Discretization issues of high-order sliding modes

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Abstract: Uncertain high-relative-degree problems of finite-time-stable output regulation are only solvable by means of high-order sliding-mode (HOSM) controllers. Output-feedback HOSM controllers make use of robust exact differentiators also based on HOSMs. It is proved that the ultimate asymptotic accuracy of output-feedback HOSM technique is preserved, if its digital implementation in controlling continuous-time systems is based on the simple zero-order-hold control and internal one-step Euler integration. At the same time due to the integration errors there is a certain performance degradation of digital HOSM differentiators solely used for signal processing.

Keywords: Sliding Mode Control, Output-Feedback Control, Differentiators, Digital Implementation.

1. INTRODUCTION

Sliding mode control is one of the main tools to cope with heavy uncertainty conditions. The corresponding approach (Emelyanov et al., 1970, Utkin, 92; Zinober, 94; Edwards et al., 1998) is based on the exact keeping of a properly chosen constraint by means of high-frequency control switching. Although very robust and accurate, the standard sliding mode requires the relative degree of the constraint to be 1, also the frequent control switching may cause dangerous vibrations (chattering effect, Boiko et al., 2005; Fridman, 2001, 2003).

The chattering issues can be settled by high-gain control with saturation (Slotine et al., 1991) and the sliding-sector method (Furuta et al., 2000). The sliding-mode order approach (Levant, 1993) successfully treats both the chattering and the relative-degree restrictions, while preserving the sliding-mode features and improving the accuracy.

High order sliding mode (HOSM) is a motion on a discontinuity set of a dynamic system understood in Filippov's sense (1988). Let the task be to make some smooth function \( \sigma \) vanish, keeping it at zero afterwards. Then successively differentiating \( \sigma \) along trajectories, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes \( \sigma = 0 \) may be classified by the number \( r \) of the first successive total derivative \( \sigma^{(r)} \) which is not a continuous function of the state space variables or does not exist due to some reason, like trajectory nonuniqueness. That number is called the sliding order (Levant, 1993, Emelyanov et al., 1993, Bartolini et al. 1998). The resulting sliding mode motion satisfies the condition

\[
\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0. \tag{1}
\]

The words "\( r \)th order sliding" are often abridged to "\( r \)-sliding". The term "\( r \)-sliding controller" replaces the longer expression "finite-time-convergent \( r \)-sliding mode controller".

The \( r \)-sliding controllers (Levant, 2003, 2005, 2006) produce a control a discontinuous function of \( \sigma \) and of its real-time-calculated successive derivatives \( \dot{\sigma}, \ldots, \sigma^{(r-1)} \). The standard sliding mode (Emelyanov et al., 1970, Utkin, 92) is of the first order (\( \dot{\sigma} \) is discontinuous). In order to practically remove the dangerous chattering effect, one just needs to consider a control derivative of some order as a new control input (Levant, 1993, 2010; Bartolini et al., 1998).

Such \( r \)-sliding controllers provide for the highest possible asymptotic accuracy (Levant, 1993) in the presence of measurement noises, delays and discrete measurements. Thus, with \( \tau \) being the sampling interval, the accuracy \( \sigma = O(\tau^r) \) is attained; with a measurement noise of the magnitude \( \varepsilon \), the accuracy \( \sigma = O(\varepsilon) \) is got (Levant, 2005). These asymptotical features are preserved, when a robust exact differentiator of the order \( r - 1 \) (Levant, 2003) is applied as a standard part of the output-feedback \( r \)-sliding controller. It was assumed in this consideration that while the sensor outputs are available at discrete times, the whole system, including the control devices, functions in continuous time.

Thus, formally the computer-based implementation of the HOSM controllers and differentiators requires that the dynamic parts of controllers be integrated with infinitesimally small integration step, while the control be fed to the system continuously. In practice it very much complicates the realization, and even leads to increased chattering with uneven or long sampling periods. Indeed, in such a case the differentiator input is a piece-wise constant function, and its outputs are in a persistently renewed transient to zero.

