Optimal Servomechanism Control of Plants
With Fewer Inputs Than Outputs

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Abstract: This paper deals with multivariable linear control in situations where the plant
has fewer input channels than output channels. Our control approach is based on the classical
robust servomechanism problem (RSP), in which exact asymptotic tracking and disturbance
rejection are required despite uncertainty in the plant. We show that, for the class of systems
dealt with here, it is possible to approximately satisfy the RSP specifications. More precisely, we
construct a multivariable controller so that the asymptotic performance is optimal in the sense
that the norm of the steady-state tracking error is minimized. The new scheme also exhibits
some robustness, although not to the same extent as RSP controllers built for plants with at
least as many inputs as outputs. We apply the new results to two classical industrial problems.

Keywords: Linear multivariable systems; distillation columns; furnace control;
servomechanisms; robust control.

1. INTRODUCTION

Control engineers generally accept that, to obtain good
closed-loop performance in multivariable systems, at least
one control channel should be introduced for every plant
output channel; likely the most common situation is where
the number of control channels is chosen to be equal to
the number of plant output channels. This general rule is
mathematically well justified. For example, in the problem
of finding a controller such that the step response exhibits
no overshoot, Schmid and Ntogramatzidis (2010) prove that
such a controller can be found for all multivariable
plants except in the case where the plant has fewer inputs
than outputs. Another example arises in the well-known
robust servomechanism problem (RSP), which focuses on
achieving exact asymptotic tracking and disturbance re-
jection despite a certain class of plant uncertainty. A
solution to the RSP exists only if the number of plant
inputs is at least as great as the number of plant outputs
(see Davison and Goldenberg (1975), Davison (1975), and
Davison (1976)). As a third example, Smith and Davison
(1972) make arguments for using one control channel for
each plant output channel in regulation problems.

Nevertheless, there are practical scenarios in which engi-
neers might care about the control of plants that have
fewer inputs than outputs. For example, actuators must
be taken offline periodically for maintenance. The usual
approach for dealing with such a scenario is to either
turn off the entire process, or, in cases where such action
is deemed to be too expensive or unsafe, switch to a
backup (redundant) actuator. However, if it were possible
to control the system with only a small loss in performance,
despite losing an actuator, then an alternative solution is
to keep the process running with a deficiency of actuators
until the actuator can be brought online again. As another
example, in applications where cost or a desire for simplic-
ity is of paramount performance, reducing the number of
actuators outright is a very attractive possibility if the
resulting loss in performance is small. Motivated by such
scenarios, we consider in this paper the control of plants
where the number of plant inputs is less than the number
of plant outputs, and we ask if it is possible to design
a controller that achieves good performance. Our overall
approach is to solve a RSP for an artificial plant for which
the number of inputs equals the number of outputs.

Assume the plant is linear time-invariant with the model

\[ \dot{x} = Ax + Bu + Ew \]  
\[ y = Cx + Du + Fw, \]

where \( u \) is the \( m \)-channel control signal, \( y \) is the \( r \)-channel plant output, \( w \) is the \( \Omega \)-channel unmeasurable disturbance signal, and \( x \) is the \( n \)-component state. Denote the \( m \)-channel reference signal by \( y_{\text{ref}} \) and the associated tracking error by

\[ e = y - y_{\text{ref}}. \]

The basic RSP control objective is to drive \( e \) to zero
asymptotically, i.e., reject the disturbance \( w \) and track
the signal \( y_{\text{ref}} \). In the following section, assumptions,
the standard RSP formulation, and the standard RSP
solution are reviewed. A key condition for a solution to
exist is that \( m \geq r \), i.e., there must exist at least one
input channel for each output channel. In Section 3 we
determine, for plants where \( m < r \), a bound on the

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best possible steady-state performance achievable by any control scheme. Then, in Section 4, we construct, for any plant where \( m < r \), an artificial plant for which the number of inputs equals the number of outputs. We subsequently find a controller that solves the RSP for the artificial plant, and determine how such a controller performs on the actual plant. In particular, we prove that any such controller achieves optimal steady-state performance when applied to the actual plant. Two industrial examples, a distillation column system and a furnace system, are then examined in Sections 5 and 6.

