Spectral Conditions for the Negative Imaginary Property of Transfer Function Matrices

Mohamed Mabrok, [*] **Abhijit G. Kallapur,** [*] **Ian R. Petersen,** [*] **and Alexander Lanzon** [**]

[*] *School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia, email:abdallamath@gmail.com,abhijit.kallapur@gmail.com, i.r.petersen@gmail.com*
[**] *Control Systems Center, School of Electrical and Electronic Engineering, University of Manchester, Manchester M60 1QD, United Kingdom, email:Alexander.Lanzon@manchester.ac.uk*

Abstract: This paper derives some necessary and sufficient conditions for linear time invariant systems to have the negative imaginary property in both the single-input-single-output as well as the multi-input-multi-output cases. The conditions for a system to be negative imaginary are described in terms of spectral conditions obtained for a given transfer function matrix.

Keywords: Negative imaginary systems, Positive real systems, Hamiltonian matrix.

1. INTRODUCTION

Positive real (PR) linear time invariant (LTI) systems and passivity theory have been well researched in the control theory literature; e.g., see Anderson and Vongpanitlerd [1973], Brogliato et al. [2007]. However one of the drawbacks of the PR theory is the requirement for the relative degree of the system transfer function to be either zero or one Brogliato et al. [2007]. This limits the application of the positive real theory and it can not be applied to applications such as those involving flexible structures with collocated force actuators and position sensors; e.g., see Petersen and Lanzon [2010].

Lanzon and Petersen introduced a new class of linear systems in Lanzon and Petersen [2007, 2008] called negative imaginary (NI) systems. In the single-input single-output (SISO) case, such systems are defined by considering the properties of the imaginary part of a transfer function $G(s) = D + C(sI - A)^{-1}B$, where $G(s)$ belongs to the set of real-rational stable transfer functions. This work was extended by Xiong et. al. in Xiong et al. [2009a,b, 2010] by allowing for simple poles on the imaginary axis of the complex plane except at the origin. Furthermore, NI controller synthesis has also been discussed in Lanzon and Petersen [2007, 2008]. A feature of NI systems is that they do not have the same restrictions on the relative degree as PR transfer function matrices and can be applied to a different range of applications including flexible structures with collocated force actuators and position sensors; e.g., see Petersen and Lanzon [2010].

One topic of interest in the theory of PR systems has been to test for the positive realness of a given transfer function matrix; e.g., see Wen [1988], Bai and Freund [2000], Gao and Zhou [2003], Shorten and Narendra [2003], Shorten et al. [2008b]. An efficient method to test for the positive realness of a proper transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D + D^T > 0$ is to check the eigenvalues of its corresponding Hamiltonian matrix Wen [1988], Bai and Freund [2000], Gao and Zhou [2003]. However, this method fails when $D + D^T$ is singular. Shorten et. al., propose strict spectral conditions to test the positive realness of a given transfer function matrix in the case in which $D + D^T$ is singular for both the SISO Shorten and King [2004] as well as multi-input multi-output (MIMO) cases Shorten et al. [2008a]. In this paper, we extend the ideas in Shorten et al. [2008a] to test if a MIMO transfer function matrix is NI. This is achieved by checking the eigenvalues of a Hamiltonian matrix associated with the transfer function matrix under consideration. We also present an alternative method to check whether a given transfer function matrix is NI in the SISO case, which is a modification of the spectral method proposed in Shorten and King [2004] for PR systems. These results may be useful in both the analysis of NI systems, and in the synthesis of NI controllers using optimization techniques.

The remainder of the paper is organized as follows: Section 2 introduces the concept of positive real and negative imaginary systems and presents a relationship between them. Section 3 describes the main results of the paper, discusses the construction of the Hamiltonian matrix, and provides conditions under which a given system transfer function matrix is NI for both MIMO as well as SISO systems. A numerical example is presented in Section 4

⋆ This work was supported by the Australian Research Council. this paper has submitted to IFAC congeries 2010

and finally, the paper is concluded with remarks and future work outlined in Section 5.

