Maximal compact positively invariant sets of discrete-time nonlinear systems

Alexander P. Krishchenko∗ Anatoly N. Kanatnikov ∗∗

∗ Bauman Moscow State Technical University, Moscow, 105005 Russia
(e-mail: apkri@bmstu.ru)

∗∗ Bauman Moscow State Technical University, Moscow, 105005 Russia
(e-mail: skipper@bmstu.ru)

Abstract: In this paper we examine the existence problem for maximal compact positively invariant sets of discrete-time nonlinear systems. To find these sets we propose the method based on the localization procedure of compact positively invariant sets. Analysis of a location of compact positively invariant sets and maximal compact positively invariant set of the Chatala system is realized for all values of its parameters.

Keywords: Discrete-time system, maximal positively invariant set, localization

1. INTRODUCTION

During the past years, the interest of many researchers has been attracted to the idea of finding some geometrical bounds for attractors, periodical orbits and chaotic dynamics of a nonlinear autonomous differentiable right-side system

\[ \dot{x} = f(x), \quad f \in C^1(\mathbb{R}^n). \] (1)

Mainly, in the presented literature this problem was solved by Lyapunov-type functions, see e.g. articles of Doering & Gibbon [1995], Leonov et al. [1996], McMillen [1998], Swinnerton-Dyer [2001], Pogromsky et al. [2003], Li et al. [2005]. The method for finding families of semipermeable surfaces was proposed in Giacomini & Neukirch [1997] and then developed in other publications of these authors. The functional method was proposed in Krishchenko [1995, 1997] to the localization problem of periodical solutions of the system (1) and developed in Coria & Starkov [2009], Kanatnikov & Krishchenko [2009], Krishchenko [2005], Krishchenko & Starkov [2006a,b, 2008], Starkov [2007, 2009a,b,c, 2010] in the case of compact invariant sets.

The theory of set invariance plays a fundamental role in the control of constrained discrete-time systems and has been a subject of research by many authors — see for instance Gilbert & Tan [1991], Blanchini [1999], Kolmanovsky & Gilbert [1998], Kerrigan & Maciejowski [2000], Rakovic et al. [2004, 2006] and the references therein. In this paper we consider the case of compact positively invariant sets.

The localization method of compact invariant sets in the case of discrete-time nonlinear systems

\[ x_{k+1} = F(x_k), \quad F : M \to M, \quad M \subset \mathbb{R}^n \] (2)

with the continuous right-side was proposed in Kanatnikov et al. [2010, 2011]. Using results obtained in these papers we suggest a method to find the maximal compact positively invariant sets of discrete-time nonlinear systems.

In this paper, when we talk about a localization problem for the discrete-time system (2) we have in mind the same problem as in the case of continuous-time systems: find the localizing set.

The set \( V \subset Q \subset M \) is the localizing set for compact positively invariant sets of system (2) contained in \( Q \) if and only if \( V \) contains all compact positively invariant sets of system (2) contained in \( Q \). The localizing set of a discrete-time system (2) is a localizing set for compact positively invariant sets of system (2) contained in \( M \).

This paper presents the methods for computation of a localizing set as well as the computation of decrescent outer approximations of the maximal compact positively invariant set. These approximations are achieved by computing a sequence of compact localizing sets.

This paper is organized as follows.

In Sec. 2 and 4 we prove the main results describing the maximal compact positively invariant sets.

Sec. 3 contains results of localization method in the case of discrete-time nonlinear systems. These results point out how to find the localizing sets of a discrete-time nonlinear system and the localizing sets for compact positively invariant sets of a discrete-time nonlinear system contained in some subset of state space.

In Sec. 5 we show the computation of localizing sets of the Chatala system and find outer approximations of the attractor of this system. Sec. 6 contains conclusions.

2. MAIN RESULTS

A subset \( K \subset M \) is said to be positively invariant for the discrete-time nonlinear system (2) if \( F(K) \subset K \).
The positively invariant sets of a discrete-time system include equilibria (stationary points), periodic orbits, and attractors.

Let $G$ be any set in $M$, we introduce

$$G_0 = G, \ G_i = G_{i-1} \cap F^{-1}(G_{i-1}), \ i = 1, 2, \ldots, (3)$$

and

$$G_\infty = \bigcap_{i=0}^{\infty} G_i.$$  

For these sets we get

$$G = G_0 \supset G_1 \supset \cdots \supset G_i \supset \cdots; \ x \in G_\infty \text{ if } x \in G_{i-1} \text{ and } x \in F^{-1}(G_{i-1}), \ i = 1, 2, \ldots$$

**Theorem 1.** The set $G_\infty$ is a positively invariant set of the discrete-time system (2).

**Proof.** Let $x \in G_\infty$. Then for any $i > 0$ we have $x \in F^{-1}(G_{i-1})$, i.e. $F(x) \in G_{i-1}$. Therefore, $F(x) \in \bigcap_{i=1}^{\infty} G_{i-1} = G_\infty$, i.e. the set $G_\infty$ is a positively invariant set of the discrete-time system (2).

