The Value Function as a Solution of Hamiltonian Systems in Linear Optimal Control Problems with Infinite Horizon *

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Abstract: The paper deals with analytical construction of the value function for a linear control problem with infinite horizon arising in problems of economic growth. The proposed algorithm is based on analysis of asymptotic properties of the Hamiltonian system in the Pontryagin maximum principle. The method of indeterminate coefficients is applied for identification of parameters of the value function. Sensitivity analysis of parametric solutions is implemented with respect to qualitative properties of steady states of the Hamiltonian system. The structure of optimal feedbacks is outlined and asymptotic behavior of optimal trajectories is analyzed. Applications to economic growth modeling are discussed.

Keywords: Optimal control theory, control problems under conflict or uncertainties, Hamiltonian trajectories in optimal control.

1. INTRODUCTION

The research is focused on the construction procedure for the value function of an optimal control problem arising in economic growth modeling. Describing the considered economic growth model it is worth to mention that it is based on the classical approach of the growth theory (see Solow (1970), Shell (1969)). Let us note that a modification of the model under consideration is suggested in Krasovskii (2006). The main output variable is the gross domestic product (GDP) defined as the market value of all final goods and services produced within a country in a year. In parallel with the GDP trajectory, we consider dynamics of two production factors: the capital stock $K$ and the labor force $L$. In the original model the GDP level is generated by the exponential production function of the Cobb-Douglas type. Extending the model, we analyze the limit case with the linear production function.

The reason of such extension is twofold. The first idea is that the linear case provides the possibility to construct the value function analytically basing on asymptotic properties of the Hamiltonian system in the Pontryagin maximum principle. For this purpose, we base our research on methods of the optimal control theory (see Pontryagin et al. (1962)), and transversality conditions (see Aseev and Kryazhimskiy (2007)) for optimal control problems with infinite horizon. In our analysis we use also results of the theory of control problems under uncertainty and differential games Krasovskii and Subbotin (1988), Krasovskii and Krasovskii (1995). Particularly, we refer on papers Dolcetta (1983), Feichtinger and Veliov (2007), Adiatulina and Tarasyev (1987), Subbotin (1995), Nikol’ski (2007), devoted to construction of value functions in optimal control problems and differential games with infinite horizon. Let us mention also papers Falcone (1987), Falcone (1991), Ushakov and Latushkin (2006), Tarasyev et al. (1994), Subbotina (2000) on finite-difference schemes for constructing value functions.

The second circumstance is that in growth models with strictly concave production functions of the Cobb-Douglas type the essential role for construction of optimal trajectories belongs to the existence result of a steady state with positive coordinates and local properties of the Hamiltonian system in its neighborhood (see, for example, Krasovskii and Tarasyev (2008)). A remarkable fact is that in the limit case of the linear production function steady states of the Hamiltonian system do not exist. It means that positive coordinates of steady states for strictly concave problems tend in the limit case of the linear production function either to infinity or to zero depending on relations of model parameters.

In this connection, we analyze asymptotic behavior of solutions of growth models with strictly concave production function of the Cobb-Douglas type when their parameters tend to limit cases generating the linear production function. In this analysis, we pay special attention to asymptotics of steady states, corresponding values of optimal control and value functions.

It is important to note that in econometrics the parameters of growth models are calibrated with rather large confi-
dence intervals, and contain a lot of uncertainties. Therefore, the implemented sensitivity analysis with respect to parameters of productions functions plays crucial role in the forecasting procedure based on growth modeling.

Another important feature of the implemented research is that it combines analysis of the Hamiltonian systems arising in the Pontryagin maximum principle with the backward algorithms of construction of the value function and provides a stabilized procedure of solution of optimal control problems with infinite horizon.

2. MODEL DESCRIPTION

The model

Our research is connected with the paper Krasovskii and Tarasyev (2008) in which two-factors economic growth model is investigated. The model is focused on the analysis of GDP $Y(t)$ of a country. Two production factors are considered in the model: capital stock $K$ and labor $L$.