2. PRELIMINARIES

In this section, we introduce the concept of PR and NI systems in terms of previously established definitions and lemmas. We also present a lemma describing the relationship between PR and NI systems which will be used in deriving spectral conditions for NI systems in Section 3.

2.1 Positive Real Systems

The definition of PR systems is motivated by the study of linear electric circuits composed of resistors, capacitors, and inductors. The same definition applies for analogous mechanical and hydraulic systems. This idea can be extended to study electric circuits with nonlinear passive components and magnetic couplings. Here, we present definitions and a lemma describing PR systems in terms of their transfer function matrix. For a detailed discussion on PR systems, see Anderson and Vongpanitlerd [1973], Brogliato et al. [2007] and references therein.

Definition 1. A transfer function $f(s)$ is said to be positive real if:

1. $f(s)$ is analytic in $Re[s] > 0$.
2. $Re(f(s)) \geq 0$ for all $Re[s] > 0$.
3. $f(s)$ is real for positive real s.

Definition 2. A square transfer function matrix $F(s)$ is positive real if:

1. $F(s)$ has no pole in $Re[s] > 0$.
2. $F(s)$ is real for all positive real s.
3. $F(s) + F(s)^* \geq 0$ for all $Re\ [s] > 0$.

Here $F(s)^*$ denotes the complex conjugate transpose of $F(s)$.

The Hamiltonian matrix provides a convenient method for checking whether or not a given transfer function matrix is PR Shorten et al. [2008b]. The following lemma describe the relationship between the transfer function matrix $F(s)$ and the corresponding Hamiltonian matrix N. Consider an LTI system with the state-space representation,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad (1)$$

$$y(t) = Cx(t) + Du(t), \qquad (2)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. The corresponding Hamiltonian matrix is given by,

$$N = \begin{bmatrix} -A + BV^{-1}C & BV^{-1}B^T \\ -C^TV^{-1}C & A^T - C^TV^{-1}B^T \end{bmatrix},$$

where $V = D + D^T$ is assumed to be non-singular.

Lemma 1. Let Ω be the distinct set of frequencies for which $\det [F(j\omega) + F(j\omega)^*] = 0$, with the elements of $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ listed in strictly increasing order. These frequencies are the eigenvalues of the Hamiltonian matrix N that are on the imaginary axis. Then $F(s)$ is PR if and only if:

1. N has no eigenvalues on the imaginary axis with odd multiplicity.
2. $F(j\eta_i)+F(j\eta_i)^*$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$.

Proof. See Shorten et al. [2008b].

■

2.2 Negative Imaginary Systems

Definition 3. Lanzon and Petersen [2007, 2008], Xiong et al. [2009a] A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1. $G(s)$ has no pole at the origin and in $Re[s] > 0$.
2. For all $\omega > 0$, such that $j\omega$ is not a pole of $G(s)$, and $j\,(G(j\omega) - G(j\omega)^*) \geq 0$.
3. If $j\omega_0$, is a pole of $G(j\omega)$, it is at most a simple pole and the residue matrix $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)sG(s)$ is positive semidefinite Hermitian.

2.3 Relationship between Negative Imaginary and Positive Real Systems

Since the theory of PR systems is well-researched, it is useful to establish a relationship between PR and NI systems to further develop the theory for NI systems. The following lemma Xiong et al. [2009a] provides one such relationship which will be used to derive the spectral conditions for NI systems in Section 3.

Lemma 2. Given a real rational strictly proper transfer function matrix $G(s)$ with minimal state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and, define the transfer function matrix $\tilde{G}(s) = G(s) - D$. The transfer function matrix $G(s)$ is negative imaginary if and only if,

(1) $G(s)$ has no poles at the origin.
(2) The transfer function matrix $F(s) = s\tilde{G}(s)$ is positive real.

Proof. See Xiong et al. [2009a].

■

Remark 1. The first condition in Lemma 2 is required to avoid a pole zero cancellation.