The natural way of the computer-based implementation is to keep the outputs of observers and integrators constant between the measurements, and to apply the one-step Euler-integration method in the dynamic parts of the controllers and observers. The approach remains the same also with variable sampling intervals. The approach is for the first time formulated and established in this paper. The above asymptotic system accuracy is proved to be preserved.
2. SUMMARY OF HOSED CONTROL THEORY

Problem statement. Consider a dynamic system of the form

\[ \dot{x} = a(x) + b(x)(u - \sigma) = a(t, x) + b(t, x)u, \quad \sigma = \sigma(t, x), \]

(2)

where \( x \in \mathbb{R}^n \), \( a, b \) and \( \sigma: \mathbb{R}^{n+1} \to \mathbb{R} \) are unknown smooth functions, \( u \in \mathbb{R} \), the dimension \( n \) might be also uncertain. Only measurements of \( \sigma \) are available in real time. The task is to provide in finite time for exactly keeping \( \sigma = 0 \).

The relative degree \( r \) of the system (Isidori, 1989) is assumed to be constant and known. In other words, for the first time the control explicitly appears in the \( r \)th total time derivative of \( \sigma \) and

\[ \sigma^{(r)} = h(t, x) + g(t, x)u, \]

(3)

where \( h(t, x) = \sigma^{(r)}(t, x) \), \( g(t, x) = \frac{\partial}{\partial t} \sigma^{(r)} \neq 0 \). It is supposed that for some \( K_{m^r}, K_{m^r} \), \( C > 0 \)

\[ 0 < K_{m^r} \leq \frac{\sigma^{(r)}}{\partial t} \leq K_{m^r}, \quad |\sigma^{(r)}|_{L^0} \leq C, \]

(4)

which is always true at least locally. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control \( u(t, x) \).

Since the problem uncertainty prevents it, the control

\[ u = \varphi(x, \sigma, \ldots, \sigma^{(r-1)}) \]

(5)

has to be discontinuous at least on the set (1) (Levant, 2005). Hence, the \( r \)-sliding mode \( \sigma = 0 \) is to be established. As follows from (3), (4)

\[ \sigma^{(r)} \in [-C, C] + [K_{m^r}, K_{m^r}] u. \]

(6)

The obtained inclusion does not “remember” anything on system (2) except the constants \( C, K_{m^r}, K_{m^r} \). Thus, provided (4) holds, the finite-time stabilization of (6) at the origin simultaneously solves the stated problem for all systems (3).

Note that the realization of this plan requires real-time differentiation of the output. The controllers, which are considered in this paper, are \( r \)-sliding homogeneous (Levant, 2005).

Homogeneous sliding modes. A function \( f_j: \mathbb{R}^n \to \mathbb{R} \) (respectively a vector-set field \( F(x) \subset \mathbb{R}^n, x \in \mathbb{R}^n \), or a vector field \( f_j: \mathbb{R}^n \to \mathbb{R} \)) is called homogeneous of degree \( q \in \mathbb{R} \) with the dilatation

\[ d_m: (x_1, x_2, \ldots, x_n) \mapsto (k^m_1 x_1, k^m_2 x_2, \ldots, k^m_n x_n) \]

(Bacciotti et al., 2005), where \( m_1, \ldots, m_n \) are some positive numbers (weights), \( m = \deg x \), if for any \( k > 0 \) the identity \( f(x) = k^m d_m f(d_m(x)) \) holds (respectively \( F(x) = k^m d_m F(d_m(x)), \) or \( f(x) = k^m d_m f(d_m(x)) \)). The non-zero homogeneity degree \( q \) of a vector field can always be scaled to \( \pm 1 \) by an appropriate proportional change of the weights \( m_1, \ldots, m_n \).

Note that the homogeneity of a vector field \( f(x) \) (a vector-set field \( F(x) \)) can equivalently be defined as the invariance of the differential equation \( \dot{x} = f(x) \) (differential inclusion \( \dot{x} \in F(x) \)) with respect to the combined time-coordinate transformation \( G_{c^t}: (t, x) \mapsto (k^c t, d_m x) \), where \( p, p = -q \), might naturally be considered as the weight of \( t \).

