2-D polynomial approach to control of leader following vehicular platoons

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Abstract: This paper formulates the problems of stabilization and asymptotic following in infinite platoons of vehicles within the 2-D polynomial framework, that is, the dynamics of the problem are described using a fraction of two bivariate polynomials. In contrast to some previous works, the platoon here assumes a leader (and an infinite number of followers), therefore the often used bilateral \(z\)-transform should not be used here since it assumes a doubly infinite vehicular strings. The unilateral \(z\)-transform seems better suited. However, it brings about the need to take the boundary conditions into consideration; among other, the leader vehicle comes into the scene. The necessary formalism is introduced in the paper and used to provide elegant alternative proofs of some well-known facts about the platooning problem.

Keywords: Automated guided vehicles, multidimensional systems, multivariable polynomials, matrix polynomial equations, polynomial methods.

1. INTRODUCTION

The goal of this paper is to build a mathematical formalism needed to model and analyze an infinite platoon of vehicles following their leader, and demonstrate the elegance of the approach by solving a few classical platooning problems.

Even though a platoon with an infinite number of vehicles constitutes an unrealistic model of reality, it can be used to infer the asymptotic properties of long but finite platoons. This approximation was proposed in 1970s by Melzer and Kuo (1971) and Chu (1974) and then reexamined three decades later by Jovanović and Bamieh (2005). Distinguished feature of this paper is that while majority of the papers rely on state-space formalism, here the preference is given to input-output description, that is, models are given in the form of a fraction of two bivariate polynomials. The first sketch was presented by Hurak and Sebek (2010). To save space, we also refer to that paper for more comprehensive survey of more than three decades of development in the area of control of vehicular platoons.

Another feature of this paper is that the platoon is assumed to have a leader, that is, the cars are indexed by natural numbers. Joint unilateral Laplace and \(z\)-transform, formally denoted here as \(\mathcal{LZ}_1\)-transform, is used to model the problem at hand by a fraction of bivariate polynomials. This brings the platooning problem on the same ground as numerous problems in the broad and well studied domain of 2-D signals and systems.

2. PLATOON DESCRIPTION

Semi-infinite one-dimensional platoon studied in the paper is in Fig. 1. The leading vehicle is labeled by 0 and the follow-up cars are numbered by 1, 2, . . . . The vehicles keep their original indices even when exchanging their positions. The leader is driven externally while the followers are controlled by the algorithms discussed in the paper.

Fig. 1. Platoon of vehicles with a leader.

Variables in the platoon, such as positions and velocities, are described by spatial sequences of time functions

\[
\{f(t, k)\} = f(t, 0), f(t, 1), f(t, 2), \ldots, t \in [0, \infty),
\]

corresponding to the equally indexed vehicles.

3. \(\mathcal{LZ}_1\)-TRANSFORM AND ITS PROPERTIES

To prepare the ground, a joint unilateral Laplace and (shifted) unilateral \(z\)-transform denoted by \(\mathcal{LZ}_1\) is defined as

\[
\mathcal{LZ}_1 \{f(t, k)\} = \int_{0^-}^{\infty} \left( \sum_{k=1}^{\infty} f(t, k)z^{-k} \right) e^{-st} \, dt. \quad (1)
\]

In contrast to the common \(z\)-transform definition, the discrete-space part of the \(\mathcal{LZ}_1\) transform "starts" with the vehicle indexed by \(k = 1\). This keeps the leader outside the support allowing the \(\mathcal{LZ}_1\)-transform to describe just the...
controlled vehicles. The movement of the leading vehicle becomes a boundary condition.

The \( \mathcal{LZ}_1 \)-transform of the sequence \( \{f(t, k)\} \) expands \(^1\) into

\[
f(s, z) = f(s, 1)z^{-1} + f(s, 2)z^{-2} + \ldots
\]

which is a formal power series in \( z^{-1} \) having polynomials or fractions in \( s \) as coefficients.