3. MAIN RESULTS

As mentioned earlier, the Hamiltonian matrix provides a convenient method for checking whether or not a given transfer function matrix is PR. In this section, we use the Hamiltonian matrix to determine spectral conditions for the NI property in the MIMO case. Indeed, this Hamiltonian method can be employed in the SISO case as well. However, as an alternative approach, we modify the results in Shorten and King [2004] for PR systems to determine spectral conditions for the NI property of SISO LTI systems.

3.1 MIMO Systems

Theorem 1. Consider an LTI system with minimal state-space realization (1)-(2). Also, suppose A is a Hurwitz matrix, $Q = CB + B^TC^T > 0$, and $D = D^T$.

The transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is NI if and only if the following conditions are satisfied:

1. The Hamiltonian matrix,
$$N = \begin{bmatrix} -A + BQ^{-1}CA & BQ^{-1}B^T \\ -A^TC^TQ^{-1}CA & A^T - A^TC^TQ^{-1}B^T \end{bmatrix} \quad (3)$$
has no pure imaginary eigenvalues with odd multiplicity.
2. $j\eta_i\,(G(j\eta_i) - G(j\eta_i)^*)$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$, where $\omega_i \in \Omega$, and $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ is the set of frequencies listed in strictly increasing order, such that $\det[F(j\omega_i) + F(j\omega_i)^*] = 0$. Here, $F(s)$ is defined as in Lemma 2.

Proof. Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is NI. Then
$$\tilde{G}(s) = G(s) - D = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad (4)$$
is also NI (e.g., see Lanzon and Petersen [2007, 2008]). As in Lemma 2, $F(s) = s\tilde{G}(s)$. Then a state space realization of $F(s)$ is given by $\left[\begin{array}{c|c} A & B \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, where,
$$\tilde{C} = CA, \quad \tilde{D} = CB \quad (5)$$
and it follows from Lemma 2 that $F(s)$ is PR with $\tilde{D} + \tilde{D}^T > 0$. Also Lemma 1 implies that the following conditions are satisfied:

(i) The Hamiltonian matrix,
$$N = \begin{bmatrix} -A + B(\tilde{D}+\tilde{D}^T)^{-1}\tilde{C} & B(\tilde{D}+\tilde{D}^T)^{-1}B^T \\ -\tilde{C}^T(\tilde{D}+\tilde{D}^T)^{-1}\tilde{C} & A^T - \tilde{C}^T(\tilde{D}+\tilde{D}^T)^{-1}B^T \end{bmatrix}$$
has no pure imaginary eigenvalues with odd multiplicity.
(ii) $F(j\eta_i)+F(j\eta_i)^*$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$.

By substituting for $\tilde{C}$ and $\tilde{D}$ from (5) into the Hamiltonian (6) we get,
$$N = \begin{bmatrix} -A + BQ^{-1}CA & BQ^{-1}B^T \\ -A^TC^TQ^{-1}CA & A^T - A^TC^TQ^{-1}B^T \end{bmatrix}$$
which has no pure imaginary eigenvalues with odd multiplicity. Also, from condition (ii) it follows that $F(j\eta_i) + F(j\eta_i)^* = j\eta_i\,(G(j\eta_i) - G(j\eta_i)^*)$, has only positive real eigenvalues for all $\eta_i = \frac{(w_i+w_{i+1})}{2}$, $i \in \{1, k-1\}$. This proves the necessity part of the theorem.

In order to prove the sufficiency part of the theorem, suppose N has no pure imaginary eigenvalues with odd multiplicity. Also, suppose A is Hurwitz, $j\eta_i\,(G(j\eta_i) - G(j\eta_i)^*)$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$, and $CB + B^TC^T > 0$. Then using Lemma 1 it follows that $F(s) = s\tilde{G}(s)$ is PR, where $\tilde{G}(s)$ is defined as in (4). Furthermore, Lemma 2 implies $G(s)$ is NI. This completes the proof.

