Closed-Loop Identification using Routine Operating Data: the Effect of Time Delay

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Abstract: In industry, in order to store the reams of data that are collected from all the different flow, level, and temperature sensors, the fast-sampled data are very often downsampled before being stored in a data historian. This downsampled or even compressed data are, then, used by process engineers to recover the appropriate process parameters. However, little has been written about the effects of the sampling on the quality of the model obtained. Therefore, in this paper, the effects of sampling time are investigated from both a theoretical and practical perspective using results that come out of the theory of closed-loop system identification with routine operating data. It is shown that, if the ratio between the time delay and sampling time is sufficiently large, then it is possible to recover the true system parameters. The most common industrial processes that fulfill this constraint are temperature control loops. On the other hand, for processes, such as flows, pressures, or levels, with almost no time delay, then the sampling time must be extremely small in order to identify the process parameters. These results suggest that the sampling time has an important bearing on the quality of the model estimated from routine operating data. Using an experimental set-up with a heated tank, the effect of time delay on the identification of the true continuous time parameters was considered for different sampling times. It was shown that increasing the sampling time above a given threshold resulted in identifying an incorrect model.

Keywords: model identification, closed-loop, routine operating data, time delay, discrete time, continuous time.

1. INTRODUCTION

In industry, data obtained from sampling sensors every second or faster can quickly accumulate and become an issue to store. In order to lighten the storage requirements, the accumulated data are downsampled or even compressed. Different compression algorithms are used industrially. In the process industry, it is common to use the compressed historical routine operating data in order to determine process models. In this paper, the effect of downsampling on closed-loop system identification is considered. These data can be characterised as closed-loop, discrete, routine operating, process data that have been obtained without any external excitation. Thus, there is a need to consider the conditions under which the data are usable for determining a process model.

The earliest work in discrete closed-loop identification, which includes (Box & MacGregor, 1974; Box & MacGregor, 1976; Söderström & Stoica, 1989), focused on theoretically determining the restrictions on the delay that guaranteed consistency of the parameter estimates in the absence of a reference signal or external excitation for an autoregressive moving average model with exogenous input (ARMAX). A solution for the (ARMAX) with at least a single sample delay was determined in (Söderström & Stoica, 1989) as a function of the sample delay. A similar, more general result was also considered in (Söderström et al., 1975), where it was assumed that pole-zero cancellations were a priori known. In addition to the above theoretical results, various qualitative statements have been shown to apply for closed-loop identification. For example, it was shown that if the controller is of higher order than the process or with significant nonlinearities, then it is possible to identify the process successfully (Ljung, 1999). General quantitative conditions for identification of closed-loop process were presented in (Gévers et al., 2008; Gévers et al., 2009) for different models assuming that there were no closed-loop, pole-zero cancellations using two different approaches: expectation-based and information matrix-based approaches. More recently, the conditions for identifiability that include closed-loop pole-zero cancellations were developed for the case of ARMAX models (Shardt & Huang, 2011).

The objectives of this paper are: 1) to extend the routine-operating-data-based, closed-loop identification results from the ARMAX case to the Box-Jenkins (BJ) model case; 2) using the conditions, to derive the general relationship for identifiability between the time delay, the BJ model, and a proportional, integral, and derivative (PID) controller; and 3) to perform experiments to verify the experimental results.

2. CLOSED-LOOP IDENTIFICATION WITH ROUTINE OPERATING DATA

2.1 Theoretical Results

Assume that the process of interest can be described as a closed-loop Box-Jenkins system without any external excitations, similar to that shown in Figure 1, that is,

\[
G_i = \frac{X(z^{-1})}{Y(z^{-1})}, \quad G_p = \frac{z^{-n}B(z^{-1})}{F(z^{-1})}, \quad G_s = \frac{C(z^{-1})}{D(z^{-1})}
\]  

(1)

where the \(X\)-, \(Y\)-, \(C\)-, \(D\)-, and \(F\)-polynomials are given as
\[ F(z^{-1}) = 1 + \sum_{i=1}^{n_f} f_i z^{-i} \]  
\( n_f \) is the order of the polynomial, the \( B \)-polynomial is given as
\[ B(z^{-1}) = \sum_{i=1}^{n_b} \beta_i z^{-i} \]  
\( n_b \) is the order of the \( B \)-polynomial, and \( n_b \) is the time delay in the process, which excludes the one sample time delay introduced by the sampler. For simplicity of presentation, the backshift operator, \( z^{-1} \), will be dropped in the following sections.