2. REVIEW OF THE STANDARD RSP

For simplicity, we restrict the class of disturbances \( w \) and reference signals \( y_{\text{ref}} \) to constant signals. The disturbance \( w \) is assumed to be unknown and unmeasurable. The standard RSP involves finding a dynamic linear time-invariant controller with inputs \( y \) and \( y_{\text{ref}} \), and with output \( u \), so that the following hold:

(a) the resulting closed-loop system is stable;
(b) asymptotic tracking and disturbance rejection are achieved, i.e.,
\[
\lim_{t \to \infty} e(t) = 0, \quad \forall x(0) \in \mathbb{R}^n, \quad \forall w \in \mathbb{R}^p, \quad \forall y_{\text{ref}} \in \mathbb{R}^r,
\]
and for all controller initial conditions;
(c) condition (b) holds for all perturbations of the plant model that do not cause the resulting perturbed closed-loop system to become unstable.

The well-established necessary and sufficient conditions for the existence of a solution to the standard RSP, specialized to the case were \( w \) and \( y_{\text{ref}} \) are constants, are established in Davison and Goldenberg (1975), Davison (1975), and Davison (1976):

**Theorem 1.** A solution to the RSP exists if and only if
(i) \( (C, A, B) \) is stabilizable and detectable, and
(ii) \( \text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + r \).

If the necessary and sufficient conditions (i) and (ii) are met, then Davison and Goldenberg (1975), Davison (1975), and Davison (1976) show how to construct a controller that satisfies (a)–(c). Specifically, construct a servo-compensator, which in the special case of constant \( w \) and \( y_{\text{ref}} \) reduces to a collection of integrators,

\[
\hat{\eta} = e, \quad (4)
\]
and then use standard state-space methods (e.g., an observer-based state-feedback controller) to stabilize the following system, which is the plant augmented by the servo-compensator (where \( y_m \) is the collection of measurable outputs of the system):

\[
\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u + \begin{bmatrix} E & 0 \\ F & -I \end{bmatrix} \begin{bmatrix} w \\ y_{\text{ref}} \end{bmatrix} \quad (5)
\]

\[
y_m = \begin{bmatrix} C & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \\ y_{\text{ref}} \end{bmatrix} + \begin{bmatrix} D \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} F & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w \\ y_{\text{ref}} \end{bmatrix}. \quad (6)
\]

Fig. 1. (a) Implementation of a controller that solves the standard RSP where \( m \geq r \); (b) implementation of the proposed controller for \( m < r \).

The final controller that solves the RSP for the plant (1)–(2) consists of the servo-compensator (4) and the controller that stabilizes (5)–(6). Figure 1(a) shows the controller implementation, assuming an observer-based constant-gain control law \( u = K_0 \hat{x} + K_1 y \) is used to stabilize (5)–(6).

We emphasize that condition (ii) in Theorem 1 implies \( m \geq r \). That is, if the plant has fewer inputs than outputs, then it is impossible to satisfy (ii), and therefore impossible to meet the three objectives (a)–(c). The main purpose of this paper is to establish what can be done in the case where \( m < r \). Although it is impossible to satisfy properties (a)–(c), we show that it is possible to approximately satisfy (a)–(c) in the sense that condition (b) is relaxed so that, instead of driving \( e(t) \) to zero, it is driven to some minimum value. We also lose some robustness, in a sense specified later. We start, in the next section, by determining the optimal asymptotic performance that is achievable by any controller when \( m < r \).

3. OPTIMAL ASYMPTOTIC PERFORMANCE

Consider plant (1)–(2) under the following assumptions:

**Assumption 1:** \( m \leq r \).

**Assumption 2:** \( A \) is invertible.

**Assumption 3:** \( (C, A, B) \) is stabilizable and detectable.

**Assumption 4:** \( \text{rank}[D - CA^{-1}B] = m \).

**Assumption 5:** Both \( y_{\text{ref}} \) and \( w \) are constant (step) signals.

Note that Assumption 2 is not restrictive: if \( A \) is singular, then a pre-compensation scheme \( u = -KY + v \) can be applied to the plant so that \( A - BK \) is invertible: such a \( K \) exists by Assumption 3. Assumption 4 simplifies some of the analysis below, and it turns out to be a necessary condition for results in the next section to hold.