Let \( \dot{x} \in F(x) \) be a finite-time stable homogeneous Filippov differential inclusion (Levant, 2005), \( p = \deg t > 0 \). Assume that each coordinate \( x_i \) is measured with an error not exceeding \( \beta_i \tau_i^m \) in its absolute value, and a delay not exceeding \( \tau^m \). Then the inequalities \( |x_i| < \gamma_i \tau_i^m \) are established in finite time with some positive constants \( \gamma_i \), independent of \( \tau \) (Levant, 2005).

The inclusion (5), (6) is called \( r \)-sliding homogeneous if for any \( k > 0 \) the combined time-coordinate transformation

\[ G_{c^t}: (t, \sigma, \sigma, \ldots, \sigma^{(r-1)}) \mapsto (k\beta, k\sigma, k^{r-1} \sigma, \ldots, k\sigma^{(r-1)}) \]

preserves the closed-loop inclusion (5), (6). In other words the homogeneity weights of \( t, \sigma, \ldots, \sigma^{(r-1)} \) are \( 1, r, r - 1, \ldots, 1 \) respectively. It is easy to check that (5), (6) is \( r \)-sliding homogeneous if

\[ \varphi(k\beta, k^{r-1} \sigma, \ldots, k\sigma^{(r-1)}) = \varphi(k, \sigma, \ldots, \sigma^{(r-1)}). \]

Controller (5) is called \( r \)-sliding homogeneous (\( r \)-th-order-sliding homogeneous) if (7) holds for any \( (\sigma, \sigma, \ldots, \sigma^{(r-1)}) \) and \( k > 0 \). The corresponding sliding mode is also called homogeneous (if exists).

Such a homogeneous controller is inevitably uniformly bounded and discontinuous at the origin \( 0, \ldots, 0 \), unless \( \varphi \) is a constant function. Almost all known \( r \)-sliding controllers are at least approximately \( r \)-sliding homogeneous in a vicinity of the sliding mode. The only important exception is the terminal 2-sliding controller (Man et al., 1994).

Arbitrary order sliding mode controllers. Most well-known sliding controller families (Levant, 2003, 2006) have the form \( u = -\alpha (\sigma + |\sigma|^{1/2} \sigma + |\sigma|^{1/2}) \), have magnitude \( \alpha > 0 \) and are defined by recursive procedures. Following are quasi-continuous controllers (Levant, 2006) for \( r = 1, 2, 3, 4 \):

1. \[ u = -\alpha \sigma \]

2. \[ u = -\alpha (\sigma + |\sigma|^{1/2} \sigma + |\sigma|^{1/2}) \]

3. \[ u = -\alpha (|\sigma| + |\sigma|^{3/2} \sigma + |\sigma|^{3/2} \sigma) / \]

4. \[ \varphi_{3, 4} = \sigma + 3[\sigma + (|\sigma| + 0.5 |\sigma|^{3/2})] \sigma + 0.5 |\sigma|^{3/2} \sigma] \]

5. \[ N_{3, 4} = |\sigma| + 3 [\sigma + (|\sigma| + 0.5 |\sigma|^{3/2})] \sigma + 0.5 |\sigma|^{3/2} \sigma] \]

The parameters of the controllers can be chosen in advance for each relative degree \( r \), and can be adjusted to provide for the needed convergence rate. Only the magnitude \( \alpha \) is to be adjusted for any fixed \( C, K_{m^r}, K_{m^r} \) most conveniently by
computer simulation, avoiding complicated and redundantly large estimations. Obviously, $\alpha$ is to be negative with $(\partial\alpha/\partial)\sigma^{(i)} < 0$.

Asymptotic accuracies of these controllers are readily obtained from the above homogeneity theory. In particular $\sigma^{(i)} = O(\varepsilon^{(i)}), i = 0, 1, \ldots, r-1$, if the measurements are performed with the sampling interval $\tau$.

**Chattering attenuation.** Chattering vibrations are naturally considered dangerous, if their energy cannot be neglected; i.e., if the energy does not vanish with the gradual vanishing of chattering-producing factors (noises, delays, small singular perturbation parameters, etc.). Corresponding formal notions were introduced in (Levant, 2010). The standard chattering attenuation procedure is to consider the control derivative as a new control input, increasing the relative degree (Levant, 1993). It is proved (Levant, 2010) that the resulting systems are robust with respect to the presence of unaccounted-for fast stable actuators and sensors, and no dangerous chattering is generated neither by such additional dynamics, nor by the presence of noises and delays. That also remains true when the output feedback is constructed, as in the next Subsection.