![Diagram of a generic closed-loop process](image)

**Figure 1: Generic Closed-loop Process**

Since it has been assumed that there are no external excitations in the system, the closed-loop transfer function between \( e_t \) and \( y_t \) can be written as
\[ y_t = \frac{G_i}{1 + G_i G_e} e_t = \frac{C F Y}{D (F Y + z^{-n_b} B X)} e_t \]  
(4)

For closed-loop identifiability of the process without external excitation, it has been shown that the following relationship must hold, that is, the two models as defined below must be the same (Gevers et al., 2008):
\[ \Delta W_y = G\Delta W_x \]  
(5)

where
\[ W_x = G_i^{-1} G_p = \frac{D B}{C F} z^{-n_b}, W_y = 1 - G_i^{-1} = \frac{C - D}{C} \]  
(6)

and \( \Delta \) represents the difference between two models 1 and 2, that is,
\[ \Delta W = W_1 - W_2 \]  
(7)

Substituting the results from (6) into (5) gives
\[ (F_Y + z^{-n_b} B_X) D_1 F_1 C_1 Y = (F_Y + z^{-n_b} B_X) D_2 C_2 Y \]  
(8)

Re-arranging (8) gives the following relationship that should be investigated to determine the closed-loop conditions for identifiability,
\[ (F_Y + z^{-n_b} B_X) D_1 = (F_Y + z^{-n_b} B_X) D_2 \]  
\[ C_1 F_Y C_1 F_Y \]  
(9)

In order to solve for the closed-loop conditions, assume that

1) There are cancellations between \( D_1 \) and \( F_1 \), so that \( D_1 = HE \) and \( F_1 = H F \), where \( H \) is a polynomial with order \( n_H \) so that \( D_1 \) and \( F_1 \) are coprime.

2) There are cancellations between \( C_1 F_Y \) and \( F_Y \), so that \( C_1 F_Y = T C_1 F_Y \) and \( F_Y \) are arbitrary polynomials resulting from a re-arrangement of the original polynomial, and \( T \) is a polynomial with order
\[ n_T = \min(n_c + n_p + n_r + n_s + n_y, n_b + n_x) \]  
(10)

and \( T C_1 F_Y \) and \( T \bar{C}_1 \bar{F}_Y \) are coprime. This takes into consideration any potential pole-zero cancellations in the closed-loop system transfer function as given in (4) and extends the results presented in (Gevers et al., 2008) to include the case where there are pole-zero cancellations in the closed-loop system transfer function. It should be noted that pole-zero cancellations are common in closed-loop systems running advanced control systems. Note that, in the notation, the number of overbars placed over the polynomials represents the number of potential reductions in the order of the polynomial due to non-coprime-ness of the given polynomials.

**Theorem 1:** Assume that the Box-Jenkins model is being used with the assumptions described above, then the system can be identified without any external excitation if the following relationship holds among the orders of the polynomials:
\[ \max(n_x - n_y, n_t + n_r + n_b) \geq n_H + \min(n_c + n_p + n_r + n_s, n_b + n_x) \]  
(11)

**Proof:** The proof of this theorem is presented in Appendix I.

This result extends the previous results published in (Gevers et al., 2008) by considering the effect of time delay and closed-loop pole-zero cancellations on the identifiability of the process using routine operating data.

Since it is desired to investigate the effect of the discrete time delay on identifiability, (11) will be solved for the time delay.

**Theorem 2:** The constraints on the time delay, obtained by solving (11), are
\[ \begin{cases} n_x \geq n_y + n_s + 2n_p + n_r + n_s - n_t \\ + \min(0, n_x - n_y, n_t - n_y) \end{cases} \]
\[ n_t \geq n_r - n_t + n_y - n_b \]  
(12)

**Proof:** The proof of this theorem can be found in Appendix II.
Equation (12) shows some interesting conditions on the ability to identify the process. Firstly, it can be seen that the discrete time delay plays an important role in determining the identifiability of the system. In fact, if the discrete time delay is large enough, then it is possible to identify the system even if a proportional controller is being used. Secondly, it can be seen that the relative degree, which is basically the difference \( n_x - n_f \), of the controller, is important in determining when the process can be identified. Thirdly, the first constraint can be seen to represent the constraints on identifiability when the controller is not of sufficiently high order to identify the system, while the second constraint gives the relationship that must be satisfied for the system to be identifiable irrespective of the time delay. Finally, it can be seen that the relative degree of the process model, which is the difference \( n_g - n_f \), is also important in determining the identifiability of the process.