A couple of \( \mathcal{LZ}_1 \)-transform properties are listed here that are used in the paper. The proofs are straightforward.

**Theorem 1. (\( \mathcal{LZ}_1 \)-transform properties).**

Given \( f(t, k) \) and its \( \mathcal{LZ}_1 \)-transform \( f(s, z) \), then

\[
\mathcal{LZ}_1 \left\{ \frac{\partial f}{\partial t} \right\} = sf(s, z) - f_0-(z),
\]

\[
\mathcal{LZ}_1 \left\{ \frac{\partial^2 f}{\partial t^2} \right\} = s^2 f(s, z) - sf_0-(z) - \dot{f}_0-(z),
\]

assuming that the derivatives exist. Here

\[
f_0-(z) = \sum_{k=1}^{\infty} f(0^-, k)z^{-k},
\]

\[
\dot{f}_0-(z) = \sum_{k=1}^{\infty} \dot{f}(0^-, k)z^{-k},
\]

are \( \mathcal{LZ}_1 \)-transforms of the spatial sequences of (pre)initial conditions \( f(0^-, k) \) and \( \dot{f}(0^-, k) \), respectively. Moreover, \( \mathcal{LZ}_1 \{f(t, k-1)\} = \frac{1}{z-1}f(s, z) + \frac{1}{z}f_0(s) \) (7) where

\[
\mathcal{LZ}_1 \{f(t, k-1)\} = \mathcal{LZ}_1 \{f(t, k) \} - \frac{1}{z} \mathcal{LZ}_1 \{f(t, k) \}.
\]

is the \( \mathcal{L} \)-transform of the function related to the leader.

4. PLATOON AS A GENERAL 2-D SYSTEM

Platoons and their controls are modeled here in a compact general form using fractions of real bivariate polynomials. The two variables are denoted by \( s \) and \( z \), corresponding to time and the spatial index of the vehicle, respectively.

A variety of platoons is described by the general 2-D plant

\[
a(s, z)y(s, z) = b(s, z)\dot{y}(s, z) + c(s, z),
\]

(9)

Here \( y(s, z) \) and \( u(s, z) \) stand for \( \mathcal{LZ}_1 \)-transforms of the plant output and input, respectively. Writing them as formal power series in \( z^{-1} \) with rational coefficients in \( s \)

\[
y(s, z) = y_1(s)z^{-1} + y_2(s)z^{-2} + \ldots,
\]

\[
u(s, z) = u_1(s)z^{-1} + u_2(s)z^{-2} + \ldots,
\]

nicely reveals that particular coefficients \( y_k(s) \) and \( u_k(s) \) represent the output and input at the position number \( k \).

Furthermore, \( a(s, z) \) and \( b(s, z) \) are 2-D polynomials encountered in the plant transfer function, while \( c(s, z) \) is a 2-D polynomial or fraction incorporating the information about the initial and boundary conditions in the plant. Their roles become evident from rewriting (9) into

\[
y(s, z) = \frac{b(s, z)}{a(s, z)}u(s, z) + \frac{c(s, z)}{a(s, z)}.
\]

(12)

Correspondingly, a general 2-D controller

\[
p(s, z)u(s, z) = q(s, z)e(s, z) + d(s, z),
\]

(13)

which is driven by error signal

\[
e(s, z) = y_{\text{ref}}(s, z) - y(s, z),
\]

(14)

covers a miscellany of control schemes. The role of the polynomials \( p(s, z), q(s, z) \) and \( d(s, z) \) is clear from

\[
u(s, z) = \frac{q(s, z)}{p(s, z)}e(s, z) + \frac{d(s, z)}{p(s, z)}.
\]

(15)

Use of these general 2-D models is now demonstrated on typical control policies.

