Strict Anisotropic Norm Bounded Real Lemma in Terms of Inequalities

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Abstract: This paper is aimed at extending the $H_\infty$ Bounded Real Lemma to stochastic systems under random disturbances with imprecisely known probability distributions. The statistical uncertainty is measured in entropy theoretic terms using the mean anisotropy functional. The disturbance attenuation capabilities of the system are quantified by the anisotropic norm which is a stochastic counterpart of the $H_\infty$ norm. A state-space sufficient criterion for the anisotropic norm of a linear discrete time invariant system to be bounded by a given threshold value is derived. The resulting Strict Anisotropic Norm Bounded Real Lemma involves an inequality on the determinant of a positive definite matrix and a linear matrix inequality. As is shown, these convex constraints can be approximated by two linear matrix inequalities.

Keywords: Linear systems, stochastic, uncertainty, norms, relative entropy

1. INTRODUCTION

The anisotropy of a random vector and the anisotropic norm of a system are the main concepts of the anisotropy-based approach to robust stochastic control originally developed by I.G. Vladimirov and presented in (Semyonov et al. (1994)), (Vladimirov et al. (1995, 1996-1)).

The anisotropy functional considered there is an entropy theoretic measure of the deviation of a probability distribution in Euclidean space from Gaussian distributions with zero mean and scalar covariance matrices. The mean anisotropy of a stationary random sequence is defined as the anisotropy production rate per time step for long segments of the sequence. In application to random disturbances, the mean anisotropy describes the amount of statistical uncertainty which is understood as the discrepancy between the imprecisely known actual noise distribution and the family of nominal models which consider the disturbance to be a Gaussian white noise sequence with a scalar covariance matrix.

The $\alpha$-anisotropic norm of a linear discrete time invariant (LDTI) system quantifies the disturbance attenuation capabilities by the largest ratio of the power norm of the system output to that of the input provided that the mean anisotropy of the input disturbance does not exceed a given nonnegative parameter $\alpha$.

In the context of robust stochastic control design aimed at suppressing the potentially harmful effects of statistical uncertainty, the anisotropy-based approach offers an important alternative to those control design procedures that rely upon a precisely known specific probability law of the disturbance.

Minimization of the anisotropic norm of the closed-loop system as a performance criterion results in internally stabilizing dynamic output feedback controllers that are less conservative than the $H_\infty$ controllers and more efficient for attenuating the correlated disturbance than the LQG controllers. A state-space solution to the anisotropic optimal control problem derived by Vladimirov et al. (1996-2) involves the solution of three cross-coupled algebraic Riccati equations, an algebraic Lyapunov equation and a mean anisotropy equation on the determinant of a related matrix. Solving this complex equation system requires application of specially developed homotopy-based numerical algorithms.

The suboptimal anisotropic controller design is the natural extension of this approach. Instead of minimizing the anisotropic norm of a system, a suboptimal controller is only required to keep it below a given threshold value. Rather than resulting in a unique controller, the suboptimal design yields a family of controllers, thus providing freedom to impose some additional specifications on the closed-loop system. One of such specifications, for example, may be a particular pole placement to achieve desirable transient performance.

The suboptimal anisotropic control design requires a state-space criterion for testing if the anisotropic norm of a system does not exceed a given value. An Anisotropic Norm Bounded Real Lemma (ANBRL) for the anisotropic norm as a stochastic counterpart of the $H_\infty$ norm for LDTI systems under statistically uncertain stationary Gaussian random disturbances with limited mean anisotropy was presented in (Kurdyukov et al. (2010)). The resulting criterion has the form of an inequality on the determinant...
of a matrix associated with an algebraic Riccati equation which depends on a scalar parameter. This paper aims at improving numerical tractability of ANBRL by representing the criterion as a convex optimization problem. These results are supposed to be applied to design of suboptimal anisotropic controllers by means of convex optimization and semidefinite programming.

The paper is organized as follows. Section 2 recalls the minimum necessary background on the anisotropy of signals and anisotropic norm of systems. Section 3 establishes the Strict Anisotropic Norm Bounded Real Lemma which constitutes the main result of the paper. Section 4 provides an illustrative numerical example. Concluding remarks are given in Section 5.