The output $Y(t)$ at time $t$ is given by equation $Y(t) = F[K(t), L(t)]$, where symbol $F$ denotes the production function. In the paper, we consider the production function of the Cobb-Douglas type: $Y(t) = \alpha K(t)^{\beta} L(t)^{1-\beta}$. Here parameters $\beta$ and $1-\beta$ are the positive elasticity coefficients, i.e. $0 < \beta < 1$. Let us note that the production function has the homogeneity property of degree one. This property allows to introduce relative variables: $y = Y/L$, $k = K/L$, where $y$ is the GDP per one worker and $k$ is the capital per one worker. The relative production function looks as follows: $y(t) = f(k(t)) = \alpha k(t)^{\beta}$.

Assume that symbols $C(t) \geq 0$ and $I(t) \geq 0$ denote the consumption level and investments in capital, respectively. Hence, for an aggregative closed economy, which is supposed in this model, the output can be evaluated by the formula:

$$Y(t) = C(t) + I(t) = (1 - s(t))Y(t) + s(t)Y(t). \quad (1)$$

Dynamics of the capital stock $K$ satisfies to the Solow model:

$$\dot{K}(t) = s(t)Y(t) - \mu K(t), \quad K(t_0) = K_0.$$

Here parameter $\mu > 0$ is the rate of the capital depreciation.

It is assumed that the labor input grows exponentially $\dot{L}(t) = nL(t)$, with a constant growth rate $n > 0$. Hence, dynamics of the capital per worker is described by the equation

$$\dot{k}(t) = s(t)y(t) - \lambda k(t), \quad k(t_0) = K_0/L_0 = k_0. \quad (2)$$

where $\lambda = \mu + n$ is the capital decay, and $n$ is the capital dilution.

Let us suppose that the production function per one worker $f(k)$ has the following properties:

$$f'(k) > 0, \quad f''(k) < 0 \quad \text{for} \quad k \in K^0 \subset (0, +\infty).$$

Here $f'(k)$ is the marginal productivity of the capital per worker. The symbol $K^0$ stands for a nonempty set which is called the economic domain (see Intriligator (1971)).

The optimal control problem

Let us consider the optimal control problem for growth of the capital stock. The utility functional is represented as the integral of the logarithmic consumption index discounted on the infinite horizon

$$J = \int_{t_0}^{+\infty} e^{-\delta t} \left( \ln(1 - s(t)) + \ln(f(k(t))) \right) dt, \quad (3)$$

where the symbol $\delta > 0$ denotes the constant rate of discount. It is assumed that a central planner starts his investment process with the initial level $k(0) = k_0$ and aims at maximization of the utility functional under the dynamic constraints (2), where a measurable control parameter $s(t)$ in time $t$ is subject to restrictions $0 \leq s(t) \leq a < 1$.

It should be noted that the condition of compactness of control restrictions $s \in [0, a]$ is important for the accurate application of the Pontryagin maximum principle (see Pontryagin et al. (1962)). Parameters $\alpha, \delta, \lambda = \mu + n, k_0$ are the given positive numbers, and parameter $0 < a < 1$ is a number which separates the right bound of control parameter from unit.

The problem is to find the optimal investment level $s^0(t)$ and the corresponding trajectory $k^0(t)$ of the capital stock per worker $k$ subject to dynamics (2) for maximizing the functional of consumption per worker (3). Let us note that the stated optimal control problem is investigated in Krasovskii and Tarasyev (2008) in the framework of the Pontryagin maximum principle. In the paper, we continue this research using the concept of the value function as the generalized solution of the Hamilton-Jacobi equation.

The linear model

The purpose of the paper is to construct the value function in the limit case when the elasticity coefficient $\beta$ tends to unit. Let us introduce variables of the linear model: $K_1(t), k_1(t)$ - capital and relative capital at the moment $t$, $Y_1(t), y_1(t)$ - GDP and GDP per worker, $c_1(t)$ - consumption level per worker, and $s_1(t)$ - a part of GDP invested in capital.