That standard procedure is successfully applied (Bartolini et al. 1998; Bartolini et al., 2003, etc), though formally the presence of noises and delays. That also remains true when the energy does not vanish with the gradual vanishing of chattering-producing factors (noises, delays, small singular perturbation parameters, etc.). Corresponding formal notions were introduced in (Levant, 2010). The standard chattering attenuation procedure is to consider the control derivative as a new control input, increasing the relative degree (Levant, 1993). It is proved (Levant, 2010) that the resulting systems are robust with respect to the presence of unaccounted-for fast stable actuators and sensors, and no dangerous chattering is generated neither by such additional dynamics, nor by the presence of noises and delays. That also remains true when the output feedback is constructed, as in the next Subsection.

**Differentiation and output-feedback control.** All noises in this paper are supposed to be bounded Lebesgue-measurable functions. Let the input signal $f(t)$ be a function defined on $[0, \infty)$ and consisting of a bounded noise with the unknown magnitude $\varepsilon$, and of an unknown base signal $f_0(t)$, whose $k$th derivative has a known Lipschitz constant $L > 0$.

Following (Haimo, 1986) introduce the notation $\text{sign}(w) = |w| \text{sign } w$. The problem of finding real-time robust estimations of $f_0(t), f_0(t), \ldots, f_0^{(k)}(t)$ being exact in the absence of measurement noises is solved by the differentiator (Levant, 2003)

$$
\begin{align*}
\dot{z}_0 &= v_0, \quad v_0 = -\lambda_k L^{1/(k+1)} \text{sig}(z_0 - f(t))^{k(k+1)} + z_1, \\
\dot{z}_1 &= v_1, \quad v_1 = -\lambda_{k-1} L^{1/2} \text{sig}(z_1 - v_0)^{(k-1)/2} + z_2, \\
& \vdots \\
\dot{z}_k &= v_{k-1}, \quad v_{k-1} = -\lambda_1 L^{1/2} \text{sig}(z_{k-1} - v_{k-2})^{1/2} + z_k, \\
\dot{z}_k &= -\lambda_0 \text{ } L \text{ } \text{sign}(z_k - v_{k-1}).
\end{align*}
$$

The parameters $\lambda_0, \lambda_1, \ldots, \lambda_k > 0$ being properly chosen, the following equalities are true in the absence of input noises after a finite time of the transient process:

$$
\begin{align*}
z_0(t) = f_0(t); \quad z_i(t) = f_i(t), \quad i = 1, \ldots, k.
\end{align*}
$$

Note that the differentiator has a recursive structure. Once the parameters $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ are properly chosen for the $(k-1)$th order differentiator, only one parameter $\lambda_k$ is needed to be tuned for the $k$th order differentiator. The parameter $\lambda_k$ is just to be taken sufficiently large. Any $\lambda_0 > 1$ can be used to start this process. Such differentiator can be used in any feedback, trivially providing for the separation principle (Atassi et al., 2000; Levant, 2005). A suitable choice of the differentiator parameters with $k \leq 5$ is $\lambda_0 = 1.1, \lambda_1 = 1.5, \lambda_2 = 3, \lambda_3 = 5, \lambda_4 = 8, \lambda_5 = 12$ (Levant, 2006), or $\lambda_0 = 1.1, \lambda_1 = 1.5, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 5, \lambda_5 = 8$ (Levant, 2010).

The accuracy $|z(t) - f_0^{(i)}(t)| = O(\varepsilon^{(i-1)/2(k+1)})$ is obtained. That accuracy is shown to be the best possible (Kolmogoroff, 1962; Levant, 1998). With discrete sampling and $\varepsilon = 0$ the accuracy $|z(t) - f_0^{(i)}(t)| = O(\varepsilon^{1/2(k+1)})$ is obtained.