2. BASIC CONCEPTS OF ANISOTROPY-BASED ROBUST PERFORMANCE ANALYSIS

Let us recall a minimum necessary background material on the anisotropy of signals and anisotropic norm of systems. Full information on the anisotropy-based robust performance analysis, developed originally by Vladimirov et al. (1995, 1996-1), can be found in more details in (Diamond et al. (2001)), (Vladimirov et al. (2006)).

Let $L_2^m$ denote the class of square integrable $\mathbb{R}^m$-valued random vectors distributed absolutely continuously with respect to the $m$-dimensional Lebesgue measure $\text{mes}_m$. For any $W \in L_2^m$ with PDF $f: \mathbb{R}^m \to \mathbb{R}_+$, the anisotropy $A(W)$ is defined by Vladimirov et al. (2006) as the minimal value of relative entropy $D(f||p_{m,\lambda})$ with respect to the Gaussian distributions $p_{m,\lambda}$ in $\mathbb{R}^m$ with zero mean and scalar covariance matrices $\Lambda_m$:

$$A(W) = \min_{\lambda > 0} D(f||p_{m,\lambda}) = \frac{m}{2} \ln \left( \frac{2\pi e |E(|W|^2)|}{m} \right) - h(W),$$

(1)

where $h(W)$ denotes the differential entropy of $W$ with respect to $\text{mes}_m$; see Cover & Thomas (1991). The minimum in (1) is achieved at $\lambda = |E(|W|^2)|/m$; see Vladimirov et al. (2006).

Let $W \triangleq (w_k)_{-\infty < k < +\infty}$ be a stationary sequence of vectors $w_k \in L_2^m$ interpreted as a discrete-time random signal. Assemble the elements of $W$ associated with a time interval $[s,t]$ into a random vector $W_{s,t} \triangleq \text{col}[w_s, \ldots, w_t]$.

(2)

It is assumed that $W_{0,N}$ is distributed absolutely continuously for every $N \geq 0$. The mean anisotropy of the sequence $W$ is defined by Vladimirov et al. (2006) as the anisotropy production rate per time step by $\bar{A}(W) \triangleq \lim_{N \to +\infty} \frac{A(W_{0,N})}{N}$. (3)

Let $G^m(\mu, \Sigma)$ denote the class of $\mathbb{R}^m$-valued Gaussian random vectors with mean $E(w_k) = \mu$ and nonsingular covariance matrix $\text{cov}(w_k) \triangleq E((w_k - \mu)(w_k - \mu)^T) = \Sigma$. Let $V \triangleq (v_k)_{-\infty < k < +\infty}$ be a sequence of random vectors $v_k \in G^m(0, I_m)$, i.e. an $m$-dimensional Gaussian white noise sequence. Suppose $W = GV$ is produced from $V$ by a stable shaping filter with transfer function $G(z) \in H_2^{m \times m}$. Then the spectral density of $W$ is given by

$$S(\omega) \triangleq G(\omega)^* G(\omega)^*, \quad -\pi \leq \omega < \pi,$$

(4)

where $\hat{G}(\omega) \triangleq \lim_{t \to -1} G(e^{i\omega t})$ is the boundary value of the transfer function $G(z)$. As is shown by Vladimirov et al. (1996-1), Diamond et al. (2001), mean anisotropy (3) can be computed in terms of spectral density (4) and the associated $H_2$ norm of the shaping filter $G$ as

$$\bar{A}(W) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \det \frac{mS(\omega)}{\|G\|^2_2} d\omega.$$ (5)

Since the probability law of the sequence $W$ is completely determined by the shaping filter $G$ or by the spectral density $S$, the alternative notations $\bar{A}(G)$ and $\bar{A}(S)$ are also used instead of $\bar{A}(W)$.

Mean anisotropy functional (5) is always nonnegative. It takes a finite value if the shaping filter $G$ is of full rank, otherwise, $\bar{A}(G) = +\infty$; see Vladimirov et al. (1996-1), Diamond et al. (2001). The equality $\bar{A}(G) = 0$ holds true if and only if $G$ is an all-pass system up to a nonzero constant factor. In this case, spectral density (4) is described by $S(\omega) = \mathcal{L} \mathbf{1}_m$, $-\pi \leq \omega < \pi$, for some $\lambda > 0$, so that $W$ is a Gaussian white noise sequence with zero mean and a scalar covariance matrix.