If the parameter $\beta$ equals to unit then the production function is linear $y_1 = f(k_1) = \alpha k_1$. It means that the output is proportional to the capital: $Y_1 = F[K_1] = \alpha K_1$. Here parameter $\alpha$ is a positive coefficient of proportionality. The dynamics of the capital per worker (2) for the limit case looks as follows:

$$\dot{k}_1(t) = (\alpha s_1(t) - \lambda)k_1(t), \quad k_1(t_0) = K_0/L_0 = k_0. \quad (4)$$

Based on the balance equation (1), the per worker consumption level $c_1(t)$ for the linear production function $y_1 = f(k_1)$ has the following form:

$$c_1(t) = (1 - s_1(t))y_1(t) = \alpha(1 - s_1(t))k_1(t).$$

The utility function can be rewritten as follows:

$$J_1 = \int_{t_0}^{+\infty} e^{-\delta t} \left( \ln(1 - s_1(t)) + \ln(\alpha k_1(t)) \right) dt, \quad (5)$$

3. ANALYSIS OF THE LINEAR MODEL

The optimal control problem (4), (5) can be analyzed within the optimal control theory for problems with infinite horizon (see Aseev and Kryazhimskiy (2007), Krasovskii and Tarasyev (2008)).
The Hamiltonian function

The Hamiltonian function of the optimal control problem is defined by the following relation:

\[ H(t, k_l, s_l, \tilde{\psi}_l) = e^{-\delta t} \ln (\alpha (1 - s_l) k_l) + \tilde{\psi}(\alpha s_l - \lambda) k_l. \]

The adjoint variable \( \tilde{\psi}_l \) stands for “shadow prices” (model prices) for capital \( k_l \).

Let us make substitution of the Hamiltonian and the adjoint variable by excluding the exponential time term:

\[ \tilde{H}(k_l, s_l, \psi_l) = H(t, k_l, s_l, \tilde{\psi}_l)e^{\delta t}, \quad \psi_l = \tilde{\psi}e^{\delta t}. \]

In new variables the Hamiltonian has the following form

\[ \tilde{H}(k_l, s_l, \psi_l) = \ln (\alpha (1 - s_l) k_l) + \psi_l (\alpha s_l - \lambda) k_l. \tag{6} \]

Existence of the Optimal Solution and Necessary Conditions of Optimality

It should be noted that for the control problem (4), (5) all conditions of the existence theorem (see Aseev and Kryazhimskii (2007)) are fulfilled. Moreover, one can formulate the necessary conditions of optimality for problems with infinite horizon in the form of the Pontryagin maximum principle (see Krasovskii and Tarasiev (2008)).

**Theorem 1.** Let \((s_l^0, k_l^0)\) be an optimal process. Then there exists an adjoint variable \( \tilde{\psi} \) corresponding to process \((s_l^0, k_l^0)\) and satisfying the adjoint equation

\[ \tilde{\psi}_l = -\frac{\partial \tilde{H}_t}{\partial k_l}(t, k_l^0(t), s_l^0(t), \tilde{\psi}_l(t)) \]

such that

1. the process \((s_l^0, k_l^0)\) satisfies the condition of the Pontryagin maximum principle together with the adjoint variable \( \tilde{\psi} \)

\[ \tilde{H}_t(t, k_l^0, s_l^0, \tilde{\psi}_l) = \max \left\{ \tilde{H}_t(t, k_l, s_l, \tilde{\psi}_l), s_l \in [0, a] \right\}, \]

2. the process \((s_l^0, k_l^0)\) and the adjoint variable \( \tilde{\psi}_l \) meet the stationarity condition

\[ \tilde{H}_t(t, k_l^0, s_l^0, \tilde{\psi}_l) = \delta \int_0^\infty e^{-\delta \tau} \left( \ln (1 - s_l(\tau)) + \ln \alpha k_l(\tau) \right) d\tau, \]

3. \( \tilde{\psi}_l(t) > 0, \quad \forall t \geq 0. \)

4. the adjoint variable \( \tilde{\psi}_l \) satisfies the transversality condition

\[ \lim_{t \to \infty} \tilde{\psi}_l(t) k_l^0(t) = 0. \tag{7} \]

**Optimal control and the maximized Hamiltonian**

Let us indicate properties of the Hamiltonian.

**Lemma 2.** The Hamiltonian \( H_t(k_l, s_l, \psi_l) \) (6) is a strictly concave function in variables \( k_l \) and \( s_l \).

