A Novel Estimate of The Domain of Attraction of an IDA-PBC of a Ball and Beam System

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Abstract: The ball and beam system belongs to the class of underactuated mechanisms. Recently, several control design techniques have been proposed to control underactuated systems. The Interconnection and Damping Assignment (IDA)—a formulation of Passivity–Based Control (PBC)—is one of these control design tools. The IDA-PBC technique applied to the stabilization of the ball and beam has been recently reported. The contribution of this paper is a novel approach to obtain an estimate of the domain of attraction of the IDA-PBC ball and beam system that resolves the practical motivated formulation of ensuring simultaneously that the system remains within the beam length. Numerical simulations are presented to illustrate these results.

Keywords: Ball and beam, Stability, Underactuated mechanical systems, Friction.

1. INTRODUCTION

The control of underactuated systems has attracted the attention of researchers during the last years due to the interesting theoretical challenges and practical applications. Several approaches have been introduced to control system design of underactuated systems, see e.g., [1],[2],[4]. The Interconnection and Damping Assignment (IDA)—a formulation of Passivity–Based Control (PBC)—is one of these control design tools. The IDA-PBC technique applied to the stabilization of the ball and beam has been recently reported. The contribution of this paper is a novel approach to obtain an estimate of the domain of attraction of the IDA-PBC ball and beam system that resolves the practical motivated formulation of ensuring simultaneously that the beam evolves within upward configurations and the ball remains in the beam. Numerical simulations are presented to illustrate these results.

The Hamiltonian model of the ball and beam system with viscous friction shown in Figure 1, is given by

\[
\frac{d}{dt} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1}{p_2} \\ \frac{q_1 p_2}{L^2 + q_1^2} - g \sin(q_2) - f_{v_1} p_1 \\ u - g q_1 \cos(q_2) - f_{v_2} p_2 \frac{p_2}{L^2 + q_1^2} \end{bmatrix}
\]  

(1)

where \( q_1 \) and \( q_2 \) are the ball position and the bar angle, respectively. \( p_1 \) and \( p_2 \) are the momenta of each coordinate, \( L \) is the length of the bar, the viscous friction coefficients of each coordinate are given by \( f_{v_1}, f_{v_2} \), the control input is \( u \) and \( g \) is the acceleration due to gravity. The control objective is to drive the ball and beam to rest position in the middle of the beam \( q_1 = 0 \) and the beam at its horizontal position \( q_2 = 0 \). Formally, the control aim is to ensure that

\[
\lim_{t \to \infty} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

starting from a set of initial conditions as large as possible.

This paper studies the control system analyzed in [6] to the above formulation. The contribution of this paper is a novel estimate of the domain of attraction to ensure system performance in the sense that the evolution of the ball position remains within the beam length.
2. IDA-PBC SYSTEM

The rational behind the IDA-PBC approach for control system design is to force the closed-loop system to have a desired energy function both, kinetic and potential energies. The potential energy function is chosen to have a minimum point at the desired configuration $q_1 = q_2 = 0$. According to the IDA-PBC procedure, it results that this configuration at rest $\dot{q}_1 = \dot{q}_2 = 0$ corresponds to a stable equilibrium.

In order to prove asymptotic stability, this control system has been recently analyzed in [6] invoking the Krasovski-LaSalle theorem. In this paper we propose a novel estimate of the domain of attraction to guarantee that the ball remains in the beam. Finally, the practical problem of keeping the beam in upward configurations is formulated and resolved in this paper.

The control law proposed in [4] is shown at the top of next page, where $k_p$ a positive gain, and in [6] we have proposed

$$k_p = \frac{[L^2 + q_1^2]f_p^2 + f_{q_2}^2}{2\sqrt{2}[L^2 + q_1^2]f_{q_1}}.$$  

(3)

For the sake of simplicity denote $q = [q_1 \ q_2]^T$ and $p = [p_1 \ p_2]^T$. It is shown in [4] that this controller induces a closed-loop system whose total energy function is

$$V(q, p) = \frac{1}{2} \left[ \frac{\sqrt{2}q_1^2}{L^2 + q_1^2} - \frac{2p_1 p_2}{L^2 + q_1^2} + \frac{\sqrt{2}p_2^2}{(L^2 + q_1^2)^2} \right] + V_d(q)$$

(4)

where

$$V_d = g[1 - \cos(q_2)] + \frac{k_p}{2} \left[ q_2 + \frac{1}{\sqrt{2}} \text{arcsinh} \left( \frac{q_1}{L} \right) \right]^2.$$  

(5)

The first term of the right hand side corresponds to kinetic energy and $V_d(q)$ is the potential energy.