Let $F \in H_2^{m \times m}$ be a LDTI system with an $m$-dimensional input $W$ and a $p$-dimensional output $Z = FW$. Let the random input sequence $W = GV$, where, as before, $V \in G^m(0, I_m)$. Denote by $g_a \triangleq \{G \in H_2^{m \times m} : \bar{A}(G) \leq a\}$ (6) the set of shaping filters $G$ that produce Gaussian random sequences $W$ with mean anisotropy (5) bounded by a given parameter $a \geq 0$.

The $a$-anisotropic norm of the system $F$ is defined as

$$\|F\|_a \triangleq \sup_{G \in g_a} \frac{\|GF\|_2}{\|G\|_2},$$

(7)

see Vladimirov et al. (1996-1), Diamond et al. (2001).

It is shown by Vladimirov et al. (1996-1) that the $a$-anisotropic norm of a given system $F \in H_2^{m \times m}$ is a nondecreasing continuous function of the mean anisotropy level $a$ which satisfies

$$\frac{1}{\sqrt{m}} \|F\|_2 = \|F\|_0 \leq \lim_{a \to +\infty} \|F\|_a = \|F\|_\infty.$$ (8)

These relations show that the $H_2$ and $H_{\infty}$ norms are the limiting cases of the $a$-anisotropic norm as $a \to 0, +\infty$, respectively.

3. STRICT ANISOTROPIC NORM BOUNDED REAL LEMMA

Let $F \in H_2^{m \times m}$ be a LDTI system with an $m$-dimensional input $W$, $n$-dimensional internal state $X$ and $p$-dimensional output $Z$ governed by

$$\begin{bmatrix} x_{k+1} \\ z_k \\ w_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix},$$

(9)

where $A$, $B$, $C$, $D$ are appropriately dimensioned real matrices, and $A$ is stable (its spectral radius $\rho(A) < 1$). The input sequence $W$ is supposed to be a stationary Gaussian random sequence whose mean anisotropy does not exceed $a \geq 0$, i.e. $W$ is produced from the $m$-dimensional Gaussian white noise $V \in G^m(0, I_m)$ by an unknown shaping filter $G$ which belongs to the family $g_a$ defined by (6).
3.1 Main result: a convex formulation

The theorem below provides a state-space criterion for the anisotropic norm of system (9) to be strictly bounded by a given threshold \( \gamma \).

**Theorem 1.** Let \( F \in \mathbb{R}^{p \times m} \) be a system with state-space realization (9), where \( \rho(A) < 1 \). Then its \( a \)-anisotropic norm (7) is strictly bounded by a given threshold \( \gamma > 0 \), i.e. 

\[
\|F\|_a < \gamma
\]

if there exists \( q \in \{0, \min(\gamma^{-2}, \|F\|_\infty^2)\} \) such that the inequality 

\[
-(\det(I_m - B^T R B - q D^T D))^{1/m} < -\left(1 - q \gamma^2\right)e^{2a}
\]

holds true for the real \((n \times n)\)-matrix \( R = R^T > 0 \) satisfying the linear matrix inequality 

\[
\begin{bmatrix}
A^T R A - R & A^T R B \\
B^T R A & B^T R B - I_m
\end{bmatrix} + q \begin{bmatrix}
C^T \\
D^T
\end{bmatrix}[C \ D] < 0.
\]

**Remark 2.** Note that both inequalities (11) and (12) form the convex constraints upon both variables \( q \) and \( R \). As is known, the function \(-\det(\cdot)\) is strictly one resulting in similar modification of (18).