We start, in this section, by focusing just on steady-state performance. Consider the following cost index:

\[
J_\infty := \lim_{t \to \infty} e'(t)e(t) = \lim_{t \to \infty} [e_1^2(t) + \cdots + e_r^2(t)]. \quad (7)
\]

The value of \( J_\infty \) is bounded below as follows:
Theorem 2. Under Assumptions 1–5, the asymptotic performance of any stabilizing (possibly nonlinear and/or time-varying, and possibly with access to measurements of the plant state and/or the disturbance signal) control scheme is bounded below by

$$J_{\infty} \geq (e_{\infty}^{opt})'(e_{\infty}^{opt}),$$

where

$$\Theta := D - CA^{-1}B,$$

$$\Psi := [I, CA^{-1}E - F] \begin{bmatrix} y_{\text{ref}} \\ w \end{bmatrix}.$$  

$$e_{\infty}^{opt} := [I - \Theta(\theta'\Theta)^{-1}\Theta']\Psi.$$  

Proof: In steady-state, the plant equations satisfy

$$0 = Ax_{\infty} + Bu_{\infty} + Ew$$

$$y_{\infty} = Cx_{\infty} + Du_{\infty} + Fw.$$  

Solve (12) for \(x_{\infty}\), then substitute the result into (13) to find an expression for \(y_{\infty}\). The resulting steady-state error is \(e_{\infty} = y_{\infty} - y_{\text{ref}} = \Theta u_{\infty} + \Psi \) where \(\Theta \) and \(\Psi \) are defined in (9)–(10). Hence, \(J_{\infty} = e_{\infty}'e_{\infty} = u_{\infty}'\Theta'\Theta u_{\infty} + 2u_{\infty}'\Theta'\Psi + \Psi'\Psi\). By Assumption 4, \(\Theta \theta'\Theta \) is full rank. Therefore \(J_{\infty}\), as a function of \(u_{\infty}\), has a unique minimum that can be found by differentiation. Such a calculation, after routine matrix manipulations, results in the following optimal steady-state control signal and error:

$$u_{\infty}^{opt} = (\theta'\Theta)^{-1}\theta'\Psi$$  

$$e_{\infty}^{opt} = [I - (\theta'\Theta)^{-1}\theta']\Psi.$$  

Result (8) immediately follows. □.

In the special case \(m = r\), matrix \(\Theta \) is square and invertible, and the expressions in Theorem 2 simplify so that \(e_{\infty}^{opt} = 0\), i.e., exact asymptotic tracking and disturbance rejection are not ruled out. This conclusion is consistent with Section 2, where it was shown that it is possible to achieve \(e_{\infty} = 0\) when \(m = r\). For \(m < r\), however, Theorem 2 implies it is not possible to achieve \(e_{\infty} = 0\) for arbitrary \(w\) and \(y_{\text{ref}}\).

4. MAIN RESULT

As in the previous section, consider the plant (1)–(2) under Assumptions 1–5. The disturbance \(w\) is assumed to be unknown and unmeasurable. Here we propose the following control scheme:

Step 1: Introduce the \(m\)-channel signals \(\tilde{y}, \tilde{y}_{\text{ref}}, \) and \(\tilde{e},\)

$$\tilde{y} := \theta' y$$

$$\tilde{y}_{\text{ref}} := \theta' y_{\text{ref}}$$

$$\tilde{e} := \tilde{y} - \tilde{y}_{\text{ref}},$$

where \(\Theta \) is defined in (9).