The proof follows immediately from strict negativity of the matrix of second derivatives of the Hamiltonian (6) in variables \( k_l \) and \( s_l \).

Solving the problem of maximization of the Hamiltonian \( H_t \) (6) over the control parameter \( s_l \) subject to constraints \( s_l(t) \in [0, a] \), one can find the structure of optimal control as

\[ s_l^0 = \begin{cases} 0, & k_l e^{\psi_l} \leq 1; \\ 1 - \frac{a k_l^0(\psi_l)}{\alpha k_l^0(\psi_l)}, & 1 \leq a k_l e^{\psi_l} \leq a; \\ a, & a k_l e^{\psi_l} \geq a, \end{cases} \tag{8} \]

where \( a_s = 1/(1 - a) \).

The maximized Hamiltonian is a function in variables \( k_l, \psi_l \) which is defined as maximum of the original Hamilton function in the control parameter

\[ H_t(k_l, \psi_l) = \max_{s_l \in [0, a]} H_t(k_l, s_l, \psi_l) = -\lambda k_l \psi_l + \begin{cases} \ln (\alpha k_l), & s_l^0 = 0, \\ -\ln \psi_l + a k_l e^{\psi_l} - 1, & s_l^0 = 1 - \frac{1}{\alpha k_l \psi_l}, \\ \ln (\alpha (1 - a) k_l) + a k_l e^{\psi_l}, & s_l^0 = a. \end{cases} \tag{9} \]

Basing on the Pontryagin maximum principle we construct the Hamiltonian system:

\[ \begin{align*} k_l(t) &= (\alpha s_l^0 - \lambda)k_l(t), \\ \psi_l(t) &= \delta \psi_l(t) - \frac{\partial H_t}{\partial k_l}(k_l(t), \psi_l(t)). \end{align*} \tag{10} \]

Departing from the structure of the optimal control \( s_l^0 \) (8) and the Hamiltonian system (10), one can prove the following statement.

**Proposition 3.** For the optimal trajectory \( k_l^0(t) \) and the adjoint variable \( \psi_l(t) \) the following equality is valid:

\[ k_l^0(t) \psi_l(t) = 1/\delta \quad \forall t \geq 0. \tag{11} \]

**Proof.** Let us multiply the first equation of the Hamiltonian system by variable \( \psi_l \) and the second one – by \( k_l \), and then summarize both equations. We obtain

\[ (k_l \psi_l)' = (\delta - \alpha s_l^0) k_l \psi_l - \frac{\partial H_t}{\partial k_l}(k_l(t), \psi_l(t)). \tag{12} \]

Next, we find partial derivatives of the maximized Hamiltonian:

\[ \frac{\partial H_t}{\partial k_l}(k_l(t), \psi_l(t)) = \begin{cases} \frac{1}{k_l} - \lambda \psi_l, & s_l^0 = 0, \\ (a - \lambda \psi_l), & s_l^0 = 1 - \frac{1}{\alpha k_l \psi_l}, \\ \frac{1}{k_l} + (a \alpha - \lambda \psi_l), & s_l^0 = a. \end{cases} \]

Substituting the optimal control \( s_l^0 \) and partial derivatives of the maximized Hamiltonian by the phase variable \( k \) in equation (12), we get:

\[ (k_l \psi_l)' = \delta k_l \psi_l - 1. \tag{13} \]

The last equation (13) is valid for all values of the optimal control \( s_l^0 \). Solution of this equation has the following form:

\[ k_l \psi_l = \frac{1}{\delta} + \kappa e^{\delta t}, \tag{14} \]

where \( \kappa \) is a constant which can be found from the transversality condition (7) rewritten for the adjoint variable \( \psi_l = \tilde{\psi}_l e^{\delta t} \):

\[ \lim_{t \to \infty} (k_l^0(t) \psi_l(t) e^{-\delta t}) = 0. \tag{15} \]

Using this condition for relation (14) we get

\[ 0 = \lim_{t \to \infty} (k_l^0(t) \psi_l(t) e^{-\delta t}) = \lim_{t \to \infty} (\frac{1}{\delta} e^{-\delta t} + \kappa) = \kappa. \]

Hence, \( \kappa = 0 \). The statement of the proposition is proved.