To prove the main result, first we must prove the following assertion:

**Lemma 4.** Let \( F \in \mathbb{R}^{p \times m} \) be a system with state-space realization (9), where \( \rho(A) < 1 \), and let the real positive values \( \gamma \) and \( a \) be given. Suppose that there exist a real \((n \times n)\)-matrix \( R = R^T > 0 \) and scalar value \( q \in \{0, \min(\gamma^{-2}, \|F\|_\infty^2)\} \) such that 

\[
A^T R A - R + (A^T R B + q C^T D)(I_m - B^T R B - q D^T D)^{-1}
\]

\[
\times (B^T R A + q D^T C) + q C^T C < 0,
\]

\[\text{and}
\]

\[
\text{ln det } (I_m - B^T R B - q D^T D) > m \ln (1 - q \gamma^2) + 2a.
\]

Then there exists a stabilizing solution \( \hat{R} = \hat{R}^T > 0 \) to the algebraic Riccati equation 

\[
A^T \hat{R} A - \hat{R} + (A^T \hat{R} B + q C^T D)(I_m - B^T \hat{R} B - q D^T D)^{-1}
\]

\[
\times (B^T \hat{R} A + q D^T C) + q C^T C = 0
\]

such that 

\[
I_m - B^T \hat{R} B - q D^T D > 0
\]

and 

\[
\text{ln det } (I_m - B^T \hat{R} B - q D^T D) > m \ln (1 - q \gamma^2) + 2a.
\]

Moreover, \( \hat{R} \sim R \).

**Proof.** Let us fix \( q \). From (19) it follows that there exists a real \((n \times n)\)-matrix \( Q = Q^T > 0 \) such that 

\[
A^T R A - R + (A^T R B + q C^T D)(I_m - B^T R B - q D^T D)^{-1}
\]

\[
\times (B^T R A + q D^T C) + q C^T C + Q = 0.
\]

Note that (20) also yields \( I_m - q D^T D > 0 \). Then, by virtue of Lemma 2.1 in (de Souza & Xie (1992)) there exists a real \((n \times n)\)-matrix \( \hat{R} = \hat{R}^T > 0 \) satisfying (22) such that (23) holds true and all eigenvalues of the matrix 

\[
\hat{\Sigma} \triangleq A + B(I_m - B^T \hat{R} B - q D^T D)^{-1}(B^T R A + q D^T C)
\]

lie within the closed unit disc. Furthermore, we have 

\[
0 \prec \hat{R} \preceq R.
\]

Inequalities (21) and (24) can be rewritten as 

\[
\text{det } (I_m - B^T R B - q D^T D) > (1 - q \gamma^2)^m e^{2a}
\]

\[
\text{det } (I_m - B^T \hat{R} B - q D^T D) > (1 - q \gamma^2)^m e^{2a}
\]

respectively. From (26)–(28) it can be seen that 

\[
\text{det } (I_m - B^T R B - q D^T D) > (1 - q \gamma^2)^m e^{2a}
\]

which proves (24). Now, let us show that the matrix \( \hat{\Sigma} \) is actually stable, i.e. the matrix \( \hat{R} \) is the stabilizing solution of algebraic Riccati equation (22). Denoting \( P \triangleq -R \) and 

\[
\hat{P} \triangleq -\hat{R},
\]

equations (25), (22) can be rewritten as 

\[
A^T P A - P - (A^T P B - q C^T D)(I_m - q D^T D + B^T P B)^{-1}
\]

\[
\times (B^T P A - q D^T C) - q C^T C - Q = 0,
\]

\[
A^T \hat{P} A - P - (A^T \hat{P} B - q C^T D)(I_m - q D^T D + B^T \hat{P} B)^{-1}
\]

\[
\times (B^T \hat{P} A - q D^T C) - q C^T C = 0,
\]
respectively. Applying Lemma 3.1 from (de Souza (1989)), we have that the matrix $\hat{P} - P$ must satisfy the following equation:

$$
\hat{P} - P = \hat{A}^T (\hat{P} - P) \hat{A} + \hat{A}^T (\hat{P} - P) B (I_m - q D^T D + B^T P B)^{-1} B^T (\hat{P} - P) \hat{A} + Q.
$$

(29)