**Output-feedback control.** Any $r$-sliding homogeneous controller (5) turns into an output-feedback controller in combination with an $(r-1)$th-order differentiator (8). Differentiator parameter $L \geq C + \sup |\theta|$ $K_M$ is to be taken. The output-feedback controller provides for the finite-time convergence of each trajectory to the $r$-sliding mode $\sigma = 0$.

Let $\sigma$ measurements be carried out with a sampling interval $\tau$, or let them be corrupted by a noise being an unknown bounded Lebesgue-measurable function of time of the magnitude $\varepsilon$. Then the output-feedback controller provides in the absence of measurement noises for the inequalities $|\sigma| < \gamma_1 \varepsilon^{1/2}, |\sigma| < \gamma_2 \varepsilon, \ldots, |\sigma| < \gamma_r \varepsilon^r$ for some $\gamma_1, \gamma_2, \ldots, \gamma_r > 0$.

With continuous measurements and $\varepsilon > 0$ the accuracy $|\sigma| < \delta_0 \text{ } \varepsilon^{1/2}, |\sigma| < \delta_1 \varepsilon, \ldots, |\sigma| < \delta_r \varepsilon^r$ are obtained for some $\delta_0, \delta_1, \ldots, \delta_r > 0$.

That asymptotic accuracy is the best possible with discontinuous $\sigma$ and discrete sampling (Levant, 1993). The general result in the case of discrete noisy sampling is also easily formulated (Levant, 2005, also see below).

3. DISCRETIZATION OF HOSMS

Consider discrete implementation of the differentiator (8). Let the sampling take place at the moments $t_1, t_2, \ldots, 0 < t_{i+1} - t_i = \tau_i \leq \tau$, $i = 1, 2, \ldots$ being upcoming. At the time instant $t_i$ the standard Euler discretization of (8) produces the equation

$$
\text{z}(t_i) = \text{z}(t_{i-1}) + \delta(t_{i-1}, f(t_i)) \tau_i,
$$

where $z = (z_0, \ldots, z_k)^T$ is the vector of the derivative estimations, and $\delta(z, f) \in \mathbb{R}^k, f \in \mathbb{R}$, is defined by the recursive equations

$$
\begin{align*}
v_0 &= -\lambda_k L^{1/(k+1)} \text{sign}(z_0 - f(t_i))^{k(k+1)} + z_1, \\
v_1 &= -\lambda_{k-1} L^{1/2} \text{sign}(z_1 - v_0)^{(k-1)/2} + z_2, \\
& \vdots \\
v_{k-1} &= -\lambda_1 L^{1/2} \text{sign}(z_{k-1} - v_{k-2})^{1/2} + z_k, \\
v_k &= -\lambda_0 L \text{ sign}(z_k - v_{k-1}).
\end{align*}
$$

Scheme (9) establishes the method of numerical differentiation with variable sampling step. Note that successive substitution of the expression for $v_0$ into the expression for $v_1$, of the expression for $v_1$ into the expression for $v_2$, etc., produces a non-recursive definition of $v$.
Due to (9) the sampling which is got at $t_i$ does not influence the estimation $z(t)$. This inevitably produces certain delay. In the important particular case when the sampling periods are constant or slowly changing, scheme (9) can be replaced by

$$z(t_i) = z(t_{i-1}) + v(z(t_{i-1}), f(t_i)) \tau,$$

(11)

In such a case $z(t)$ has the sense of the prediction of the derivatives of $f(t)$ for the starting sampling interval.

Sometimes if the current sampling step is too large, one needs to evaluate the derivatives between the measurements. In the cases of scheme $9$ and $(11)$ one can respectively define

$$z(t) = z(t_i) + v(z(t_i), f(t_i)) (t - t_i), \quad t \in [t_i, t_{i+1});$$

(12)

$$z(t) = z(t_i) + v(z(t_i), f(t_i)) (t - t_i), \quad t \in [t_i, t_{i+1}).$$

(13)

While the results of $(13)$ should be dismissed with $t \geq t_{i+1}$, $(12)$ stays valid also for the past sampling periods.

It is shown below that the features of the continuous-time differentiator are preserved if the differentiation order does not exceed 1. With higher differentiation orders homogeneity features are violated by the one-step Euler scheme.