It is convenient to define the set $D \subset \mathbb{R}^4$ as

$$D = \left\{ \frac{q}{p} \in \mathbb{R}^4 : q_2 \in (-\pi, \pi) \right\},$$

and

$$L_V(c) = \left\{ \left[ \begin{array}{c} q \\ p \end{array} \right] \in D : V(q, p) < c \right\}. $$  

(6)

The main stability result concerning this system is stated as Proposition 4 in [4]; for the reader convenience is quoted below in a slightly different form

Proposition 1. Consider the ball and beam system (1) under the IDA-PBC control law (2). Then, the origin of the closed-loop system state space described in terms of $[q_1 \ q_2 \ p_1 \ p_2]^T \in \mathbb{R}^4$ is an asymptotically stable equilibrium. Furthermore, an estimate of the domain of attraction is $L_V(2g)$.

A by-product of Proposition 1 is derived in this paper to achieve control system performance in two senses:

- the beam angle $q_2$ stays always upward, that is $|q_2(t)| < \frac{\pi}{2}$ for all $t > 0$.
- the ball position $q_1$ remains always in the beam length, that is, $|q_1(t)| < L$ for all $t > 0$.

New results on these issues are presented in the following Corollary 1. Consider the IDA-PBC control of the ball and beam system.

a) An estimate of the domain of attraction of the origin of the state space that ensures $|q_1(t)| < \frac{\pi}{2}$ and for all $t \geq 0$ is $L_V(c^*)$ where $g > c^* > 0$.

b) An estimate of the domain of attraction of the origin of the state space that ensures $|q_1(t)| < L$ is $L_V(c^*)$ where

$$c^* = g[1 - \cos(x)] + \frac{\sqrt{2}}{2k_p} \sin^2(x)$$

(7)

with $x \in (-\pi, \pi)$ the unique solution of

$$x + \frac{g}{k_p} \sin(x) = \frac{\arcsinh(1)}{\sqrt{2}}.$$  

(8)

c) The estimate of the domain of attraction $L_V(c^*)$ of the origin of the state space with $c^*$ in (7) ensures

$$|q_1(t)| < L \text{ and } |q_2(t)| < \frac{\arcsinh(1)}{\sqrt{2}} < \frac{\pi}{2}$$

for all $t \geq 0$.

Corollary 1 extends the previous published analysis of this system in three directions. First, the practical requirement of keeping the beam angle within upward configurations is formulated and resolved in item a. Second, a novel estimate of the domain of attraction to maintain the ball position in the beam length is provided in item b. Third, previous issues are addressed together in item c.

A remark on item a of Corollary 1 is the following. It shall be shown latter on that the estimate of the domain of attraction $L_V(c^*)$ where $c^* = g[1 - \cos(\epsilon)]$ with $\pi > \epsilon > 0$ ensures that $|q_2(t)| < \epsilon$ for all $t \geq 0$. Obviously when $\epsilon = \frac{\pi}{2}$, it results $c^* < g$ as stated in item a.

Item c of Corollary 1 establishes that if the initial condition satisfies

$$[q(0) \ p(0)] \in L_V(c^*)$$

where $c^*$ is the value in (7), then $q_1(t) \in (-L, L)$ and $q_2(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \geq 0$. Furthermore, the closed-loop system trajectories evolve in $L_V(c^*)$ and they will asymptotically tend to the origin of the state space.

It is worth noticing that if the control gain $k_p$ is sufficiently large in the sense $k_p > g$, then the unique solution $x$ of (8) can be easily obtained thanks to the contraction

$\nabla \nabla \nabla$
\[
\begin{align*}
\frac{q_1}{\sqrt{2}(L^2 + q_1^2)} & \left[ -\sqrt{L^2 + q_1^2} p_1^2 + \sqrt{2} p_1 p_2 + \frac{1}{(L^2 + q_1^2)} p_2^2 \right] \\
+ g q_1 \cos(q_2) - g \sqrt{2(L^2 + q_1^2)} \sin(q_2) - k_p \sqrt{L^2 + q_1^2} & \left[ q_2 - \frac{1}{\sqrt{2}} \arcsinh \left( \frac{q_1}{L} \right) \right] \\
+ \frac{k_c}{L^2 + q_1^2} & \left[ p_1 - \frac{2}{L^2 + q_1^2} p_2 \right]
\end{align*}
\]  

(2)

mapping theorem as the steady state solution \( (k \to \infty) \) of the difference equation

\[
x_{k+1} = \frac{\arcsinh(1)}{\sqrt{2}} - \frac{g}{k_p} \sin(x_k)
\]

(9)

for any initial condition \( x_0 \in \mathbb{R} \).

The proof of Proposition 1 has been shown in [6], while the proof of Corollary 1 is carried out in the following section.