Suppose that the matrix $\hat{A}$ is not stable, i.e. there exists a nonzero vector $\zeta \in \mathbb{R}^m$ and scalar value $\lambda, |\lambda| = 1$, such that $\hat{A} \zeta = \lambda \zeta$. Then from (29) it follows that

$$
\zeta^T \hat{A}^T (\hat{P} - P) B (I_m - q D^T D + B^T P B)^{-1} B^T (\hat{P} - P) \hat{A} \zeta = \zeta^T \hat{A}^T (R - \hat{R}) B (I_m - q D^T D - B^T R B)^{-1} B^T (R - \hat{R}) \hat{A} \zeta > 0
$$

for all nonzero $\zeta$. From (30) it follows that $\zeta^T Q \zeta = 0$. This is a contradiction, since $Q > 0$. Therefore, the matrix $\hat{A}$ is stable, i.e. the matrix $\hat{R}$ is the positive definite stabilizing solution to (22). Finally, from (29) it follows that $\hat{R} \preceq R$, which completes the proof.

**Proof of Theorem 1.** Note that by virtue of Schur Theorem (see e.g. Bernstein (2005)) linear matrix inequality (12) is equivalent to (19), (20) for all $q \in (0, \min(\gamma^{-2}, ||F||^{-2}))$. Inequality (11) can be rewritten as (21) and strict form of (14). Applying Lemma 4, we determine that in this case there exists a stabilizing solution to algebraic Riccati equation (22) such that inequality (24) holds true. Then, by virtue of Theorem 1 in (Kurdyukov et al. (2010)) (see Remark 3), inequality (10) also holds, which was to be proved.

**Remark 5.** A solution to inequalities (11), (12) of Theorem 1 can be found by means of available software packages for convex optimization that allows using the convex function $-(\det(\cdot))^{1/m}$ not only as an objective, but also in constraints; see e.g. Löfberg (2004).

### 3.2 A linear approximation

Convex but nonlinear inequality (11) of Theorem 1 can be approximated by a linear but rather conservative constraint. Now let us formulate the Strict Anisotropic Norm Bounded Real Lemma in terms of LMIs.

**Theorem 6.** In conditions of Theorem 1, an-anisotropic norm (7) of system $F$ is strictly bounded by a given threshold $\gamma > 0$, i.e. inequality (10) holds true if there exists $\hat{Q} > 0$ such that $\|F\|_2 < \gamma$. Then there exists a stabilizing solution to algebraic Riccati equation (22) such that inequality (24) holds true. Then, by virtue of Theorem 1 in (Kurdyukov et al. (2010)) (see Remark 3), inequality (10) also holds, which was to be proved.

By applying the change of variable $\frac{1}{q} \hat{R} \equiv \hat{R} > 0$, we can rewrite (32), (12) as

$$
B^T \hat{R} B + D^T D < \gamma^2 I_m.
$$

(33)

Since $\gamma^2 < \frac{1}{q} < +\infty$, from (33) it follows that

$$
B^T \hat{R} B + D^T D - \frac{1}{q} I_m < 0.
$$

Applying Schur Theorem to the last inequality together with LMI (34) yields

$$
A^T \hat{R} A - \hat{R} + C^T C < 0,
$$

(35)

while (33) implies

$$
\frac{1}{m} \text{tr} (B^T \hat{R} B + D^T D) < \gamma^2.
$$

(36)

But satisfying inequalities (35), (36) is equivalent to

$$
\frac{1}{\sqrt{m}} ||F||_2 < \gamma;
$$

(37)

see e.g. Zhou et al. (1996). Let us note that using the same reasoning it is not hard to derive (35)–(37) from more conservative conditions of Theorem 6 choosing the matrix $\hat{S} = I_m$ in inequality (31).
ineffective. In this case, by applying the change of variable \( \tilde{R} \triangleq \gamma R \), LMI (12) can be rewritten in the form

\[
\begin{bmatrix}
A^T \tilde{R} A & A^T \tilde{R} B & C^T \\
B^T \tilde{R} A & B^T \tilde{R} B - \gamma I_n & D^T \\
C & D & -\gamma I_p
\end{bmatrix} < 0
\]  

(38)

which is well-known in the context of the discrete time \( H_\infty \) control; see e.g. Doyle et al. (1991), Gahinet & Apkarian (1994). This fact closely relates to the convergence \( \lim_{a \to +\infty} \| F \|_a = \| F \|_\infty \) in (8) whereby inequality (10) approximates

\[ \| F \|_\infty < \gamma \]  

(39)

for sufficiently large values of \( a \). Thus, in the limit, as \( a \to +\infty \), both Theorems 1 and 6 become \( H_\infty \) Bounded Real Lemma establishing the equivalence between (39) and existence of a positive definite solution to LMI (38).