Step 2: Form a standard RSP using \(\tilde{y}, \tilde{y}_{\text{ref}}, \) and \(\tilde{e}\) in place of the the \(r\)-channel signals \(y, y_{\text{ref}}, \) and \(e\). That is, we aim to solve the standard RSP for the following plant that has \(m\) inputs and \(m\) outputs:

$$\dot{x} = Ax + Bu + Ew$$

$$\tilde{y} = \Theta'Cx + \Theta'Du + \Theta'Fw.$$  

Assuming the output \(y\) can be accessed by the controller, the following theorem establishes that a solution to the RSP in Step 2 always exists:

Theorem 3. Under Assumptions 1–5, the RSP for (19)–(20) is solvable.

Proof: By Theorem 1, the RSP is solvable if

(i) \((C, A, B)\) is stabilizable and detectable, and

(ii) \(\text{rank} \begin{bmatrix} A & B \\ \theta'C & \theta'D \end{bmatrix} = n + m.\)

For condition (i), we refer to the stabilizability and detectability of \((C, A, B)\) rather than \((\theta'C, A, B)\) because we assume that the true plant output, \(y\), is measurable, as shown in Figure 1(b). Condition (i) is satisfied due to Assumption 3.

The rank condition (ii) is equivalent to \(\text{rank} \begin{bmatrix} \theta'D - \theta'CA^{-1}B \end{bmatrix} = m\), which, in turn, is equivalent to \(\text{rank} \begin{bmatrix} \theta' \end{bmatrix} = m\), which, since \(m \leq r\), is equivalent to \(\text{rank} \begin{bmatrix} \theta' \end{bmatrix} = m\). However, this last condition is satisfied by Assumption 4. □

Figure 1(b) shows how the proposed controller is implemented; the implementation is no more complicated than that of the standard RSP except for the one extra gain block, \(\theta'\).

The following theorem establishes that the proposed controller achieves optimal steady-state performance:

Theorem 4. Under Assumptions 1–5, any controller that solves that RSP problem for the artificial plant (19)–(20) achieves optimal steady-state performance for the actual plant (1)–(2), in the sense that equality is achieved in (8).

Proof: In steady-state, any controller that solves the RSP for the artificial plant (19)–(20) results in

$$0 = Ax_{\infty} + Bu_{\infty} + Ew$$

$$\tilde{y}_{\infty} = \theta'Cx_{\infty} + \theta'Du_{\infty} + \theta'Fw.$$  

$$\tilde{e}_{\infty} = \tilde{y}_{\infty} - \tilde{y}_{\text{ref}} = 0.$$  

Solve for \(x_{\infty}\) from (21), substitute that result into (22) to find an expression for \(\tilde{y}_{\infty}\), and then substitute that expression into (23). This calculation leads to

$$\Theta'\Theta u_{\infty} + \theta'(F - CA^{-1}E)w - \tilde{y}_{\text{ref}} = 0.$$  

By Assumption 4, \(\theta'\Theta\) is invertible, so the above equation implies

$$u_{\infty} = (\theta'\Theta)^{-1}\tilde{y}_{\text{ref}} - (\theta'\Theta)^{-1}\theta'(F - CA^{-1}E)w$$

$$= (\theta'\Theta)^{-1}\theta'y_{\text{ref}} - (\theta'\Theta)^{-1}\theta'(F - CA^{-1}E)w$$

$$= (\theta'\Theta)^{-1}\theta'\Psi,$$  

where \(\Psi\) is defined in (10). The value of the steady-state control signal in (24) is identical to the optimizing value in (14), so therefore steady-state performance is optimal, i.e., equality is achieved in (8). □
Fig. 2. The distillation column system for Example 1, reproduced from (Davison, 1967, Fig. 2) with new labelling.

In addition to optimal steady-state performance, the proposed controller exhibits robustness in the sense that optimal asymptotic performance is guaranteed for any plant perturbation such that

(i) the closed-loop system remains stable, and
(ii) the steady-state plant gain, (9), remains unchanged.

Point (ii) arises since obtaining optimal asymptotic performance depends on knowing the value of \( \Theta = D - CA^{-1}B \), which is the steady-state gain of the plant. The standard RSP exhibits robustness even if the steady-state plant gain changes, so robustness suffers to some degree when the plant has fewer control signals than plant outputs.