The compact set $V \subset G \subset M$ is the maximal compact positively invariant set contained in $G$ if and only if $V$ is positively invariant and contains all compact positively invariant sets contained in $G$. In the case $G = M$ the compact set $V$ is the maximal compact positively invariant set of system (2).

**Theorem 2.** If the set $G$ is a compact localizing set for all compact positively invariant sets of the discrete-time system (2), then $G_\infty$ is the maximal compact positively invariant set of this discrete-time system.

**Theorem 3.** If the set $G$ is a compact localizing set for compact positively invariant sets of the discrete-time system (2) contained in $Q$, then $G_\infty$ is the maximal compact positively invariant set contained in $Q$.

**Corollary 4.** The discrete-time system (2) has the maximal compact positively invariant set iff this system has a compact localizing set for all compact positively invariant sets.

To prove Theorems 2, 3 and to find $G_\infty$ we need some results of the localization method.

### 3. THE LOCALIZATION METHOD

Let $Q$ be a subset of $M$. For a continuous function $\varphi(x)$ defined on $M$ we introduce the sets

$$\Sigma^+_\varphi(Q) = \{ x \in Q : \varphi(F(x)) - \varphi(x) \geq 0 \},$$

$$\Sigma^-_\varphi(Q) = \{ x \in Q : \varphi(F(x)) - \varphi(x) \leq 0 \}$$

and define

$$\varphi^-_\varphi(Q) = \inf_{x \in \Sigma^-_\varphi(Q)} \varphi(x), \quad \varphi^+_\varphi(Q) = \sup_{x \in \Sigma^-_\varphi(Q)} \varphi(x).$$

**Theorem 5.** Each compact positively invariant set of the discrete-time system (2) contained in $Q$ is contained in the set

$$\Omega^\varphi(Q) = \{ x \in Q : \varphi^-_\varphi(Q) \leq \varphi(x) \leq \varphi^+_\varphi(Q) \},$$

i.e. the set $\Omega^\varphi(Q)$ is a localizing set for compact positively invariant sets of system (2) contained in $Q$.

**Proof.** Let $K \subset Q$ be a compact positively invariant set. The continuous function $\varphi(x)$ attains its maximum on $K$ at some point $x^* \in K$ since $K$ is a compact set. Then $F(x^*) \in K$ is a positively invariant set. As a result, $\varphi(F(x^*)) \leq \varphi(x^*)$, i.e. $\varphi(F(x^*)) - \varphi(x^*) \leq 0$, and $x^* \in \Sigma^{-}_\varphi(Q)$ because $x^* \in Q$. Therefore for any point $x \in K$ we get

$$\varphi(x) \leq \varphi(x^*) \leq \sup_{x \in \Sigma^-_\varphi(Q)} \varphi(x) = \varphi^-_\varphi(Q).$$

In the same way, for any point $x \in K$ we get the inequalities

$$\varphi(x) \geq \varphi(x_*) \geq \inf_{x \in \Sigma^+_\varphi(Q)} \varphi(x) = \varphi^+_\varphi(Q),$$

where $x_* = \arg\min \varphi(x)$. Inequalities (7), (8) mean that the compact positively invariant set $K$ is contained in the set (6).

**Corollary 6.** If the set $G = \Omega^\varphi(Q)$ (6) is a compact set, then $G_\infty$ is the maximal compact positively invariant set contained in $Q$.

**Corollary 7.** If $\Sigma^+_\varphi(Q) = \varnothing$ or $\Sigma^-_\varphi(Q) = \varnothing$ then the discrete-time system (2) has no compact positively invariant sets contained in $Q$.

**Theorem 8.** Let $\varphi_\alpha$, $\alpha \in I$, be a family of functions defined on $M$. $\varphi_\alpha \in C(Q)$, $Q \subset M$. Then each compact positively invariant set of the discrete-time system (2) contained in $Q$ is contained in the set $\bigcap_{\alpha \in I} \Omega^\varphi_\alpha(Q)$.

**Theorem 9.** Let $\varphi$ be a function defined on $M$ and continuous on $Q \subset M$. If $\psi(t) = h(\varphi(x))$, where $h: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotone function, then $\Omega^\varphi(Q) = \Omega^\psi(Q)$. Particularly, this equality holds for $h(t) = at + b$, $a \neq 0$.