**Example 1. Predecessor Following Control.**

Consider a platoon of identical vehicles, each governed by a simple double integrator equation, where for every vehicle the distance to its predecessor is measured and used for control. In time and space, such a platoon is modeled by the equations (for \( t \in [0, \infty], k = 1, 2, 3, \ldots \))

\[
\dot{x}(t, k) = \frac{1}{m}u(t, k),
\]

\[
r(t, k) = x(t, k-1) - x(t, k),
\]

(16)

where the quantities \( x(t, k), u(t, k) \) and \( r(t, k) \) stand for the position of the \( k \)-th vehicle, its control input (driving force) and its distance from the \( (k-1) \)-th vehicle, its predecessor, respectively. Naturally, the whole sequences \( \{x(t, k)\}, \{u(t, k)\} \) and \( \{r(t, k)\} \) describe the positions of all the vehicles, all the driving forces (local inputs) and all the distances between the neighboring vehicles, respectively. To complete the model, some initial as well as boundary conditions must be known. These are the initial positions \( x(0^-, k) = x_0(k) \) and the velocities \( \dot{x}(0^-, k) = \dot{x}_0(k) \) for all \( k = 1, 2 \ldots \) as well as the leader’s position \( x(t, 0) = x_0(t) \) for all \( t \in [0, \infty). \)

The \( \mathcal{LZ}_1 \)-transform turns (16) into

\[
x(s, z) = \frac{1}{ms^2}u(s, z) + \frac{1}{s}x_0-(z) + \frac{1}{s^2}\dot{x}_0-(z),
\]

(17)

\[
r(s, z) = (z^{-1} - 1)x(s, z) + z^{-1}x_0(s).
\]

Putting this together yields

\[
ms^2r(s, z) = (z^{-1} - 1)u(s, z) + ms(z^{-1} - 1)x_0-(z) + m(z^{-1} - 1)x_0(z) + ms^2z^{-1}x_0(s),
\]

which matches the general format of (9) with the output

\[
y(s, z) = r(s, z) + c(s, z),
\]

and the corresponding polynomials

\[
a(s, z) = ms^2,
\]

\[
b(s, z) = (z^{-1} - 1),
\]

\[
c(s, z) = ms(z^{-1} - 1)x_0-(z) + m(z^{-1} - 1)x_0(z) + ms^2z^{-1}x_0(s).
\]

A natural strategy is to control each vehicle locally by a controller operating on the error of the distance to the predecessor from its desired reference value. When all the local controllers are identical, the \( \mathcal{L} \)-transform yields

\[
p(s)u_k(s) = q(s)(r_{\text{ref}, k}(s) - r_k(s)),
\]

(19)

where the role of initial conditions is neglected. Global controller, which can be viewed as a sequence of local controllers, fits into general 2-D format (42) with

\[
e(s, z) = r_{\text{ref}, k}(s) - r_k(s),
\]

\[
p(s) = p(s),
\]

\[
q(s, z) = q(s),
\]

\[
d(s, z) = 0.
\]

(20)

**Example 2. Leader Following Control.**

As another example, consider again the platoon above, where now for every vehicle its distance to the leader is...
measured and used for control. Such a platoon is described by the equations (for \( t \in [0, \infty) \), \( k = 1, 2, 3, \ldots \))

\[
\dot{x}(t, k) = \frac{1}{m} u(t, k), \quad w(t, k) = x(t, 0) - x(t, k),
\]

where \( w(t, k) \) stands for the distance between the \( k \)-th and the leading (0th) vehicle. The initial and the boundary conditions are as above. Using \( \mathcal{L} \mathcal{Z}_1 \)-transform, (21) becomes

\[
\begin{align*}
    x(s, z) &= \frac{1}{m s^2} u(s, z) + \frac{1}{s} x_0(z) + \frac{1}{s^2} \tilde{x}_0(z), \\
    w(s, z) &= \frac{z}{1 - z^{-1}} x_0(s) - x(s, z),
\end{align*}
\]

from which finally

\[
-(z^{-1} - 1)ms^2 w(s, z) = (z^{-1} - 1)u(s, z),
\]

\[
+ ms(z^{-1} - 1)\dot{x}_0(z) + m(z^{-1} - 1)\ddot{x}_0(z) = ms^2 z^{-1} x_0(s).
\]