5. EXAMPLE 1: A DISTILLATION COLUMN

Distillation columns have, for many decades, been key components of many industrial processes. Given their huge economic importance and complicated multivariable dynamics, the control of distillation columns remains an active research area. An excellent overview of distillation column control is provided by Buckley et al. (1985). Here we use a linearized distillation column model derived by Davison (1967); the model has served as a benchmark system for multivariable control.

The distillation column is shown in Figure 2, and parameters for the model (1)–(2) are given in Table 2 at the end of the paper. The plant has three actuators: the reboiler power \( u_1 \), the reflux valve \( u_2 \), and the condenser valve \( u_3 \). The plant has three outputs: top product composition \( y_1 \), bottom product composition \( y_2 \), and column pressure \( y_3 \). See Davison (1967) for information about units of measurement. The flow into the column is treated as a (scalar) disturbance, \( w \). The main control objective is to keep the output compositions of the column constant, independent of the inflow, i.e., a disturbance rejection problem. The open-loop response of the system to a unit step disturbance is shown in Figure 3.

In practice, engineers are often hesitant to use the condenser valve \( u_3 \) for control purposes. Rather, only reboiler power \( u_1 \) and reflux valve \( u_2 \) are used for active control. Here we compare two control strategies:

**S1:** Use \( u_1 \) and \( u_2 \) to achieve exact asymptotic disturbance rejection on \( y_1 \) and \( y_2 \), and hope that \( y_3 \) behaves reasonably well.

**S2:** Use \( u_1 \) and \( u_2 \) to achieve optimal asymptotic disturbance rejection on all three outputs.

Fig. 3. Simulation of the open-loop response to a unit step disturbance for the distillation column. The numbers indicate steady-state values.

Fig. 4. Simulation comparing Strategies S1 (dashed lines) and S2 (solid lines) for the distillation column. The numbers indicate steady-state values.
For Strategy $S_3$, the number of plant input channels equals the number of plant output channels ($m = r = 2$), so either the standard RSP methods of Section 2, or the new proposed method, presented in Section 4, can be used to design a controller. We choose to use the standard RSP approach to design an observer-based controller, as shown in Figure 1(a), where the observer gain matrix and feedback gain matrices ($K_0$ and $K_1$) were chosen using LQR methods. Closed-loop simulation results for a unit step disturbance on $w$ are shown in Figure 4 (dashed lines). Notice that exact asymptotic disturbance rejection is obtained for $y_1$ and $y_2$, as expected. The steady-state performance of $y_3$ is better than the open-loop performance (it drops from 4.5373 to 0.1390), but is still quite far from zero.

For Strategy $S_2$, the plant has fewer inputs than outputs ($m = 2$ and $r = 3$), so a standard RSP-based controller does not exist. However, the control approach proposed in Section 4 is applicable. Again we used an observer-based control approach with the observer and state-feedback gains both designed using LQR methods. Closed-loop simulation results are shown in Figure 4 (solid lines). Note that, consistent with Theorem 4, optimal asymptotic performance is achieved since the value of $J_{\infty}$ (computed using (7)) equals $(e_{\infty}^y)'(e_{\infty}^u)$ (computed using (9)–(11)). From an engineering perspective, both Strategy $S_1$ and Strategy $S_2$ yield good performance compared to the open-loop behaviour. However, Strategy $S_2$ is better than Strategy $S_1$ because the steady-state behaviour of $y_3$ is significantly improved (it drops from 0.1390 to 0.0004748) while the loss in steady-state performance for $y_1$ (an increase from zero to 0.003165) and $y_2$ (an increase in magnitude from zero to 0.007467) is negligible. In other words, at least for this example, the sacrifice in losing exact steady-state disturbance rejection on $y_1$ and $y_2$ is more than compensated by the improvement in performance on $y_3$. We emphasize that this improvement is attained without using the third actuator. (If the third actuator was to be used, then standard RSP methods apply and it would be possible to achieve exact asymptotic performance on all three outputs.)