■

Special Case: Now, consider the case in which $CB + B^TC^T$ is singular. Here, the Hamiltonian method needs to be modified in order to give the spectral conditions for the NI property. To this effect, we consider the following result and observation from Shorten et al. [2008a].

Lemma 3. Consider the transfer function matrix $F(s) = \tilde{C}(sI - A)^{-1}B + \tilde{D}$ with $\tilde{D} + \tilde{D}^T$ singular and $F(0) + F(0)^* > 0$. Then the transfer function matrix $F(s)$ is PR if and only if the following conditions are satisfied:

1. The Hamiltonian matrix
$$N_1 = \begin{bmatrix} -A_1 + B_1V_1^{-1}\tilde{C}_1 & B_1V_1^{-1}B_1^T \\ -\tilde{C}_1^TV_1^{-1}\tilde{C}_1 & A_1^T - \tilde{C}_1^TV_1^{-1}B_1^T \end{bmatrix}$$
has no pure imaginary eigenvalues with odd multiplicity.
2. $F(j\eta_i)+F(j\eta_i)^*$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$.

Here $A_1 = A^{-1}, B_1 = -A^{-1}B, \tilde{C}_1 = \tilde{C}A^{-1}, \tilde{D}_1 = \tilde{D} - \tilde{C}A^{-1}B$ and $V_1 = \tilde{D}_1 + \tilde{D}_1^T$.

Proof. This result follows from Theorem 3 and Theorem 4 in Shorten et al. [2008a].

■

Observation 1. Shorten et al. [2008a] Let Σ denote the locus of eigenvalues of the matrix $(F(j\omega) + F(j\omega)^*)$, for $\omega \in [-\infty, \infty]$, and let Σ^T denote the locus of eigenvalues of the matrix $\left(\widetilde{F}(j\delta) + \widetilde{F}(j\delta)^*\right)$ for $\delta \in [-\infty, \infty]$, with $\delta = \omega - \omega_0$, and $\widetilde{F}(j\delta) = F(j\delta + j\omega_0)$. Then Σ and Σ^T coincide.

Note that,
$$\begin{aligned}\widetilde{F}(j\delta) &= F(j\delta + j\omega_0) = \tilde{C}((j\delta + j\omega_0)\,I - A)^{-1}B + \tilde{D}, \\ &= \tilde{C}(j\delta I - \tilde{A})^{-1}B + \tilde{D},\end{aligned}$$
where $\tilde{A} = A - j\omega_0 I$.

Now consider the following result:

Theorem 2. Consider an LTI system with minimal state-space realization (1)-(2). Also, suppose A is a Hurwitz matrix, $CB+B^TC^T = 0, D = D^T$ and $det[F(j\omega) + F(j\omega)^*] \neq 0$ for some $\omega \in \mathbb{R}$.

Then, the transfer function matrix $G(s) = C(sI-A)^{-1}B+D$ is NI if and only if the following conditions are satisfied:

(1) The Hamiltonian matrix
$$\tilde{N} = \begin{bmatrix} -\tilde{A}_1 + B_1V_2^{-1}\tilde{C}_1 & B_1V_2^{-1}B_1^T \\ -\tilde{C}_1^TV_2^{-1}\tilde{C}_1 & \tilde{A}_1^T - \tilde{C}_1^TV_2^{-1}B_1^T \end{bmatrix}$$
has no pure imaginary eigenvalues with odd multiplicity. Here $\tilde{D}_2 = \tilde{D} - \tilde{C}\tilde{A}^{-1}B, V_2 = \tilde{D}_2 + \tilde{D}_2^T$ and $\tilde{A}_1 = \tilde{A}^{-1}$.

(2) $j\eta_i\,(G(j\eta_i) - G(j\eta_i)^*)$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i+\omega_{i+1})}{2}$, $i \in \{1, k-1\}$, where $\omega_i \in \Omega$, and $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ is the set of frequencies listed in strictly increasing order, such that $\det[F(j\omega_i) + F(j\omega_i)^*] = 0$. Here $F(s)$ is defined as in Lemma 2.

