ((\Delta_2 - \lambda)u^* - (\Delta_3 - \mu)v)
\]

From the above equation, for the differential game, it is deduced that when the optimal strategy satisfy \( v^* = v_2 \), the optimal value of \( L(u^*, v^*, \lambda, \mu, t) \) satisfy that

\[
L(u^*, v^*, \lambda, \mu, t) = \max_u \min_v L(u, v, \lambda, \mu, t).
\]

Otherwise, we can suppose that there exists the optimal strategy \( v_n \) (\( v_1 < v_n < v_2 \)) such that

\[
\min_v L(u, v, \lambda, \mu, t) = (\Delta_2 - \lambda)u_m - (\Delta_3 - \mu)v
\]

With \( (\Delta_3 - \mu)v_n \geq (\Delta_3 - \mu)v \), there exists that

\[
\min_v L(u, v, \lambda, \mu, t) \geq (\Delta_2 - \lambda)u_2 - (\Delta_3 - \mu)v
\]

Then, the assumption is not hold, that is,

\[
L(u^*, v^*, \lambda, \mu, t) \leq L(u^*, v, \lambda, \mu, t)
\]

d) According to c), we can get that when \( \Delta_3 < \mu \), the optimal force strengths complementary rate is \( v_1 \), where \( L(u^*, v^*, \lambda, \mu, t) \leq L(u, v, \lambda, \mu, t) \).

From a)- d), we can get that

\[
L(u^*, v^*, \lambda, \mu, t) \leq L(u, v, \lambda, \mu, t)
\]

Then, we can solve the integral of (15) and (16) from 0 to \( T \), and get

\[
\int_0^T L(u^*, v, \lambda, \mu, t) dt \geq \int_0^T L(u, v, \lambda, \mu, t) dt
\]

By Lemma 1, we know that

\[
f(u^*, v^*) = \max \min \left\{ f(u, v) \left| \begin{array}{l}
\int_0^T u dt \leq \int_0^T u^* dt; \\
\int_0^T v dt \leq \int_0^T v^* dt
\end{array} \right. \right\}
\]
and we also get that
\[ J(u^*, v^*) = \max_u \min_v J(u, v) \]
then, Theorem 1 is proved.

According to Theorem 1, the optimal reinforcement strategies \( u^*(t) \) and \( v^*(t) \) on both sides can be solved as follows,
\[
\begin{align*}
\quad u^*(t) &= \begin{cases} 
  u_2 & \Delta_2 \geq \lambda \\
  u_1 & \Delta_2 < \lambda 
\end{cases} \\
\quad v^*(t) &= \begin{cases} 
  v_2 & \Delta_3 \geq \mu \\
  v_1 & \Delta_3 < \mu 
\end{cases}
\end{align*}
\tag{17}
\]
Let \( \Delta_2 = \lambda \), one gets
\[
T \lambda = T - \ln \left( \frac{\lambda + \sqrt{\lambda^2 - (1 - \frac{\beta}{\alpha})}}{1 + \sqrt{\beta/\alpha}} \right) / \sqrt{\alpha \beta}
\tag{18}
\]
Similarly, let \( \Delta_3 = \mu \), then \( t_\mu \) can be obtained as
\[
t_\mu = T - \ln \left( \frac{\mu + \sqrt{\mu^2 - (1 - \frac{\beta}{\alpha})}}{1 + \sqrt{\beta/\alpha}} \right) / \sqrt{\alpha \beta}
\tag{19}
\]
Then the optimal strategies (19) can be written as follows,
\[
\begin{align*}
\quad u^*(t) &= \begin{cases} 
  u_2 & t \in [0, t_\lambda] \\
  u_1 & t \in [t_\lambda, T]
\end{cases} \\
\quad v^*(t) &= \begin{cases} 
  v_2 & t \in [0, t_\mu] \\
  v_1 & t \in [t_\mu, T]
\end{cases}
\end{align*}
\tag{20}
\]
and the objective function satisfies
\[
J(u^*, v^*) = \max_u \min_v J(u, v).
\tag{21}
\]
That is, \( u^*(t) \) and \( v^*(t) \) are the optimal strategies of the optimization model.

4. NUMERICAL EXAMPLE AND DISCUSSION

Consider the differential game (1)-(6), and suppose that \( x_0 = 1000, y_0 = 1000, T = 3, \alpha = 0.4, \beta = 0.9, u_1 = 25, u_2 = 50, v_1 = 20, v_2 = 40, X_0 = 100, Y_0 = 100. \) Then, the conditions (2), (4), (5) and (6) are satisfied, and we will solve the optimal value of \( x(T) - y(T) \) to get the optimal strategies \( u^* \) and \( v^* \).