Results of lemma (3) show that the optimal control has the constant value depending on model parameters:

\[ s_l^0 = \begin{cases} 0, & \alpha \leq \delta; \\ 1 - \delta/\alpha, & (1 - a) \alpha \leq \delta \leq \alpha; \\ a, & \delta \leq (1 - a) \alpha. \end{cases} \tag{16} \]
The maximized Hamiltonian (9) for the linear model can be rewritten as follows:

\[ H_l(k_l, \psi_l) = A/\delta + \begin{cases} 
\ln (\alpha k_l), & s_l^0 = 0, \\
\ln \delta k_l, & s_l^0 = 1 - \delta/\alpha, \\
\ln (\alpha(1-a)k_l), & s_l^0 = \alpha. 
\end{cases} \tag{17} \]

Here parameter \( A \) depends on the optimal control and equals to the following values:

\[ A = \begin{cases} 
-\lambda, & s_l^0 = 0, \\
\alpha - \delta - \lambda, & s_l^0 = 1 - \delta/\alpha, \\
\alpha \alpha - \lambda, & s_l^0 = \alpha, 
\end{cases} \tag{18} \]

\( \delta \leq (1 - a) \alpha. \)

Let us describe properties of the maximized Hamiltonian function (17).

**Lemma 4.** The maximized Hamiltonian function (17) \( H_l(k_l, \psi_l) \) is a smooth function in variables \((k_l, \psi_l)\) for all values of optimal controls.

**Lemma 5.** The maximized Hamiltonian \( H_l(k_l, \psi_l) \) is a strictly concave function in variable \( k_l \).

**Proposition 6.** Due to properties stated in lemmas 4 and 5 of the Hamiltonian function, the Pontryagin maximum principle ensures the sufficient optimality conditions in the problem (4), (5).

The proof of this statement can be found in Krasovskii and Tarasyev (2008).

**Comparison of linear and nonlinear models**

First, let us note that in the nonlinear model with the Cobb-Douglas production function the maximized Hamiltonian is defined in three domains \( D_i \) \( (i = 1, 3) \) generated by different optimal control regimes:

\[ H(k, \psi) = -\lambda \psi k + \begin{cases} 
\ln (\alpha k^3), & (k, \psi) \in D_1; \\
-\ln \psi + a k^3 \psi - 1, & (k, \psi) \in D_2; \\
\ln (\alpha(1-a)k^3) + ak^3 \psi, & (k, \psi) \in D_3. 
\end{cases} \]

\( D_1 = \{(k, \psi) : a k^3 \psi \leq 1\}, \quad s_0^0 = 0; \)

\( D_2 = \{(k, \psi) : 1 \leq a k^3 \psi \leq a_s\}, \quad s_0^0 = 1 - k^3/(\alpha \psi); \)

\( D_3 = \{(k, \psi) : a k^3 \psi \geq a_s\}, \quad s_0^0 = a. \)

Second, proposition 3 is not valid for the nonlinear model with the Cobb-Douglas production function. It means that the optimal control in the nonlinear model does not coincide with a constant and has a transient control regime \( s_0^0 = 1 - 1/ak^3 \psi \). Third, one can prove that the maximized Hamiltonian in the nonlinear model also has properties of smoothness and strict concavity.

**4. OPTIMAL TRAJECTORIES AND STEADY STATES**

In this section, we examine trajectories generated by the optimal control.

**Optimal trajectories of the linear model**

Let us consider the optimal control \( s_l^0 \) (16) for the linear model. It has constant values which depend entirely on model parameters.

The corresponding optimal trajectory is the solution of the Hamiltonian system. The first equation of the Hamiltonian system can be written as follows: \( k_l(t_1) = Ak_l, \quad k_l(t_0) = k_0 \), where parameter \( A \) is defined by formula (18). Due to these facts, one can formulate the following statement:

**Proposition 7.** The optimal trajectory of the problem (4), (5) is defined by the relation

\[ k_l(t) = k_0 e^{A(t-t_0)} \tag{19} \]

**Optimal trajectories of the nonlinear model**

Let us note that optimal trajectories for the nonlinear model in domains \( D_1 \) and \( D_2 \), where the optimal control equals to zero and the maximum value \( a \), respectively, can be found analytically. Let us consider solutions of the Hamiltonian system in these domains.