Matching this to (9) with the output \( y(s, z) = w(s, z) \) gives

\[
a(s, z) = (1 - z^{-1})ms^2, \quad b(s, z) = (z^{-1} - 1),
\]

\[
c(s, z) = ms(z^{-1} - 1)\dot{x}_0(z) + m(z^{-1} - 1)\ddot{x}_0(z) + ms^2 z^{-1} x_0(s).
\]

A convenient controller is here

\[
p(s)u(s, z) = q(s) \left( \hat{y}_{\text{ref}}(s, z) - \tilde{y}(s, z) \right),
\]

where

\[
\hat{y}_{\text{ref}}(s, z) = \frac{z^{-1}}{1 - z^{-1}} \frac{\tilde{r}_0}{s}.
\]

It is actually driven by the regulation error as

\[
e(s, z) = \hat{y}_{\text{ref}}(s, z) - \tilde{y}(s, z)
\]

\[
= z^{-1} \frac{\tilde{r}_0}{1 - z^{-1}} - (z^{-1} - 1 - \tilde{r} s) x(s, z)
\]

\[
- z^{-1} \dot{x}_0(s) - \tilde{r} x_0(z)
\]

\[
= \frac{z^{-1}}{1 - z^{-1}} \frac{\tilde{r}_0}{s} + \tilde{r} s x(s, z) - \tilde{r} x_0(z)
\]

\[
- (z^{-1} - 1) x(s, z) - z^{-1} \dot{x}_0(s)
\]

\[
= \hat{y}_{\text{ref}}(s, z) - r(s, z).
\]

Such a controller is, of course, again just a particular case of the general 2-D controller (42) with

\[
p(s, z) = p(s), q(s, z) = q(s), d(s, z) = 0.
\]

5. CONTROL FOR A GENERAL 2-D SYSTEM

This paper investigates how the distributed control schemes for vehicular platoons with a leader vehicle scale with the growing number of followers. In the input-output setting this goal is usually rephrased as a stability requirement for certain transfer functions.

Putting together the general 2-D plant and a 2-D controller equations (9) and (13), they implicitly relate certain variables that are “given” or “supplied from outside” to other variables that are to be controlled or at least taken into account. The “given” variables include the reference command \( y_{\text{ref}}(t, k) \) as well as the initial and boundary conditions in the plant. The conditions are \((x(0, k) = x_0(k),\notag \dot{x}(0, k) = \dot{x}_0(k)) \) and \((x(t, 0) = x_0(t))\notag \) and are included in \( c(s, z) \) through

\[
e(s, z) = c_1(s, z) x_0(z) + c_2(s, z) \dot{x}_0(z) + c_3(s, z) x_0(s)
\]

The initial and boundary conditions of the controller, expressed similarly by \( d(s, z) \), are also part of the game.

The controlled or otherwise notable variables \(^2\) naturally comprise the error \( e(s, z) \) as the measure of quality, the plant output \( y(s, z) \), as well as the plant input \( u(s, z) \). Their explicit expressions, assuming \( d(s, z) = 0 \), are

\[
e(s, z) = a(s, z) p(s, z) - \frac{p(s, z)}{m(s, z)} c(s, z),
\]

\[
y(s, z) = b(s, z) q(s, z) + \frac{p(s, z)}{m(s, z)} c(s, z),
\]

\[
u(s, z) = a(s, z) q(s, z) + \frac{q(s, z)}{m(s, z)} c(s, z)
\]

where we have denoted the common denominator by

\[
a(s, z) p(s, z) + b(s, z) q(s, z) = \tilde{m}(s, z).
\]

---

\(^2\) Notable variables not appearing in (9) and (13) can be computed from the particular platoon equations. So in Example 1, one gets the positions \( x(s, z) \) from \( r(s, z) \) via (16), etc.
The relations (33)-(35) consist of all the closed-loop transfer functions from the given variables to the controlled or notable variables.