**Proof.** Let us prove the equality $\Omega^\varphi(Q) = \Omega^\psi(Q)$ in the case where $h: \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function. By strictly increasing function $h$ the inequality $\varphi(x_1) \leq \varphi(x_2)$ is equivalent to $\psi(x_1) \leq \psi(x_2)$. Hence the set $\Sigma^-_\varphi(Q)$ is equal to the set $\Sigma^-_\psi(Q)$ and we get

$$\psi^-_\psi = \sup_{x \in \Sigma^-_\psi(Q)} \psi(x) = \sup_{x \in \Sigma^-_\varphi(Q)} h(\varphi(x)) = \sup_{x \in \Sigma^-_\varphi(Q)} h^\varphi = h(\varphi^-_\varphi).$$

By the same way we obtain $\psi^+_\psi = h(\varphi^+_\varphi)$. Therefore the inequalities $\psi^- \leq \psi(x) \leq \psi^+$ are equivalent to $\varphi^- \leq \varphi(x) \leq \varphi^+$, i.e. $\Omega^\varphi = \Omega^\psi$.

**Theorem 10.** Suppose that a set $G$ contains all compact positively invariant sets of the discrete-time system (2). Then the sets $F^{-1}(G)$ and $F^{-1}(G) \cap G$ contain all compact positively invariant sets of this system as well.

**Proof.** Let $K$ be a compact positively invariant set. Then $K \subset G$ and $F(K) \subset K$. Hence $F(K) \subset G$; $K \subset F^{-1}(G)$ and by Theorem 8 $K \subset F^{-1}(G) \cap G$.

In view of this theorem, given a localizing set for a discrete-time system, we can obtain new localizing sets for compact positively invariant sets by applying shifts along the orbits of the system.
4. THE PROOF OF THEOREM 2

The set $G_\infty$ is a closed set, because it is the intersection of closed sets. Therefore, the set $G_\infty$ is a compact set, because it is a closed subset of the compact set $G$. By Theorem 1 the set $G_\infty$ is a positively invariant set of the discrete-time system (2). Therefore, the set $G_\infty$ is a compact positively invariant set.

If the set $G$ is a localizing set for all positively invariant compact sets of the discrete-time system (2), then by Theorems 8, 10 the set $G_k$ from (3) is a localizing set. Therefore, the set $\bigcap_{k=0}^{\infty} G_k = G_\infty$ is a localizing set.

As a result, we get that the set $G_\infty$ is a compact positively invariant set containing all other positively invariant compact sets of the system, i.e. the set $G_\infty$ is the maximal compact positively invariant set of the discrete-time system (2).

5. THE CHATALA SYSTEM

The Chatala system (Mira et al. [1996]) is one of the discrete-time models of nonlinear dynamics exhibiting chaos. We consider this two-dimensional system in the form

$$
\begin{cases}
x_{n+1} = p_1 x_n + y_n \\
y_{n+1} = p_2 + x_n^2,
\end{cases}
$$

where $p_1, p_2 \in \mathbb{R}$ are parameters.

Under $p_1 = 1$, $p_2 = -0.5952$ the system has an attractor (Fig. 1). This attractor is a compact positively invariant set.

![Fig. 1. The attractor of the Chatala system](image)

In this section $F: \mathbb{R}^2 \to \mathbb{R}^2$ is the Chatala mapping,

$$F(x, y) = (p_1 x + y, p_2 + x^2)^T.$$  

Note that $F$ is not a reversible mapping.

Under $(p_1 - 1)^2 - 4p_2 > 0$ the Chatala system has two equilibria points $(x_1, y_1)$, $(x_2, y_2)$, where

$$
\begin{align*}
x_1 &= \frac{1 - p_1 - \sqrt{(1 - p_1)^2 - 4p_2}}{2}, \\
y_1 &= \frac{(1 - p_1)^2 - (1 - p_1)\sqrt{(1 - p_1)^2 - 4p_2}}{2}, \\
x_2 &= \frac{1 - p_1 + \sqrt{(1 - p_1)^2 - 4p_2}}{2}, \\
y_2 &= \frac{(1 - p_1)^2 + (1 - p_1)\sqrt{(1 - p_1)^2 - 4p_2}}{2}.
\end{align*}
$$

5.1 Localization of compact positively invariant sets

For the linear function $\varphi(x, y) = Ax + By$ we get

$$
\varphi(F(x, y)) = \varphi(x, y) = A(p_1 x + y) + B(p_2 + x^2) - Ax - By = Bx^2 + A(p_1 - 1)x + (A - B)y + Bp_2.
$$