Further, the variable \( z = k \psi \) is used instead of the adjoint variable \( \psi \).

**Domain \( D_1 \) with zero control regime**

Let us note that the condition \( ak(t)^3 - 1 \leq 1 \) provides the zero control regime. The Hamiltonian system with the zero control regime has the following form

\begin{align*}
\dot{k}(t) & = -\lambda k(t), \\
\dot{z}(t) & = \delta z(t) - \beta.
\end{align*}

Solution \( k_1(t) \) of the Hamiltonian system in this domain is defined by the relation: \( k_1(t) = k_0 e^{-\lambda(t-t_0)} \). It decreases to the value \( k_{12} \) which is located at the boundary between domains \( D_1 \) and \( D_2 \): \( L_{12} = \{(k, z) : ak(t)^3 - 1 = 1\} \). Let us denote by the symbol \( t_{12} \) the moment of time when solution \( k_1(t) \) achieves this boundary \( L_{12} \).

**Domain \( D_3 \) with intensive control regime**

Domain \( D_3 \) is defined by relation: \( \alpha(1-a)z(t)k(t)^{3-1} \geq 1 \). The Hamiltonian system in this domain is presented by formulas

\begin{align*}
\dot{k}(t) & = a k^3(t) - \lambda k(t), \\
\dot{z}(t) & = (\delta + \alpha(1-\beta)k(t)^{3-1})z(t) - \beta.
\end{align*}

The phase trajectory \( k_3(t) \) is given by the relation

\[ k_3(t) = \left( \frac{aa}{\lambda} \left( 1 - \frac{a}{a_α k_0^{1-\beta}} e^{-\lambda(1-\beta)(t-t_0)} \right)^{\frac{1}{\delta + \alpha(1-\beta)}} \right) \]

It grows up from the initial point \( k_0 \) to the value \( k_{23} \) at the border of domains \( D_2 \) and \( D_3 \): \( L_{23} = \{(k, z) : (1-a)ak(t)^{3-1} = 1\} \). Denote by the symbol \( t_{23} \) the moment of time when solution \( k_3(t) \) comes to the boundary \( L_{23} \).

If elasticity parameter \( \beta \) goes to unit then the value of capital \( k(t) \) converges to the optimal trajectory (19) for the case when model parameters meet the restriction \( \delta \leq (1 - a) \alpha \).

**Domain \( D_2 \) with transient control regime**

The Hamiltonian system with the transient control regime has the following form

\begin{align*}
\dot{k}(t) & = \left( ak(t)^{3-1} - \lambda - \frac{1}{\delta} \right) k(t), \\
\dot{z}(t) & = (\delta + \alpha(1-\beta)k(t)^{3-1})z(t) - 1.
\end{align*}

Denote by the symbol \( k_2(t) \) the optimal solution in domain \( D_2 \). In domain \( D_2 \) this solution can not be found analytically. Let us note that in the paper Krasovskii and Tarasyev (2008) a numerical algorithm is suggested to find it. The initial condition for the Hamiltonian system is defined by the initial point \( k_0 \) according to relations

\[ \begin{cases} 
k(t_{12}) = k_{12}, \quad k_0 \in D_1; \\
k(t_0) = k_0, \quad k_0 \in D_2; \\
k(t_{23}) = k_{23}, \quad k_0 \in D_3. 
\end{cases} \]
The second relation arises from the transversality condition (15) written in the form: \( \lim_{t \to \infty} e^{-\delta t} z(t) = 0 \).

**Steady states**

Let us indicate the important fact that steady states are absent in the linear model.

Let us remind that in the nonlinear model there exists the transient optimal regime when the optimal control is not constant. In this case, the Hamiltonian system has a saddle steady state with the following coordinates:

\[
\begin{aligned}
k^* &= \left( \frac{\alpha \beta}{\delta + \lambda} \right)^{1/\delta}, \\
z^* &= \frac{\beta}{\delta + (1 - \beta)\lambda}.
\end{aligned}
\]

**Remark 8.** The steady state is located in the domain with the transient control regime if and only if model parameters meet the condition: \( a > \beta \frac{\lambda}{\delta + \lambda} \).