According to Theorem 1, the optimal strategies \( u^* \) and \( v^* \) can be solved as follows,
\[
\begin{align*}
\quad u^*(t) &= \begin{cases} 
  25, t \in [0, 0.9667] \\
  50, t \in [0.9667, 3]
\end{cases} \\
\quad v^*(t) &= \begin{cases} 
  20, t \in [0, 1.9426] \\
  40, t \in [1.9426, 3]
\end{cases}
\end{align*}
\]
where \( t_\lambda = 0.9667 \) and \( t_\mu = 1.9426. \) And we can get that \( \lambda = 4.1600, \mu = 1.6600, \int_0^T u^* dt = 99.1684, \int_0^T v^* dt = 99.9522 \). Then, the optimal index performance \( J(u, v) \) can be obtained as follows,
\[
\max_u \min_v J(u, v) = x(T) - y(T) = 483.5
\]
Fig.1 gives the change of force strengths on both sides with the optimal strategies of force strength complementary \( u^* \)

![Fig.1. the change of force strengths on both sides with the optimal strategies of troops support \( u^* \) and \( v^* \)](image)

<table>
<thead>
<tr>
<th>( T )</th>
<th>( t_\lambda )</th>
<th>( t_\mu )</th>
<th>( \int_0^T u^* dt )</th>
<th>( \int_0^T v^* dt )</th>
<th>( X_0 )</th>
<th>( Y_0 )</th>
</tr>
</thead>
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<tr>
<td>3</td>
<td>0.9667</td>
<td>1.9426</td>
<td>99.1684</td>
<td>99.9522</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
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<td>3.4167</td>
<td>179.4877</td>
<td>149.3335</td>
<td>180</td>
<td>150</td>
</tr>
<tr>
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<td>2.4520</td>
<td>199.1684</td>
<td>149.4400</td>
<td>200</td>
<td>150</td>
</tr>
<tr>
<td>10</td>
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<td>7.4520</td>
<td>449.1684</td>
<td>349.0400</td>
<td>450</td>
<td>350</td>
</tr>
</tbody>
</table>

Table 1. The solutions of the differential game when \( T = 3, 4, 5, 10 \)

| \( T \) | \( |t_\lambda^* - t_\lambda| \) | \( |t_\mu^* - t_\mu| \) | \( \int_0^T u^* dt - X_0 \) | \( \int_0^T v^* dt - Y_0 \) |
|-----|-----|-----|-----|-----|
| 3   | 0.0333 | 0.0674 | 0.8316 | 0.4478 |
| 4   | 0.1000 | 0.0833 | 0.5123 | 0.6665 |
| 5   | 0.0333 | 0.0480 | 0.8316 | 0.5600 |
| 10  | 0.0333 | 0.0480 | 0.8316 | 0.6000 |

Table 2. The error of the true value \( t_\lambda^* \) , \( t_\mu^* \), \( \int_0^T u^* dt \) and \( \int_0^T v^* dt \) and the desired value of \( \lambda^* \), \( \mu^* \), \( X_0 \) and \( Y_0 \). The calculation errors of \( t_\lambda \) and \( t_\mu \) in the confidence intervals satisfy
\[
|t_\lambda^* - t_\lambda| \leq 0.1, |t_\mu^* - t_\mu| \leq 0.1
\tag{22}
\]
And the calculation errors of force strengths complementary satisfy the confidence intervals as follows,
\[
\int_0^T u^* dt - X_0 \leq 0.9
\tag{23}
\]
T \int_0^T v^* dt - Y_0 \leq 0.9 \tag{24}

Then, it can be observed that the strategies is feasible in the above confidence intervals.

5. CONCLUSIONS

What should be emphasized is that this work seems to be the important investigation on the warfare command decision making problem with force strengths complementary. We establish the troops support differential game model and the optimum conditions. The optimal strategies of troops support are obtained with a numerical example. the computation results are given to verify the feasibility and effectiveness of the model and the solving approach researched in this paper.

There are still some related problems to be investigated. For example, how to construct the warfare command decision making and game problems by considering the additional battle factors? such as, information war, and self-destroying factor. How to propose the better approach to solve the differential game model based on Lancaster square law equation in order to decrease the calculation error? How to construct the generalized decision making or game model also is an important deserving consideration.

REFERENCES