To find $\varphi^r_{\text{sup}}$ and $\varphi^r_{\text{inf}}$, we have to solve two problems

$$
\begin{cases}
Ax + By \to \sup, \\
Bx^2 + A(p_1 - 1)x + (A - B)y + Bp_2 \leq 0;
\end{cases} \quad (10)
$$

Consider the case $B > 0$. Then we get $\varphi^r_{\text{inf}} = -\infty$. To find $\varphi^r_{\text{sup}}$ we can suppose that $B = 1$ (see Theorem 9), i.e. we have the problem

$$
\begin{cases}
Ax + y \to \sup, \\
x^2 + A(p_1 - 1)x + (A - 1)y + p_2 \leq 0.
\end{cases}
$$

Let us note the case $A = 1$ in which we get the problem

$$
\begin{cases}
x + y \to \sup, \\
x^2 + (p_1 - 1)x + p_2 \leq 0.
\end{cases}
$$

The inequality in this problem has no solutions in the case $(p_1 - 1)^2 - 4p_2 < 0$. Under this condition the set $\Sigma^r_\varphi(\mathbb{R}^2)$ is empty and the Chatala system has no compact positively invariant sets.

In the case $A > 1$ the problem (11) has solution

$$
\varphi^r_{\text{sup}} = \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)}.
$$

Hence we get the family $G_A$ of localizing sets defined by the inequality

$$
Ax + y \leq \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)},
$$

or

$$
y \leq \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} - Ax, \quad (12)
$$

where $A > 1$.

The intersection of sets $G_A$ (Fig. 2) is described by the inequality

$$
y \leq \inf_{A > 1} \left( \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} - Ax \right), \quad (13)
$$

or, after some calculations,

$$
y \leq \min_{A > 1} \left( \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} - Ax \right). \quad (14)
$$
By Theorem 10 $F^{-1}(G_A)$ is a localizing set. To find $F^{-1}(G_A)$, we realize the substitution $x \rightarrow p_1 x + y, \ y \rightarrow p_2 + x^2$

in (12):

$$p_2 + x^2 \leq \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} - A(p_1 x + y),$$

or in the equivalent form:

$$y \leq \frac{A^2(A - p_1)^2 - 4p_2}{4A(A - 1)} \frac{x^2 + p_1 Ax + p_2}{A}, \quad A > 1.$$  

The intersection $G^1$ of these sets (Fig. 3) is described by the inequality

$$y \leq \min_{A > 1} \left( \frac{A^2(A - p_1)^2 - 4p_2}{4A(A - 1)} - \frac{x^2 + p_1 Ax + p_2}{A} \right).$$

In a similar manner one can find the set $F^{-2}(G_A) = F^{-1}(F^{-1}(G_A))$. As a result of the same substitution we obtain inequality defining the set $F^{-2}(G_A)$

$$p_2 + (p_1 x + y)^2 \leq \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} - Ap_1(p_1 x + y) - A(p_2 + x^2),$$

or

$$(p_1 x + y)^2 + Ax^2 + Ap_1(p_1 x + y) + (1 + A)p_2 - \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} \leq 0.$$  

This set is bounded by the ellipse

$$(p_1 x + y)^2 + Ax^2 + Ap_1(p_1 x + y) + (1 + A)p_2 - \frac{A^2(A - p_1)^2 - 4p_2}{4(A - 1)} = 0.$$  

Therefore we obtain a compact localization for all compact positively invariant sets of the Chatala system and this system has maximal compact positively invariant set containing in the set $F^{-2}(G_A)$.

The intersection $G^2$ of the sets $F^{-2}(G_A)$ is plotted on Fig. 4.

In a similar manner one can find $F^{-4}(G_A)$ and the set $G^4 = \bigcap_{A > 1} F^{-4}(G_A)$ (Fig. 5).
By Theorems 8, 10 the sets

\[ F^{-k}(G_A), \quad G_k = \bigcap_{A>1} F^{-k}(G_A), \quad k = 2, 3, \ldots, \]

contain the maximal compact positively invariant set of the Chatala system.

Now we can conclude that, due to Theorem 2, the maximal compact positively invariant set of the Chatala system is the limit of the sequence

\[
G_0 = F^{-k}(G_A), \\
G_i = G_{i-1} \cap F^{-1}(G_{i-1}), \quad i = 1, 2, 3, \ldots,
\]

where \( k > 1 \) can be chosen arbitrarily.

6. CONCLUSIONS

In our paper we obtain the condition of existence of maximal compact positively invariant set of the discrete-time system. We indicate the method to construct the approximation of the maximal compact positively invariant set. This method is based on the localization of all compact positively invariant sets of the discrete-time system. It allows us to find the maximal compact positively invariant set as a limit of sequence of sets. As an example, the Chatala system is examined.

REFERENCES