2.2 Practical Implications for a PID Controller

For a PID controller, where \( n_x = 2 \) and \( n_f = 1 \), and a Box-Jenkins model, (12) can be re-arranged to show that the following conditions must hold in order for the system to be identifiable without any external excitation:

\[
\begin{align*}
    n_g &\geq n_f + n_g + n_f \\
    +\min(0, n_g - n_f + 1) &\geq n_i - n_g - n_f \\
\end{align*}
\]

(13)

It should be noted that a PID controller is not sufficiently complex to identify an arbitrary process, since the second constraint in (12) does not hold in this case.

2.3 Continuous and Discrete Time Delay

Since it has been determined that the discrete time delay must be sufficiently large in order to identify the process, the question remains on how this can be obtained. Furthermore, assume that the continuous-time model structure of the process is fixed. Now, the relationship that holds between the continuous and discrete time delays is

\[
n_d = \frac{\tau_d}{\tau_s}
\]

(14)

where \( \tau_d \) is the continuous time delay, and \( \tau_s \) is the sampling time. This implies that any process with a time delay can be identified if the sampling time is fast enough. As well, it can be seen that if the data is downsampled too much, that is the sampling time is too large, then the process cannot be identified. Thus, not only must the compression algorithm preserve the information in the signal, it must be sampled sufficiently fast to permit the data to be used for system identification. Finally, it can be noted that for loops with an almost instantaneous response or no time delay, such as a level or flow control loop, the sampling time would need to be very fast in order to obtain reasonable parameter estimates from closed-loop data without any external excitation. On the other hand, loops with larger time delays, such as temperature control loops, would be easier to identify from closed-loop data without any external excitation.

3. EXPERIMENTAL PROCEDURE

In order to test the above results, a heated tank system similar to that shown in Figure 2 was used. For the heated tank system, the following parameters were used:

1. The level in the tank was set to 0.2 m and controlled using the cold water flow rate cascade controller.
2. The following nominal steady-state values were used: cold water flow rate was 6.4 kg/min, the temperature was 39°C, and the steam flow rate was 15 kg/h.
3. The temperature controller had the following parameters: \( K = 2 \) (normalised), \( \tau_1 = 50.0 \), and \( \tau_D = 4.0 \), which were determined based on the method presented in (Seborg et al., 2004).
4. The data are sampled every second.

![Process schematic](https://example.com/process_schematic.png)

Figure 2: Process schematic

3.1 Step Response Model

In order to determine the continuous time model, a step test using the steam flow rate was performed. The steam flow rate was decreased by 4.9 kg/h in open-loop conditions. The data obtained are shown in Figure 3. The level remained under cascade control. A first order plus dead time (FOPD) model was determined for the data, for which the general form is given as

\[
G_p(s) = \frac{K}{1 + \frac{\tau_p}{\tau_d}}e^{-\frac{s}{\tau_d}}
\]

(15)

The following parameter values were obtained using the method presented in (Seborg et al., 2004) for identifying FOPD models: \( K = 1.2 \text{ h°C/kg}, \tau_p = 66 \text{ s}, \) and \( \tau_d = 30 \text{ s} \). It should be noted that these parameters were estimated using a step test and may not give the best estimate of the true process dynamics. However, these open-loop estimated parameters can provide a reliable range for the actual step response and can be useful when comparing the closed-loop identification results.
3.2 Closed-Loop Identification

The data for closed-loop identification were obtained using a PID temperature controller. There was no external excitation affecting the system. A total of 1,900 seconds of data was collected, shown in Figure 4. There seems to be a small decreasing secular trend in the cold water flow rate, and the Sensor 2 temperature seems to have some type of periodic trend.