4. NUMERICAL EXAMPLE

Let us illustrate application of Theorems 1 and 6 by the numerical example. Consider an asymptotically stable system \( F \) with the state-space realization

\[ F = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

By (8), the \( a \)-anisotropic norm \( \| F \|_a \) of this system varies from \( \| F \|_0 = \| F \|_2 / \sqrt{2} = 4.7731 \) (for \( a = 0 \)) to \( \| F \|_\infty = 22.1868 \) (as \( a \to +\infty \)). Convex problem (11), (12) and linear matrix inequalities (31), (12) are infeasible whereas the \( a \)-anisotropic norm of the system \( F \) is bounded by \( \gamma \). One can see that the difference between the volumes of the ellipsoids is greater for the lesser value of \( a \) that corresponds to greater conservatism of linear approximation (31). It should be noted that the results of the considered numerical example are typical for all systems tested by the authors.

As the tables show, increasing the mean anisotropy level \( a \) results in quite successful testing for values of \( \gamma \) verging closer to the precise values of the \( a \)-anisotropic norm \( \| F \|_a \) by criterion of Theorem 6. We can see that LMIs (31), (12) become infeasible as \( a \) approaches the right boundary of the admissible interval \( (0, \min (\gamma^{-2}, \| F \|_\infty^{-2})) \) (for the considered system, \( \| F \|_\infty^{-2} = 2.0315 \cdot 10^{-3} \)). The values of the ratio \( \gamma_a / \| F \|_a \) where \( \gamma_a \) denotes the minimum value of \( \gamma \) resulting in feasible convex inequalities (11), (12) and LMIs (31), (12), for the considered values of the mean anisotropy \( a \) are presented in Table 5.

So, the considered example demonstrates some relaxation of conservatism of Theorem 6 conditions with growth of the mean anisotropy level, while the conservatism of Theorem 1 is scarcely noticeable. Difference in conservatism of two criteria is visualized in Fig. 1, where one can find the plots of two ellipsoids \( \xi^T (I_2 - B^T R B - q D^T D) \xi \leq 1 \), \( \xi \in \mathbb{R}^2 \), derived from solving inequalities (11), (12) (CP, light grey ellipsoid) and (31), (12) (LMIs, dark grey ellipsoid) for \( a = 0.01 \) (upper diagram), \( a = 0.3 \) (lower diagram) and \( \gamma = 20 \) in both cases. The volume of the first ellipsoid is greater, and the light grey regions not covered by the second ellipsoid correspond to the case when LMIs (31), (12) are infeasible whereas the \( a \)-anisotropic norm of the system \( F \) is bounded by \( \gamma \). One can see that the difference between the volumes of the ellipsoids is greater for the lesser value of \( a \) that corresponds to greater conservatism of linear approximation (31). It should be noted that the results of the considered numerical example are typical for all systems tested by the authors.

Table 1. Test results; \( a = 0.01 \), \( \| F \|_a = 5.7460 \)

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>CP</th>
<th>( q ) (CP)</th>
<th>LMIs</th>
<th>( q ) (LMIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>feas.</td>
<td>1.4414 · 10^{-3}</td>
<td>feas.</td>
<td>1.7008 · 10^{-3}</td>
</tr>
<tr>
<td>15</td>
<td>feas.</td>
<td>1.6957 · 10^{-3}</td>
<td>feas.</td>
<td>1.9720 · 10^{-3}</td>
</tr>
<tr>
<td>7.2929</td>
<td>feas.</td>
<td>1.6887 · 10^{-3}</td>
<td>feas.</td>
<td>1.2472 · 10^{-3}</td>
</tr>
<tr>
<td>7.2928</td>
<td>feas.</td>
<td>2.0047 · 10^{-3}</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>5.7467</td>
<td>feas.</td>
<td>1.4666 · 10^{-3}</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>5.7466</td>
<td>infeas.</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

5. CONCLUSION

We have presented two formulations of a Strict Anisotropic Norm Bounded Real Lemma (SANBRL) in terms of inequalities providing a state-space criteria for testing if the anisotropic norm of a LDTI system is bounded by a given threshold value. This result extends \( H_\infty \) Bounded Real
Lemma to stochastic systems where the statistical uncertainty present in the random disturbances is quantified by the mean anisotropy level.