3. PERFORMANCE: A NOVEL ESTIMATE OF DOMAIN OF ATTRACTION

An estimate of the domain of attraction is provided in the proof of Proposition 5 in [4] as \( L_V(2g) \). This set qualifies as an estimate of the domain of attraction because it is bounded. This is a consequence of the following inequalities on the terms of the Lyapunov function (4)

\[
\frac{1}{2} \left[ \frac{\sqrt{2} p_1^2}{\sqrt{L^2 + q_1^2}} - \frac{2 p_1 p_2}{L^2 + q_1^2} + \frac{\sqrt{2} p_2^2}{(L^2 + q_1^2)^2} \right] < 2g
\]

and

\[
g[1 - \cos(q_2)] + \frac{k_p}{2} \left[ q_2 + \frac{1}{\sqrt{2}} \arcsinh \left( \frac{q_1}{L} \right) \right]^2 < 2g
\]

which imply that \( q_1 \) and \( q_2 \), and \( q_1 \) and \( q_2 \), respectively, are bounded and they belong to \( D \). A trivial consequence is that the set \( L_V(c) \) for any \( 2g > c > 0 \) is also a smaller estimate of the domain of attraction.

From a practical point of view, it is enforced to keep the ball position \( q_1 \) within the beam length \( L \). Corollary 1 provides an estimate of the domain of attraction that ensures such a requirement. Let \( L_{Vq}(c) \) denote the level set of the potential energy function \( V_{d}(q) \) given by

\[
L_{Vq}(c) = \{ q_1 \in \mathbb{R}, q_2 \in (-\pi, \pi) : V_{d}(q) = V(q, 0) < c \}.
\]

According to this definition and Lyapunov function structure (4), it is easy to show that the following implication holds for \( 2g > c > 0 \)

\[
\{ q \} \in L_V(c) \Rightarrow q \in L_{Vq}(c)
\]

(10)

The level curve \( \partial L_{Vq}(c) \) of the potential energy function is defined by

\[
\partial L_{Vq}(c) = \{ q_1 \in \mathbb{R}, q_2 \in (-\pi, \pi) : V_{d}(q) = V(q, 0) = c \} = \left\{ g[1 - \cos(q_2)] + \frac{k_p}{2} \left[ q_2 + \frac{1}{\sqrt{2}} \arcsinh \left( \frac{q_1}{L} \right) \right]^2 \right\}
\]

(11)

For a given \( c \) such as \( 2g > c > 0 \), the projection on the plane \( q_2 - q_1 \) of the level curve \( \partial L_{Vq}(c) \) can be equivalently expressed by the parameterized curves

\[
q_1 = L \sinh \left( \sqrt{2} \left[ q_2 \pm \frac{2}{k_p} [c - g(1 - \cos(q_2))] \right] \right)
\]

which are well defined in the interval

\[
q_2 \in (-q_2, q_2) \text{ with } q_2 = \arccos \left( \frac{1 - c}{g} \right)
\]

(12)

Figure 2 depicts a typical form of the level curve \( \partial L_{Vq}(c) \). In virtue that \( 2g > c > 0 \), hence \( \pi > q_2 \). Notice that in the level set \( L_{Vq}(c) \) the beam position satisfies \( |q_2| < q_2 \). Furthermore, in virtue of (12) the corresponding values of \( c \) are \( c < g[1 - \cos(q_2)] \).

Fig. 2. Level curve \( \partial L_{Vq}(c) \) of the potential energy.

Thus, when the beam position \( q_2 \) is desired to evolve within upward configurations \( q_2 = \frac{\pi}{2} \), the corresponding values of \( c \) must hold \( c < g \) as stated in item a of Corollary 1.

In order to determine the minima and maxima of (11), we look at its critical points. The values of \( q_2 \), say \( q_2^* \), where the time derivative of (11) vanishes are those satisfying

\[
\mp \frac{2}{k_p} [c - g[1 - \cos(q_2^*)]] = \frac{g}{k_p} \sin(q_2^*).
\]

(13)

Therefore, the minimum and maximum values of \( q_1 \), denoted by \( q_1^* \), are obtaining by substituting above expression in (11), hence

\[
q_1^* = \pm L \sinh \left( \sqrt{2} \left[ q_2^* + \frac{g}{k_p} \sin(q_2^*) \right] \right).
\]

Since we wish that \( |q_1^*| = L \), then the value of the sinh(·) function must be one. Consequently, one obtains
\[ q^*_2 + \frac{g}{k_p} \sin(q^*_2) = \frac{\operatorname{asinh}(1) \sqrt{2}}{\sqrt{2}} \]

which corresponds to (8). In the interval \( q^*_2 \in (-\pi, \pi) \), above equation has a unique solution —regardless the positive value of the control gain \( k_p \) —which can be easily found by graphical or numerical methods. Once \( q^*_2 \) is determined, the corresponding value of \( c \), say \( c^* \), is obtained straightforwardly from (13); this produces (7) of Corollary 1.