**6. EXAMPLE 2: A FURNACE**

The second example deals with an industrial boiler furnace system that is described in (Rosenbrock, 1974, Section 4.3) and shown in Figure 5. The system has four sets of heating coils, each located above a burner. The four burners and coils are enclosed in an insulated box. The outlet temperature of each of the coils is measured, leading to a 4-input 4-output system. Parameters for the model (1)–(2) are given in Table 3 on the next page. The goal is to adjust the burners so that the coil temperatures track a desired temperature profile. Here we investigate five approaches:

A1: Use all four actuators to control the four outputs.
A2: Use $u_1$, $u_2$, and $u_3$ to try to control all four outputs.
A3: Use $u_2$ and $u_3$ to try to control all four outputs.
A4: Use just $u_2$ to try to control all four outputs.
A5: Use $u_2$ to control $y_2$, hoping the other outputs behave reasonably well.

In the first four approaches, we use the proposed control scheme of Section 4. In Approach $A_5$, we use standard RSP methods, summarized in Section 2. In all cases, observer-based controllers were designed with gains chosen using LQR methods.

The simulation outputs for the five controllers are given in Figure 6 for the reference signal $y_{\text{ref}}(t) = [1 1 1 1]'$.
The steady-state tracking performance is summarized in Table 1. Observe the following:

- As expected, only for Approach $A_1$ is exact asymptotic tracking achieved for all four channels.
- In Approaches $A_2$, $A_3$, and $A_4$, optimal steady-state performance is achieved, i.e., equality is achieved in (8). It is not possible for any control scheme to perform better.
- In comparing Approaches $A_1$–$A_4$, we see that, unsurprisingly, asymptotic performance degrades when actuators are removed.
- A comparison of Approaches $A_4$ and $A_5$ (each which uses just one actuator) shows that, by incorporating all four plant outputs into the control scheme, overall better performance can be achieved.

7. SUMMARY AND EXTENSIONS

We have proposed a technique for dealing with multivariable linear control of plants that have fewer inputs than outputs. The technique is based on formulating an artificial plant for which the number of inputs equals the number of outputs, and solving the RSP for that plant. Such an approach guarantees optimal steady-state performance for the actual plant (see Theorem 4). Based on this theory and two examples, we reach two conclusions of practical interest: (i) although the best achievable performance degrades when actuators are taken offline, the degree of degradation is not necessarily severe; and (ii) in situations where a plant has fewer inputs than outputs, it is better to incorporate into the controller design the performance of all the outputs rather than just a subset of them.

Our approach to dealing with systems that have fewer inputs than outputs can be readily extended to discrete-time systems, decentralized control problems, tuning regulator problems, and problems where transient shaping is desired.

REFERENCES


Table 2. State-space matrices for the open-loop distillation column system (Example 1).

\[
A = 10^{-2}, \quad B = 10^{-4}, \quad E = 0.01 \]

Table 3. State-space matrices for the open-loop furnace system (Example 2).

\[
A = 10^{-2}, \quad B = -0.334 -0.223 -0.4942 -0.416 \\
-0.161 -0.247 0.1345 0.330 \\
0.148 -0.329 0.0593 -0.435 \\
0.199 -0.270 -0.2105 -0.258 . \\
-0.157 0.245 0.0557 0.281 . \\
0.076 -0.048 -0.0740 -0.024 . \\
-0.020 0.050 0.0288 0.069 . \\
-0.038 0.098 0.0084 0.062 . \\
\]

\[
D = 0, F = 0. \\
\]

\[
C = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 1.0 \\ 0 & 0.000 & 0.000 & 0.0 \\ 0 & 0.000 & 0.000 & 0.0 \\ \end{bmatrix} \\
\]

\[
A = 10^{-2}, \quad B = -0.076 -0.048 -0.0740 -0.024 . \\
-0.020 0.050 0.0288 0.069 . \\
-0.038 0.098 0.0084 0.062 . \\
\]

\[
D = 0, E = 0, F = 0. \\
\]