Common denominator of all the transfer functions – the polynomial \( \bar{m}(s, z) \) – arises from (36). Given the plant, i.e. \( a(s, z) \) and \( b(s, z) \), various right hand sides can be achieved by choosing the controller, i.e. \( p(s, z) \) and \( q(s, z) \). Reversely, given the plant and the polynomial \( \bar{m}(s, z) \), (36) can be solved as a 2-D polynomial equation.

The right-hand side must vanish at all common zeros of the left-hand side polynomials \( a(s, z) \) and \( b(s, z) \). If the common zeros are stable, a stable polynomial \( \bar{m}(s, z) \) can be achieved. If they are unstable, so is every \( \bar{m}(s, z) \). See Sebek (1985) or Sebek (1992) for more on 2-D polynomial equations.

6. 2-D BIBO STABILITY AND STRING STABILITY

Two concepts of stability appear in the platooning literature: 2-D BIBO stability and string stability.

According to Kamen (1985) (Theorem 4.3, pp. 126), a spatially distributed 2-D system with a coprime transfer \( f(s, z) = b(s, z)/a(s, z) \) is BIBO stable if

\[
a(s, e^{j\omega}) \neq 0 \quad \forall s \in \mathbb{C}, \omega \in \mathbb{R} : \Re(s) \leq 0, \omega \in [0, 2\pi]
\]

(37)

In other words, if it is a stable polynomial in \( s \) after substituting for \( z \) any complex number from the unit circle.

Note that the stability condition above is not necessary. The stability can sometimes be saved by nonessential singularities of the second kind at the distinguished stability boundary (see Goodman (1977)). To avoid this subtlety, all relevant coprime transfer functions are required to have a stable denominator (not vanishing on the distinguishing boundary of the stability domain). Every polynomial satisfying (37) is called 2-D stable.

All the transfer functions in (33)-(35) have the same denominator \( \bar{m}(s, z) \). If it is stable, then all the transfer functions are BIBO stable. Even if \( \bar{m}(s, z) \) is not stable, its unstable factor may happen to cancel in relevant transfer functions. In some experiments, also the denominators of \( y_{ref}(s, z) \) or \( c(s, z) \) can be unstable.

Most papers that appeared in this domain were from the early days based on the concept of a string stability, which was introduced by Cosgriff (1969) under the name asymptotic stability and essentially means that spacing errors between neighboring vehicles (induced by disturbances, noise, or changes in the reference signals) are not amplified when propagated down the platoon. When the \( L_2 \) norm is used to measure the error signals, the necessary condition is

\[
\| \tilde{e}(s, e^{j\omega}) \|_{\infty} \leq 1
\]

This concept was later extended for nonlinear systems by Swaroop and Hedrick (1996), who actually coined the term string stability, and derived sufficient conditions as well. One of the early interesting results referring to this notion of stability is by Peppard (1974) who shows that it is impossible to achieve string stability when only measurements of relative distance from the vehicle ahead are measured and PID controller is used locally. Seiler et al. (2004) later argues that not only PID but every linear controller is incapable of string-stabilizing a platoon with such a measurement configuration simply because the achievable \( H_{\infty} \) norm is always above 1. Interestingly enough, when the relative distance from the vehicle ahead is measured as well as the absolute velocity of the vehicle, string stability can be achieved with proportional distance and velocity controllers.