If the elasticity coefficient \( \beta \) approaches the unit value then the steady state tends to zero or infinity, namely,

\[
\lim_{\beta \to 1 - \alpha} \left( \frac{\alpha \beta}{\delta + \lambda} \right)^{1/\delta} = \begin{cases} 0, & \alpha < \delta + \lambda \\ +\infty, & \alpha > \delta + \lambda. \end{cases}
\]

**Remark 9.** If the equality \( \alpha = \delta + \lambda \) takes place then the relative capital does not leave the initial point.

On Fig. 1.(a), (b) the phase coordinate of the steady state is depicted as a function of parameter \( \beta \).

The optimal control at the steady state is determined by the formula: \( s^* = \frac{\lambda}{\delta + \beta} \). If parameter \( \beta \) is close to unit then

\[
\begin{aligned}
s^* &< s^0, & \alpha > \delta + \lambda \\
s^* &\geq s^0, & \alpha < \delta + \lambda.
\end{aligned}
\]

Optimal trajectories with different values of parameter \( \beta \) are shown on Fig. 1.(c) for the case when model parameters satisfy conditions: \( a > \beta \frac{\lambda}{\delta + \lambda} \), and \( \alpha > \delta + \lambda \). All trajectories start from the initial point \( k_0 \) located in domain \( D_3 \) with the intensive control regime \( s^0 = a \) and grow up to the steady state.

5. VALUE FUNCTIONS

**The value function for the linear model**

The value function for the linear problem can be found analytically.

**Proposition 10.** The value function for the optimal control problem (4), (5) is given by the relation

\[
V[l_0, k_0] = \frac{1}{\delta} e^{-\delta t_0} \left( \ln (ak_0) - \frac{\lambda}{\delta} + B \right),
\]

where parameter \( B \) is defined by model parameters and can be evaluated as follows:

\[
B = \begin{cases} 0, & \alpha \leq \delta \\ -\ln (\alpha/\delta) + \alpha/\delta - 1, & (1 - a)\alpha \leq \delta \leq \alpha \\ \ln (1 - a) + \alpha a/\delta, & \delta \leq \alpha (1 - a). \end{cases}
\]

To find the value function we use the method of indeterminate coefficients. To apply this method it is necessary to substitute the trajectory \( k^0_2(t) \) (19) into the utility functional (5) and determine coefficients of the value function.

**The value function in the nonlinear model**

Let us consider the value function \( V[l_0, k_0] \) for the nonlinear control problem. Due to the piecewise construction of optimal trajectories the value function for the nonlinear control problem can be presented in the following form:

\[
V[l_0, k_0] = \begin{cases} I_1(t_{23}, k_0) + I_2(t_{23}, k_0), & k_0 \in D_1; \\ I_3(t_{23}, k_0), & k_0 \in D_2; \\ I_3(t_{23}, k_0) + I_2(t_{23}, k_0), & k_0 \in D_3. \end{cases}
\]

where

\[
\begin{aligned}
I_1(\tau, k_0) &= \int_{\tau_0}^{\tau} e^{-\delta t} \left( \ln (ak_0^2(\beta, k_0, t)) \right) dt, \\
I_2(\tau, k_0) &= \int_{\tau_0}^{\tau} e^{-\delta t} \left( \ln k_2(\beta, k_0, t) - \ln z_2(\beta, k_0, t) \right) dt, \\
I_3(\tau, k_0) &= \int_{\tau_0}^{\tau} e^{-\delta t} \left( \ln (\alpha(1 - a)k_0^2(\beta, k_0, t)) \right) dt.
\end{aligned}
\]

Here, functions \( k_i(\beta, k_0, t) \), \( z_i(\beta, k_0, t) \) \((i = 1, 3)\) are solutions of the Hamiltonian systems in domains \( D_i \) \((i = 1, 3)\), respectively. These solutions depend on the initial position \( k_0 \). Integrals \( I_1(\tau, k_0) \) and \( I_3(\tau, k_0) \) can be estimated analytically.