Proof. Suppose that $G(s)$ is NI. It follows from Lemma 2 that $F(s)$ is PR which implies from Observation 1 that $\widetilde{F}(s)$ is PR. Also, the case that $det\,[F(j\omega) + F(j\omega)^*] \neq 0$

for some $\omega \in \mathbb{R}$, means that there must exist an $\omega_0 \in \mathbb{R}$ such that,

$$F(j\omega_0) + F(j\omega_0)^* > 0$$

and it follows from the definition of $\widetilde{F}(j\delta)$ that $\widetilde{F}(0) + \widetilde{F}(0)^* > 0$.

Now, since $\widetilde{F}(s)$ is PR with $\widetilde{F}(0) + \widetilde{F}(0)^* > 0$ it follows from Lemma 3 that the following conditions are satisfied:

1. The Hamiltonian matrix
$$\tilde{N} = \begin{bmatrix} -\tilde{A}_1 + B_1 V_2^{-1}\tilde{C}_1 & B_1 V_2^{-1} B_1^T \\ -\tilde{C}_1^T V_2^{-1}\tilde{C}_1 & \tilde{A}_1^T - \tilde{C}_1^T V_2^{-1} B_1^T \end{bmatrix}$$
has no pure imaginary eigenvalues with odd multiplicity.
2. $j\eta_i\left(G(j\eta_i) - G(j\eta_i)^*\right)$ has only positive real eigenvalues for all $\eta_i = \frac{(\omega_i + \omega_{i+1})}{2}$, $i \in \{1, k-1\}$.

On the other hand, suppose that conditions 1 and 2 are satisfied, it follows from Lemma 3 that $\widetilde{F}(s)$ is PR, which implies from Observation 1 that $F(s)$ is PR. Hence, by using Lemma 2 we conclude that $G(s)$ is NI.

■

3.2 SISO systems

In this section, we consider spectral conditions for the NI property in the case of SISO systems.

Assumption 1. Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}, D \in \mathbb{R}$, and suppose the transfer function $G(s) = C(sI - A)^{-1}B + D$ has all of its poles and zeros in the closed left half of the complex plane excluding poles at the origin. Also, any pole on the imaginary axis is assumed to be simple.

Theorem 3. The SISO transfer function $G(s) = C(sI - A)^{-1}B + D$ with $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal state space realization, $CB > 0$, and satisfying Assumption 1 is NI if and only if the following conditions are satisfied:

1. The matrix $(I - \frac{1}{CB}BC)A^2$ has no eigenvalues of odd (algebraic) multiplicity on the open negative real axis.
2. All residues of $F(s) = s\left(G(s) - D\right)$ corresponding to poles on the imaginary axis are positive.

Proof. Suppose $G(s)$ is NI, then it follows from Lemma 2 that $F(s)$ is PR. Also, we have:

$$\begin{aligned} &2\,\mathrm{Re}\left[F(j\omega)\right] \\ &= (F(j\omega) + F(j\omega)^*), \\ &= CA\left(j\omega I - A\right)^{-1}B + CA\left(-j\omega I - A\right)^{-1}B + 2CB, \\ &= CA((j\omega I - A)^{-1} + (-j\omega I - A)^{-1})B + 2CB, \\ &= 2CB - 2CA^2\left(\omega^2 I + A^2\right)^{-1}B, \\ &= 2CB\left(1 - \frac{1}{CB}CA^2\left(\omega^2 I + A^2\right)^{-1}B\right). \end{aligned} \tag{6}$$