Figure 3: Step response model data

Figure 4: Closed-loop identification data

The previous open-loop results suggest that the ideal sampling time for identification is roughly between 6 (0.1τp) and 13 (0.2τp) s. In order to investigate the properties of the model obtained, larger sampling times will also be considered, which may cause some aliasing effects. First order, Box-Jenkins models that can be described as follows

\[ y(t) = \frac{\beta z^{-n_k}}{1-\alpha z^{-1}} u(t) + \frac{1}{1-d_z z^{-1}} e(t) \]  

(16)

will be used throughout this section. Assuming an exact first-order discretisation (Huang & Kadali, 2008), the continuous time parameters can be recovered as follows:

\[ \hat{\beta} = \frac{\beta z^{-n_k}}{\alpha} \Rightarrow K = \frac{\hat{\beta}}{1-\hat{\alpha}}; \tau_p = -\frac{n_k}{\ln(\hat{\alpha})} \]  

(17)

where \( \tau_p \) is the sampling time, \( \hat{\beta} \) is the estimated value of \( \beta \), and \( \hat{\alpha} \) is the estimated value of \( \alpha \).

Table 1 shows the estimated parameter values for different sampling times ranging from 6 s to 30 s. The models are named as “model,” where \( \alpha \) represents the sampling time. The total discrete time delay is calculated using

\[ n_k = 1 + n_k \]  

(18)

where the time delay, \( n_k \), for the process is determined using (14), and the 1 represents the sample delay due to the zero order hold. The models were determined using MATLAB®’s bode function. For all the final models obtained, both the cross-correlation and autocorrelation tests for system identification were passed (Ljung, 1999; Söderström & Stoica, 1989). As well, for the selected values of \( n_k \), both the Akaike’s Information Criteria (AIC) and the Final Prediction Criteria (FPC) were smallest. All estimated parameter values in Table 1 include their standard deviation.

Table 1: Discrete time properties of the model for different sampling times

<table>
<thead>
<tr>
<th>Model</th>
<th>( \tau_p )</th>
<th>( n_k )</th>
<th>( \beta )</th>
<th>( \alpha )</th>
<th>( c_1 )</th>
<th>( d_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>model6</td>
<td>6</td>
<td>5</td>
<td>0.07 ±0.01</td>
<td>0.94 ±0.02</td>
<td>0.18 ±0.06</td>
<td>-0.991 ±0.005</td>
</tr>
<tr>
<td>model10</td>
<td>10</td>
<td>3</td>
<td>0.11 ±0.02</td>
<td>0.89 ±0.03</td>
<td>0.1279 ±0.08</td>
<td>-0.985 ±0.008</td>
</tr>
<tr>
<td>model15</td>
<td>15</td>
<td>2</td>
<td>0.16 ±0.02</td>
<td>0.84 ±0.05</td>
<td>0.11 ±0.09</td>
<td>-0.97 ±0.01</td>
</tr>
<tr>
<td>model30</td>
<td>30</td>
<td>1</td>
<td>-0.03 ±0.08</td>
<td>-0.03 ±0.2</td>
<td>0.4 ±0.2</td>
<td>-0.73 ±0.08</td>
</tr>
</tbody>
</table>

From Table 1, it can be seen that for the first 3 models, the parameters for the disturbance model, that is, \( c_1 \) and \( d_1 \), are very similar, while for the last model, which is obtained using a very large sampling time, completely different parameter values are obtained. Furthermore, it can be seen that the sign of \( \beta \) for the last model is different from the other 4 models, and that the confidence interval is rather large. This suggests that something is probably wrong with this model.

Table 2 shows the corresponding continuous time model parameter values obtained using (17). From Table 2, it can be seen that, for the first 3 models, the value of \( K \) is close to the value determined in the open-loop step response, while the process time constant is slightly larger than that estimated from the step response. On the other hand, for the fourth model, where the sampling time was 30 s, it can be seen that the continuous time parameter estimates are completely wrong, with the gain having an opposite sign to that observed, and the process time constant is not even recoverable. This suggests that too slow sampling produces data that are insufficient for modelling a system. This result may also be influenced by the fact that aliasing is present in the system, as the sampling time selected may be too large. Finally, it can be noted that, in this example, when \( n_k = 2 \), the continuous time system parameters that were obtained are reasonable.

According to Theorem 1, in this experimental case, \( n_k \geq 2 \) will be sufficient to make the system identifiable using...
closed-loop data without external excitation. This sufficient condition has been verified by the results presented in Table 2.