In sum, under this choice of \( c^* \), the points \([q_1, q_2]\) of the level curve \( \partial L_{V_2}(c^*) \) satisfy \( |q_1| < L \) and therefore all points \([q_1, q_2]\) of the level set \( L_{V_2}(c^*) \) also hold \( |q_1| < L \).

Therefore, in virtue that \( 2g > c^* > 0 \), the level set \( L_V(c^*) \) is an invariant set and an estimate of the domain of attraction, then from implication (10) it follows

\[
\begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} \in L_V(c^*) \Rightarrow \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \in L_V(c^*)
\]

\[ \Rightarrow q(t) \in L_{V_2}(c^*) \Rightarrow |q_1(t)| < L \]

for all \( t \geq 0 \). In words, for all initial conditions starting off in \( L_V(c^*) \), the trajectories remain inside \( L_V(c^*) \), \( |q_1(t)| < L \) for all \( t \geq 0 \), and according to (12) we have \( |q_2(t)| < q_2 < \pi \) for all \( t \geq 0 \). This proves item \( b \) of Corollary 1.

In order to prove item \( c \) of Corollary 1, we shall show that \( c^* \) computed through (7) holds \( g > c^* \). Since by design \( k_p > 0 \), from (8) we obtain

\[ k_p = \frac{g \sin(x)}{\operatorname{arcsinh}(1)} \sqrt{2} - x \]

as far as \( x \in \left(0, \frac{\operatorname{arcsinh}(1)}{\sqrt{2}}\right) \). Plugging in (7) leads to

\[ c^* = g \left[ 1 - \cos(x) + \frac{1}{2} \left[ \frac{\operatorname{arcsinh}(1)}{\sqrt{2}} - x \right] \sin(x) \right]. \]

It is easy to check that \( \partial c^*/\partial x > 0 \) for \( x \in \left(0, \frac{\operatorname{arcsinh}(1)}{\sqrt{2}}\right) \). This means that \( c^* \) is an increasing function of \( x \) in this interval and has its maximum value at \( x = \frac{\operatorname{arcsinh}(1)}{\sqrt{2}} \), therefore it ensures that

\[ c^* < g \left[ 1 - \cos \left( \frac{\operatorname{arcsinh}(1)}{\sqrt{2}} \right) \right] \]

where the right hand side is smaller than \( g \). As previously shown, it is enough to guarantee that

\[ |q_2(t)| < \frac{\operatorname{arcsinh}(1)}{\sqrt{2}} \]

for all \( t \geq 0 \). Finally, this together with item \( b \) prove item \( c \) of Corollary 1.

4. SIMULATIONS

Numerical simulation are presented to illustrate the main theoretical results. The numerical value of the ball and beam system (1) and (2) are \( L = 1 \), \( f_{v_1} = 1 \), \( f_{v_2} = 1 \) and \( g = 9.8 \). The gains of the IDA-PBC control law (4) are \( k_p = 20 \) and \( k_c \) is given by (3).

Since the gain \( k_p \) satisfies \( k_p > g \), above values allow to compute the solution of (8) following the numerical method (9); this leads to \( x = 0.4224 \). Plugging this value in (7) produces \( c^* = 1.2648 \). For simulation purposes we have utilized a bit smaller value \( c^* = 1.2 \) which is smaller than \( g \).

According to item \( c \) of Corollary 1, if the initial state lies in \( L_V(1.2) \), then the trajectory remains in this set, the ball and beam positions satisfy \( |q_1(t)| < 1 \) and \( |q_2(t)| < \frac{\pi}{2} \) for all \( t \geq 0 \), and they tend to zero.

Two set of initial condition were considered for simulation: \( q_1(0) = 0.25, q_2(0) = -0.12 \), and \( q_1(0) = 0.96, q_2(0) = 0.4 \), both with zero initial velocities. It is easy to check that these initial conditions are in \( L_V(1.2) \). The corresponding simulation results are depicted in Figures 3 and 4, respectively. These figures also show the level curve \( \partial L_{V_2}(1.2) \), that is, the border of a region in the plane \( q_2 - q_1 \) where the initial configuration \( [q_1(0) \ q_2(0)]^T \) can start off (assuming zero initial velocities) guaranteeing that the projection in such a plane of the trajectories will remain inside and finally approach asymptotically the origin. This behavior is demonstrated in both Figures.

5. CONCLUSION

This paper has provided a novel approach to obtain an estimate of the domain of attraction that ensures control system performance in the sense of maintaining the beam in upward configurations and the ball position within the beam length.
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