Furthermore,

$$\begin{aligned} &(\ j\omega I - A)^{-1} + (-j\omega I - A)^{-1} \\ &= (j\omega I - A)^{-1}\left[I + (j\omega I - A)\left(-j\omega I - A\right)^{-1}\right], \\ &= (j\omega I - A)^{-1}\left[(-j\omega I - A) + (j\omega I - A)\right]\left(-j\omega I - A\right)^{-1}, \\ &= (j\omega I - A)^{-1}\left[-2A\right]\left(-j\omega I - A\right)^{-1}, \\ &= \left[-2A\right](j\omega I - A)^{-1}\left(-j\omega I - A\right)^{-1}, \\ &= \left[-2A\right]\left[(j\omega I - A)\left(-j\omega I - A\right)\right]^{-1}, \\ &= \left[-2A\right]\left[\left(-j^2 w^2 I + A^2\right)\right]^{-1}, \\ &= \left[-2A\right]\left[\left(\omega^2 I + A^2\right)\right]^{-1}. \end{aligned}$$

By defining $v^T = \frac{-1}{CB}CA^2, u = \left(\omega^2 I + A^2\right)^{-1}B$ and using the identity $\det[I + uv^T] = 1 + v^T u$ Kailath [1980], we get,

$$\begin{aligned} \mathrm{Re}\left[F(j\omega)\right] &= CB\det\left(1 - \frac{1}{CB}CA^2\left(\omega^2 I + A^2\right)^{-1}B\right) \\ &= CB\frac{\det\left[w^2 I + \left(I - \frac{1}{CB}BC\right)A^2\right]}{\det[w^2 I + A^2]} \geq 0. \end{aligned} \tag{7}$$

Indeed, $\det[\omega^2 I + A^2] = |\det[j\omega I + A]|^2 \geq 0$, which implies $\det\left[\omega^2 I + \left(I - \frac{1}{CB}BC\right)A^2\right]$ can change sign if and only if $-\omega^2$ is an eigenvalue of $\left(I - \frac{1}{CB}BC\right)A^2$ with odd (algebraic) multiplicity. This proves Condition 1 in the theorem. Since $G(s)$ is NI, it follows from Lemma 2 that $F(s)$ is PR and hence Condition 2 is satisfied.

On the other hand, if Conditions 1 and 2 in the theorem are satisfied, this implies that $\mathrm{Re}\left[F(j\omega)\right]$ does not change sign along the imaginary axis and is positive. This means $F(s)$ is PR which from Lemma 2 implies that $G(s)$ is NI.

■

4. NUMERICAL EXAMPLE

Consider a system with transfer function

$$G(s) = \frac{s^3 + 3s^2 + 8s + 10}{2s^4 + 7s^3 + 17s^2 + 2s + 1} \tag{8}$$

whose minimal state-space realization is given by the matrices ,

$$A = \begin{bmatrix} -3.500 & -2.125 & -0.2500 & -0.25000 \\ 4.000 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 \\ 0 & 0 & 0.500 & 0 \end{bmatrix};\ B = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0.2500\ 0.18750\ 0.500\ 1.2500];\ \ D = 0. \tag{9}$$

For the transfer function in (8) to be NI, it should satisfy Conditions 1 and 2 from Theorem 1. The Hamiltonian matrix of the above state-space system computed using (3) has the following eigenvalues,

$$[1.1254 \pm 2.08836j, -1.1254 \pm 2.08836j, \pm 0.4352455, 0, 0],$$

none of which are pure imaginary eigenvalues with odd multiplicity. Thus Condition 1 in Theorem 1 is satisfied.

In order to prove Condition 2 in Theorem 1, we compute the distinct set of frequencies Ω at which $det[F(j\omega) + F(j\omega)^*] = 0$. For $F(s) = sG(s)$, this gives $\Omega = \{\}$. Hence,

$$j\eta\left(G(j\eta) - G(j\eta)^*\right) > 0, \tag{10}$$

for all $\eta \in (0, \infty)$. This implies $G(s)$ in (8) is NI. To verify this fact, a plot of the imaginary part of $G(j\omega)$ is shown in Fig. 1 which verifies that $G(s)$ is NI.