Table 2: Continuous Time Properties of the Models

<table>
<thead>
<tr>
<th>Discrete Models</th>
<th>$K$ (h·C/kg)</th>
<th>$\tau_p$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step Test</td>
<td>1.2</td>
<td>66</td>
</tr>
<tr>
<td>model6</td>
<td>1.14</td>
<td>93</td>
</tr>
<tr>
<td>model10</td>
<td>1.07</td>
<td>93</td>
</tr>
<tr>
<td>model15</td>
<td>1.05</td>
<td>88</td>
</tr>
<tr>
<td>model30</td>
<td>-0.0301 (!)</td>
<td>not recoverable</td>
</tr>
</tbody>
</table>

4. CONCLUSIONS

In this paper, the conditions for closed-loop identification of a process that can be represented using the Box-Jenkins model and without any external excitation were developed. Based on the conditions, the effect of the process time delay on the system was investigated. It was shown that the relative order, or the difference between the numerator and the denominator of the controller, influences the identifiability of the system. Furthermore, given a fixed continuous time delay, the discrete time delay can be changed by changing the sampling time. This suggests that selecting a too slow sampling time can have a negative effect on the utility of data for future use.

The theoretical results on the time delay were compared with those obtained from a heated tank experiment for which a continuous model was obtained from an open-loop step test. The recovered continuous time parameters from the discrete models that were determined at different sampling times were compared. It was shown that a too slow sampling time can impair the usefulness of the resulting model.

Practically, these results suggest that control loops with larger time delays, such as temperature control, are easier to identify than processes, such as flow or pressure control loops, that have almost no time delay.

5. REFERENCES


APPENDIX I: PROOF OF THEOREM 1

Taking into account the cancellations mentioned previously and substituting into (9) gives

$$\frac{\hat{M}, \hat{N} + \hat{P}O}{C, F_Y} \circ \input{\hat{D}} = \frac{(M, N + PO) \circ \input{D}}{C, F_Y}$$

(19)

Therefore, since the terms on the left are coprime by the above 2 assumptions, based on the theory of Diophantine equations, the general solution can be written as

$$C, F_Y = \lambda C, F_Y$$

$$\left(M, N + PO \circ \input{D} = \lambda \left(\hat{M}, \hat{N} + \hat{P}O \circ \input{D}\right) \circ \input{O}\right)$$

(20)

where $\lambda$ is a polynomial of order $n_i = n_d + n_f$. The second equation of (20) can be rewritten as

$$\left(T\hat{M}, \hat{D} - \lambda \hat{M}, \hat{D} \circ \input{O}\right) \circ \input{D} + \left(T\hat{P}, \hat{D} - \lambda \hat{P}O \circ \input{D}\right) \circ \input{O} = 0$$

(21)

Since $O$ and $\hat{N}$ are assumed coprime, (21) can be written as

$$\left(T\hat{M}, \hat{D} - \lambda \hat{M}, \hat{D}\right) \circ \input{O} = \gamma O$$

(22)

$$\left(T\hat{P}, \hat{D} - \lambda \hat{P}O \circ \input{D}\right) = -\gamma \hat{N}$$