7. SIMULATION EXPERIMENT

To compare different control strategies, a simulation experiment is conducted. At the beginning, the platoon is traveling at a constant speed \( \bar{x}_0 \) with the vehicles evenly spaced by \( r_{0-} \). These initial conditions are described by

\[
x_0 - (z) = -r_{0-} \frac{z^{-1}}{1 - z^{-1}} - 100 \frac{z^{-1}}{1 - z^{-1}};
\]

\[
\dot{x}_0 - (z) = \bar{x}_0 - \frac{z^{-1}}{1 - z^{-1}} = 30 \frac{z^{-1}}{1 - z^{-1}}.
\]

The vehicles should maintain their original intervals \( r_{ref} = r_{0-} \), which is expressed by the reference command

\[
r_{ref}(s, z) = r_{ref}(s) \frac{z^{-1}}{1 - z^{-1}} = \frac{100}{s} \frac{z^{-1}}{1 - z^{-1}}.
\]

Besides, the vehicles should follow their leader. At the beginning, the leading vehicle is moving at the same constant speed, but then it slows down for a while and finally returns to its original velocity. This maneuver, serving as boundary condition, is described by

\[
x_0(s) = \frac{30}{s^2} + \frac{10}{s^2} - \frac{10}{s^{15}} + \frac{10}{s^{15}}.
\]

and visualized in Fig. 2.

![Fig. 2. Leader’s maneuver \( x_0(t) \) to be followed.](image-url)
The common closed-loop denominator
\[ \bar{m}(s, z) = ms^2 p(s, z) + (z - 1)^2 q(s, z) \] (41)
is clearly unstable as seen by substituting \( z = 1 \). In fact, this is caused by the unstable common zero \((s, z) = (0, 1)\) in the plant transfer function. Even worse, no "unstable part" can be factored out of \( \bar{m}(s, z) \) so that one cannot hope to cancel it. This results went unnoticed by Melzer and Kuo (1971) so that three decades later Jovanović and Bamieh (2005) show that the original scheme actually did not provide a stabilizing solution. It should be emphasized at this point that the concept of string stability used in most papers on strings or platoons of vehicles is different from the concept of 2-D BIBO stability considered here, since the latter admits spatial steps in reference signals whereas the former only assumes spatial impulses. In other words, the BIBO stability assumes bounded but persistent spatiotemporal signals whereas the string stability assumes local disturbance and studies how it propagates downstream.

Continued Example 1. Predecessor Following Control. When substituting particular experiment conditions, the above relations can be expanded into formal power series in \( z^{-1} \). Particular terms of the series then describe behaviors of the corresponding vehicles. The terms are polynomial fractions in \( s \) which for increasing powers of \( z^{-1} \) can be shown to have increasing powers of the polynomial \( \bar{m}(s, 0) = ms^2 p(s) - q(s) \) in their denominators.

Running the experiment with \( m = 1 \) and with a PD controller
\[ \frac{q(s)}{p(s)} = -0.2 - s \] (42)
results in the distances \( r(t, k) \) and positions \( x(t, k) \) shown in Fig. 3 and Fig. 4, respectively.

Fig. 3. Distances \( r(t, k) \) in predecessor following.

Fig. 4. Positions \( x(t, k) \) in predecessor following.

The polynomial \( \bar{m}(s, 0) = s^2 + s + 0.2 \) is stable so that each vehicle behaves locally well. Yet the spatial propagation of the behavior is rather ugly. This is a demonstration of string instability. Note that by increasing the damping of the system (by increasing the coefficient at the first power of \( s \) in the denominator polynomial), this nasty propagation can be attenuated. Yet the polynomial \( \bar{m}(s, z) \) remains 2-D unstable, which demonstrates that the system is not BIBO stable and the platoon response will not scale well for a large number of vehicles.