**Theorem 1.** Let $k = 1$, the input $f(t)$ be measured with a noise not exceeding $\gamma_1 \tau$ in its absolute value and sampling steps not exceeding $\tau$. Then after finite-time transient the accuracy

$$|z_i(t) - f_0(t)| \leq \mu_1 \tau, \quad |z_i(t) - f_0(t)| \leq \mu_2 \tau^2$$

is ensured by the scheme $(9)$, $(12)$. Here parameters $\mu_1 > 0$ only depend on the parameters of the differentiator and $\gamma_1$. If $\tau = \tau = \text{const}$ the same asymptotic accuracy is got with scheme $(11)$ at $t = t_{i}$, $(13)$ requires doubling $\tau$ within $[t_i, t_{i+1})$.

**Theorem 2.** Let $k > 1$, the input $f(t)$ be measured exactly with sampling steps not exceeding $\tau$. Then for any initial conditions on any sufficiently large time interval after finite-time transient the accuracy

$$|z_i(t) - f_0(t)| \leq \eta_j \tau, \quad j = 0, \ldots, k$$

is ensured by the scheme $(9)$, $(12)$. Here parameters $\eta_j > 0$ only depend on the parameters of the differentiator and upper bounds of $|f|$, $|f^{(k-1)}|$ on this time interval. The same asymptotic accuracy is got with the scheme $(11)$, $(13)$ if $\tau = \tau = \text{const}$.

Recall that in any case $|f^{(k+1)}| \leq L$. Note that parameters $\eta_j$ are constant, if the derivatives are uniformly bounded. Moreover, the derivatives are uniformly bounded, provided $|f|$, $|f^{(k-1)}|$ are bounded (Kolmogoroff 1962, also proved in (Levant 1998)). The standard asymptotics from the previous section is trivially preserved with sufficiently small $\tau$ in the presence of bounded input noises. Note also that from the practical point of view the differentiator remains very accurate.

Consider now a HOSM implementation. Introduce the notation $\sigma = (\sigma, \sigma, \ldots, \sigma)$. First consider the direct implementation of a discontinuous controller:

$$u = \phi_r(\sigma_r(t)), \quad t \in [t_i, t_{i+1}).$$

The controlled system is still described by (2) in continuous time, which means that the differential inclusion (6) is still applicable. The corresponding asymptotic accuracy is readily achieved in (Levant, 2005) and is listed in the previous section.

Now consider the output-feedback chattering-attenuation procedure. Let the system relative degree be artificially increased by $l$, and the zero hold method be used. The $\sigma$-subsystem has now the hybrid form

$$\sigma^{(r)} = h(t_x) + g(t_x)u(t), \quad t \in [t_i, t_{i+1}),$$

$$u(t_i + 1) = u(t_i) + \phi_r(c(t_i)), \quad t_i \in [t_i, t_{i+1}) ,$$

$$u(t_i + 1) = u(t_i) + \phi_r(c(t_i)), \quad t \in [t_i, t_{i+1}).$$

where $\phi(c)$ is the corresponding $(r+l)$-sliding controller, and $z$ is obtained from (9) (or (11) with a constant sampling period) with $k = r + l - 1$. Note that case $l = 0$ (no chattering attenuation) is also included.

**Theorem 3.** Let the output $f(t)$ be measured with a noise not exceeding $\gamma_1 \tau$ and sampling step not exceeding $\tau$. Then after a finite-time transient the accuracy

$$|\sigma^{(r)}| \leq \mu_2 \tau^{r+1}$$

is ensured, where with $l = 0$ parameters $\mu_j > 0$ are only defined by the parameters of the controller, the problem statement and $\gamma_1$.

Thus, the accuracy of the continuous-time case is completely preserved in spite of the degraded performance of the digitalized differentiators (Theorem 2). The explanation is that when the system enters the desired sliding mode, all derivatives of the differentiator input $v$ vanish. With $l > 0$ the Theorem has the local sense (Levant et al., 2007).

Note that also robustness of the digital implementation with respect to the influence of unaccounted for sensors, actuators and general small system perturbations influencing the relative degree can be proved (see Levant 2010, Levant et al., 2010).

Due to the lack of space only the main points of the proofs are presented and only the scheme (9), (12) is considered.