The derived criteria implies solving a LMI and an inequality on the determinant of a matrix or two LMIs with respect to a positive definite matrix and a positive scalar parameter. The criterion formulated in terms of LMIs is characterized by much greater conservatism in comparison with the nonlinear convex problem. SANBRL in terms of inequalities seems to be applicable to design of suboptimal controllers which ensure a specified upper bound on the anisotropic norm of the closed-loop system, possibly combined with additional specifications which may include pole placement to provide desirable transient performance of the system, by means of convex optimization and semidefinite programming.

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REFERENCES


Table 2. Test results; $\alpha = 0.5$, $\|F\|_{\alpha} = 14.6499$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>CP</th>
<th>$q$ (CP)</th>
<th>LMIs</th>
<th>$q$ (LMIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>feas.</td>
<td>$1.0427 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$1.0179 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>20</td>
<td>feas.</td>
<td>$1.9571 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$1.0691 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>16</td>
<td>feas.</td>
<td>$1.9793 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$1.7459 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>15.1950</td>
<td>feas.</td>
<td>$2.0011 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$2.0229 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>15.1949</td>
<td>feas.</td>
<td>$2.0265 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>14.6500</td>
<td>feas.</td>
<td>$2.0264 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>14.6499</td>
<td>infeas.</td>
<td>—</td>
<td>infeas.</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 3. Test results; $\alpha = 1.5$, $\|F\|_{\alpha} = 19.7529$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>CP</th>
<th>$q$ (CP)</th>
<th>LMIs</th>
<th>$q$ (LMIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>feas.</td>
<td>$1.6600 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$1.6000 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>20</td>
<td>feas.</td>
<td>$2.0314 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$2.0229 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>19.9086</td>
<td>feas.</td>
<td>$2.0297 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>19.9085</td>
<td>feas.</td>
<td>$2.0314 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>19.7530</td>
<td>feas.</td>
<td>$2.0313 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>19.7529</td>
<td>infeas.</td>
<td>—</td>
<td>infeas.</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 4. Test results; $\alpha = 3$, $\|F\|_{\alpha} = 21.6675$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>CP</th>
<th>$q$ (CP)</th>
<th>LMIs</th>
<th>$q$ (LMIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>feas.</td>
<td>$1.6600 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$1.5987 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>22</td>
<td>feas.</td>
<td>$2.0314 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$2.0315 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>21.6995</td>
<td>feas.</td>
<td>$2.0313 \cdot 10^{-3}$</td>
<td>feas.</td>
<td>$2.0315 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>21.6994</td>
<td>feas.</td>
<td>$2.0315 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>21.6676</td>
<td>feas.</td>
<td>$2.0314 \cdot 10^{-3}$</td>
<td>infeas.</td>
<td>—</td>
</tr>
<tr>
<td>21.6675</td>
<td>infeas.</td>
<td>—</td>
<td>infeas.</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 5. Dependence of ratio $\gamma_{\alpha}/\|F\|_{\alpha}$ on $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\gamma_{\alpha}/|F|_{\alpha}$ (CP)</th>
<th>$\gamma_{\alpha}/|F|_{\alpha}$ (LMIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.000122</td>
<td>1.269213</td>
</tr>
<tr>
<td>0.5</td>
<td>1.000007</td>
<td>1.037208</td>
</tr>
<tr>
<td>1.5</td>
<td>1.000005</td>
<td>1.007882</td>
</tr>
<tr>
<td>3</td>
<td>1.000004</td>
<td>1.001477</td>
</tr>
</tbody>
</table>