where $\gamma$ is a polynomial whose leading term is zero, since all the polynomials on the left hand side have a leading term of 1, which upon subtraction will be zero. If the order of $\gamma$ is less
than or equal to 0, then it can be concluded that the system is identifiable. Thus,
\[
\min (n_\omega, n_x, n_y - n_z) + n_p + n_c \leq 0 \tag{23}
\]
If it is assumed that (23) is satisfied, then (22) reduces to
\[
\begin{bmatrix}
\hat{T}_D \hat{D}_2 = \lambda \hat{T}_D \\
\hat{T}_D \hat{D}_2 = \lambda \hat{T}_D
\end{bmatrix}
\tag{24}
\]
Combining (24) with (20) gives that
\[
\begin{bmatrix}
(M_N + P_O) \hat{D}_2 = \hat{T}_D \hat{N} + \hat{T}_D \hat{P} \\
(M_N + P_O) \hat{D}_2 = (M_N + P_O) \hat{D}_2
\end{bmatrix}
\tag{25}
\]
Thus, it is clear that \(D_2 = D_2, M_2 = M_2, N_2 = N_2,\) and \(P_2 = P_1.\) Since it has been established that \(D, M,\) and \(P\) are the same, it can be concluded that their factorisation, that is, the factors that can be removed, will be the same. Thus, \(A = M_2\) and \(N = N_2\) and from either of the subequations in (24), it is easy to verify that
\[
\hat{T}_D \hat{D}_2 = \hat{T}_D \hat{H} \hat{D}_2 = \lambda \hat{T}_D \hat{D}_2 \tag{26}
\]
Thus, (20), can be rewritten as
\[
C_2 F_Y = \lambda \hat{C}_2 \hat{F}_Y, C_2 F_Y = TH \hat{C}_2 \hat{F}_Y \tag{27}
\]
which implies that \(C_2 F_Y = C_2 F_Y,\) and since \(F_1 = F_2,\) it can be concluded that \(C_2 = C_1.\) This shows that, under the condition given by (23), the system is identifiable. In order to determine conditions using the definitions of the original polynomials, note the following relationship between the orders of the polynomials
\[
\max (n_x + n_y - n_z + n_c + n_x + n_y) = \max (n_\omega, n_x, n_y + n_c) \tag{28}
\]
Since \(\max A = -\min A\) and re-arranging, (28) can be rewritten as
\[
\min (n_x - n_x, n_x - n_z - n_y) = \min (n_x - n_x, n_x - n_y) + (n_x - n_y) \tag{29}
\]
It can be noted that, in (29), by the construction of the polynomials
\[
n_x + n_y - n_x = 0, \quad n_x - n_y = 0 \tag{30}
\]
Combining (30) with (29) gives
\[
\min (n_x - n_x, n_x + n_y - n_z - n_y) = \min (n_x - n_x, n_x - n_y) \tag{31}
\]
Taking into consideration the relationships given by (31) and (10), (23) can be written as
\[
\min (n_x + n_y - n_x, n_y + n_y, n_x, n_y, n_x + n_y) + n_p + \min (n_x + n_y, n_y + n_y, n_x, n_x + n_x) \leq 0 \tag{32}
\]
It is easy to re-arrange (32) into the form given by (11).

\[Q.E.D.\]

APPENDIX II: PROOF OF THEOREM 2

From Equation (11), the constraints on the discrete time delay can be obtained by considering two cases that depend on the relationship between the terms on the left-hand side of the equation. The first case considers when
\[
n_x \geq n_x - n_x + n_y + n_y, n_x, n_y, n_x + n_y \tag{33}
\]
holds, while the second case considers when Equation (33) does not hold. Assume that Equation (33) holds, then Equation (11) can be written as
\[
n_x + n_x - n_x + n_y \geq n_p + \min (n_x + n_y, n_x, n_y, n_x, n_y, n_x + n_y) \tag{34}
\]
Simplifying and re-arranging Equation (34) gives
\[
n_x - n_x \geq n_p + \min (n_x + n_y, n_x, n_y, n_x, n_y, n_x + n_y) \tag{35}
\]
On the other hand, if Equation (33) does not hold, then Equation (11) can be written as
\[
n_x - n_x \geq n_p + \min (n_x - n_x, n_x, n_x, n_y, n_x, n_x + n_x) \tag{36}
\]
Simplifying and re-arranging Equation (36) gives
\[
0 \geq n_x + n_y + n_y + \min (n_x - n_x, n_x, n_x, n_y, n_x, n_x + n_x) \tag{37}
\]
The only way that Equation (37) can be satisfied is if the third term in the minimisation on the right is negative, since all the other terms are positive or equal to zero. Under this assumption, the solution to Equation (37) can be obtained as
\[
n_x - n_x \geq n_x + 2n_y \tag{38}
\]
Thus, in the second case, the only solution is given by Equation (38). This implies that the relative difference between the orders of the process model and the controller can play an important role in determining whether the system is identifiable in the second case.

Therefore, the conditions for identifiability in terms of the discrete time delay can be written as follows
\[
\begin{cases}
\begin{aligned}
n_x \geq n_x + n_y + n_x + n_y, n_x, n_y, n_x + n_y \\
+ \min (0, n_x - n_x + n_y, n_y, n_x, n_y, n_x, n_x + n_x)
\end{aligned}
\end{cases} \quad n_x \geq n_x - n_x + n_y - n_y \tag{39}
\]
which is identical to Equation (12). \[Q.E.D.\]