7.2 Leader Following Control

Leader following control described in Example 2 gives rise, via (36), to closed-loop common denominator polynomial
\[ \bar{m}(s, z) = (1 - z^{-1}) ms^2 p(s) + (z^{-1} - 1) q(s) . \] (43)
Even though \( \bar{m}(s, z) \) is again 2-D unstable, it is factorable into the product of an unstable factor \((1 - z^{-1})\) with another factor
\[ \bar{m}(s) = ms^2 p(s) - q(s) . \] (44)
The unstable factor \((1 - z^{-1})\) cancels in all the terms of \( e(s, z) \) and \( r(s, z) \) but unfortunately not in \( u(s, z) \).

Continued Example 2. Leader Following Control.

With the same experiment and local controller design (42), the results are quite different. Surprisingly, the inter-vehicular distances \( r(t, k) \) are identical for all vehicles and any \( t \) so that just one function appears in Fig. 5. That is why no amplifying propagation appears in the positions \( x(t, k) \) shown in Fig. 6. And yet the system is not 2-D BIBO stable when the control variable is considered.

Fig. 5. Leader following: distances \( r(t, k) \).

Fig. 6. Leader following: positions \( x(t, k) \).

7.3 Constant Time-Headway Policy

In the constant time-headway policy (Example 3), the desired distances between the vehicles are not prescribed by the reference command \( r_{\text{ref}}(s, z) \) specified by (39) but
they result from the policy parameters $\bar{r}_0, \bar{r}$ in (26). Here the common closed-loop denominator (36) is

$$m(s,z) = ms^2p(s) + (z^{-1} - \bar{r} s - 1) q(s), \quad (45)$$

which is again 2-D unstable because, after substitution, it converts into 1-D unstable $\bar{m}(s,1) = s (msp(s) - \bar{r} q(s))$.

**Continued Example 3. Constant time-headway policy.**

For the policy parameters $\bar{r}_0 = 10, \bar{r} = 3$, resulting intervehicular distances $r(t,k)$ and positions $x(t,k)$ are shown in Fig. 7 and Fig. 8, respectively.

![Fig. 7. Constant time-headway policy: distances $r(t,k)$.](image1)

![Fig. 8. Constant time-headway policy: positions $x(t,k)$.](image2)

7.4 Spatial IIR controller

All the controllers mentioned so far were of local nature having both the denominator $p(s,z) = p(s)$ and the numerator $q(s,z) = q(s)$ free of the spatial operator $z^{-1}$. Now consider a controller governed by

$$u_k(s) - u_{k-1}(s) = p(s)q(s) + z^{-1}u_0(s).$$

**Its transform**

$$(1 - z^{-1}) p(s) u(s, z) = q(s) e(s,z) + z^{-1}p(s) u_0(s).$$

clearly matches (42) for

$$p(s,z) = (1 - z^{-1}) p(s), \quad q(s,z) = q(s), \quad u_0(s) = z^{-1}u_0(s). \quad (46)$$

This controller possesses a spatially (semi-)infinite impulse response (IIR), that is, all the local controller instances use the outcomes of the predecessors. The resulting closed-loop common denominator polynomial then reads as (43)

$$m(s,z) = (1 - z^{-1}) ms^2p(s) + (z^{-1} - 1) q(s). \quad (47)$$

Hence it is 2-D unstable but factorable into the product of unstable and stable factors

$$m(s,z) = (1 - z^{-1}) (ms^2p(s) + q(s)). \quad (48)$$

Not only that this denominator is identical to (43) but also its unstable part cancels in some closed-loop transfer functions but not in the one relating the references (and disturbances) and the control variables (outputs of the controllers). The the overall performance then closely resemble leader following control.

8. Conclusions

This paper introduces a new formalism to the control of semi-infinite platoons of vehicles following their leader. The approach is based on 2-D polynomials and their fractions resulting from a joint unilateral Laplace and $z$-transform. This makes it possible to model a variety of platoons and controllers in a unified manner as well as to apply diverse control policies such as predecessor following, leader following, constant time-headway etc. Both the string instability and 2-D BIBO instability can be defined in the proposed framework.

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