**Proposition 11.** The series of the value functions \( V[l_0, k_0] = V[l_0, k_0] \) of nonlinear models converges to the value function \( V[l_0, k_0] \) of the linear model when parameter \( \beta \) tends to unit.

Fig. 1.(d) shows the series of plots of the value functions. These plots are obtained for different values of the model parameter \( \beta \) in the case when the initial point \( k_0 \) is located in domain \( D_3 \). The value functions are evaluated for the case when model parameters satisfy restrictions:

\[
a > \beta \frac{\lambda}{\delta + \lambda}, \quad \alpha > \delta + \lambda.
\]

The first inequality implies that the steady state belongs to domain \( D_2 \), and the second condition means that the phase coordinate \( k^* \) of the steady state tends to infinity when \( \beta \) goes to unit. On the plot, the following symbols are used: \((k_{00}, v_{00})\) is the initial position of the value function, where \( v_{00} = V[l_0, k_{00}] \) and \( k_{00} = 9.0 \); \((k_{23}, v_{23})\) is the point at which the value function leaves the domain \( D_3 \) of the intensive control regime \( s^0 = a \) and enters the domain \( D_2 \) of the transient control defined by the relation \( s^0 = 1 - k^{1-\beta}/(\alpha \delta) \), here \( v_{23} = V[l_0, k_{23}]; (k^*, v^*) \) is the plot’s point corresponding to the steady state, i.e. \( k^* \) is the phase coordinate of the steady state, and \( v^* = V[l_0, k^*] \).

**Remark 12.** Asymptotics of the value functions (21) for nonlinear models demonstrates analytically the convergence to the value of the linear model when parameter \( \beta \) tends to unit. Numerical results of the convergence are shown in Table 1. Magnitudes of the value function are evaluated at characteristic points which are depicted on Fig.1.(c), (d) as \((k_{00}, v_{00}), (k_{23}, v_{23})\) and \((k^*, v^*)\). One can see that the magnitudes of the value function grow when the parameter \( \beta \) tends to unit.

<table>
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<th>( \beta )</th>
<th>( v_{00} )</th>
<th>( v_{23} )</th>
<th>( k_{23} )</th>
<th>( v_{23} )</th>
<th>( k^* )</th>
<th>( v^* )</th>
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<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Table 1. Series of numerical results for the value functions \( V[l_0, k_0] \) at characteristic points
The value function at the steady state

Let us estimate the value functions of nonlinear models at steady states.

**Proposition 13.** The value function $V[t_0, k_0]$ (21) at the steady state is determined by the relation

$$V^* = e^{-\delta t_0} \frac{1}{\delta} \ln \left( \frac{\delta + (1 - \beta)\lambda}{\beta (\delta + \lambda)} \right).$$

**Proof.** Since the steady state $k^*$ belongs to domain $D_2$ then the value function equals to

$$V[t_0, k^*] = \int_{t_0}^{+\infty} e^{-\delta t} \ln k_2^*(k^*, t) dt = e^{-\delta t_0} \frac{1}{\delta} \ln k^*.$$ 

Substituting coordinates of the steady state (20) to the expression for the value function, we get the necessary relation.

6. CONCLUSION

In this paper we consider two variants of an economic growth model – nonlinear and linear. The first one is based on the production function of the Cobb-Douglas type $y = \alpha k^\beta$, $0 < \beta < 1$. The second model employs the linear production function $y_l = \alpha k_l$ obtained as the limit case of the Cobb-Douglas production functions when the elasticity parameter $\beta$ tends to unit. We compare solutions of optimal control problems formulated for both models.

It is shown that the solution of the nonlinear model has the steady state for the Hamiltonian system arising in the Pontryagin maximum principle, while steady states are absent in the linear variant. However, convergence of solutions of nonlinear models, including the optimal trajectories and the value functions, to the solution of the linear model is demonstrated. Asymptotic behavior is analyzed for the steady states of nonlinear models and the corresponding optimal control regimes when the elasticity parameter $\beta$ tends to unit.

REFERENCES