5. CONCLUSION

In this paper some necessary and sufficient conditions were presented to check for the negative imaginary property of

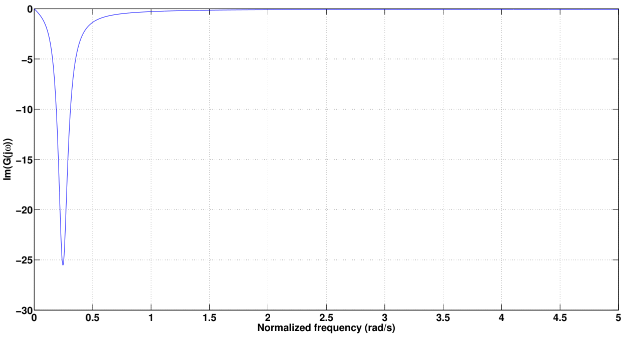

Fig. 1. Plot of the imaginary part of the transfer function $G(j\omega)$ for example 2.

an LTI system. Results were presented for both SISO as well as MIMO cases, by extending to negative imaginary systems the results concerning spectral conditions for positive real systems proposed by Shorten et. al.,. A numerical example was provided to support the results in the paper. As a note for future work, the results in the paper can be used in the synthesis of NI controllers for a given strictly NI plant.

REFERENCES

B. D. O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis: A Modern Systems Approach*. Prentice-Hall, Englewood Cliffs, N.J., USA, 1973.

Z. Bai and R. W. Freund. Eigenvalue-based characterization and test for positive realness of scalar transfer functions. *IEEE Transactions on Automatic Control*, 45(12):2396–2402, 2000.

B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and Control*. Communications and Control Engineering. Springer, London, UK, 2nd edition, 2007.

Weiguo Gao and Yishao Zhou. Eigenvalue-based algorithms for testing positive realness of SISO systems. *IEEE Transactions on Automatic Control*, 48(11):2051–2055, November 2003.

T. Kailath. *Linear Systems*. System Sciences. Prentice-Hall Inc., Englewood Cliffs, N.J., USA, 1980.

A. Lanzon and I. R. Petersen. A modified positive-real type stability condition. In *Proceedings of the European Control Conference*, pages 3912–3918, Kos, Greece, July 2007.

A. Lanzon and I. R. Petersen. Stability robustness of a feedback interconnection of systems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, 53(4):1042–1046, 2008.

I. R. Petersen and A. Lanzon. Feedback control of negative imaginary systems: Flexible structures with colocated actuators and sensors. *to appear in the IEEE Control Systems Magazine.*, 2010.

R. N. Shorten and C. King. Spectral conditions for positive realness of single-input-single-output systems. *IEEE Transactions on Automatic Control*, 49(10):1875–1879, 2004.

R. N. Shorten and K. S. Narendra. On common quadratic Lyapunov functions for pairs of stable LTI systems whose system matrices are in companion form. *IEEE Transactions on Automatic Control*, 48(4):618–621, 2003.

R. N. Shorten, P. Curran, K. Wulff, C. King, and E. Zeheb. On spectral conditions for positive realness of transfer function matrices. In *Proceedings of the American Control Conference*, pages 1076–1079, Seattle, Washington, USA, June 2008a.

R. N. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *IEEE Transactions on Automatic Control*, 53(5):1258–1261, 2008b.

J. T. Wen. Time domain and frequency domain conditions for strict positive realness. *IEEE Transactions on Automatic Control*, 33(10):988–992, 1988.

J. Xiong, I. R. Petersen, and A. Lanzon. Stability analysis of positive feedback interconnections of linear negative imaginary systems. In *Proceedings of the American Control Conference*, pages 1855–1860, St. Louis, Missouri, USA, June 2009a.

J. Xiong, I. R. Petersen, and A. Lanzon. On lossless negative imaginary systems. In *Proceedings of the 7th Asian Control Conference*, pages 824–829, Hong Kong, August 2009b.

J. Xiong, I. R. Petersen, and A. Lanzon. Finite frequency negative imaginary systems. In *Proceedings of the American Control Conference*, Baltimore, MD, USA, 2010.