**Proof of Theorems 1, 2.** Rewrite $(12)$ as differential equations

$$\dot{z} = v(z(t), f(t)), \quad t \in [t_i, t_{i+1}).$$

Subtracting $f_0(t)$ from both sides of the equation for $z_i$, $i = 0, \ldots, k + 1$, dividing by $L$ and denoting $\xi = (z_i - f_0(t))/L$,

$$\Delta f = (f(t)) - f_0(t))/L,$$

get

$$\dot{\xi}_0 = -\lambda_1 \text{sig}(\xi_0(t))\xi_0^{(k+1)} + \xi_0(t) - \Delta f,$$

$$\dot{\xi}_1 = -\lambda_1 \text{sig}(\xi_1(t))\xi_1^{(k+1)} + \xi_1(t) - \Delta f,$$

$$\dot{\xi}_{k+1} = -\lambda_1 \text{sig}(\xi_{k+1}(t))\xi_{k+1}^{(k+1)} + \xi_{k+1}(t) - \Delta f,$$

$$\dot{\xi}_k = -\lambda_1 \text{sign}(\xi_k(t))\xi_k^{(k+1)} + \xi_k(t) - \Delta f,$$

$$\dot{\xi}_k = -\lambda_1 \text{sign}(\xi_k(t))\xi_k^{(k+1)} + \xi_k(t) - \Delta f.$$
Using the Lagrange Theorem get that \( \Delta f^{(j)} \leq \max |f^{(j+1)}|/L = M_j \) with respective constants, \( M_j = L \), also \( |f^{(j+1)}|/L \leq 1 \). Thus obtain

\[
\hat{e}_0 = -\lambda_2 \text{sign}(\hat{e}_2(t)) + \xi(t) + \tau M_1[-1,1],
\]

\[
\hat{e}_1 = -\lambda_1 \text{sign}(\hat{e}_2(t) - \hat{e}_0) + \xi(t) + \tau M_1[-1,1],
\]

\[
\ldots = (14)
\]

\[
\hat{e}_k = -\lambda_0 \text{sign}(\hat{e}_2(t) - \hat{e}_0) + \xi(t) + \tau M_1[-1,1],
\]

\[
\hat{e}_k = -\lambda_0 \text{sign}(\hat{e}_2(t) - \hat{e}_0) + \xi(t) + \tau M_1[-1,1] + [-1,1].
\]

A disturbed finite-time stable homogeneous differential inclusion is obtained. With \( k = 1 \) also the disturbed inclusion is homogeneous under the homogeneity transformation

\[
(t, \xi_0, \xi_1) \mapsto (\kappa t, \kappa \xi_0, \kappa^2 \xi_1).
\]

The latter means that with parameter \( \kappa > 0 \) solutions of (14) with sampling parameter \( \tau \) transfer into solutions of (14) with the sampling parameter \( \kappa \tau \). The further proof of Theorem 1 is similar to (Levant, 2005). ■

With \( k > 1 \) the disturbance is not homogeneous. The set of inclusions starting from the inclusion for \( \hat{e}_j \) describes a subsystem which is a disturbed finite-time stable inclusion with \( \hat{e}_{j+1} \) entering the first inclusion as a noise. A chain convergence to an \( O(\tau) \) region is shown to take place similarly to (Levant, 2003) by induction on \( k \). ■

**Proof of Theorem 3.** Let for simplicity the system relative degree be artificially increased by 1. The larger relative-degree of inclusion is similarly considered. The system has now the hybrid form

\[
\sigma = \dot{x} = h(t,x) + g(t,x)u(t), \quad t \in [t_{j+1}, t_j),
\]

\[
u(t_{j+1}) = u(t_{j+1}) + \varphi_{r+1}(\hat{e}_{r+1}(t_{j+1})),
\]

Introduce \( H_j(t), u_j(t) = u(t), \hat{u}_j = \varphi_{r+1}(\hat{e}_{r+1}(t)), j = 1, \ldots, r-1; \)

Let \( \hat{u}_j = \hat{u}_{r+1}, j = 0, \ldots, r - 1; \)

Obviously, \( \hat{e}_{r+1}(t) = \hat{e}_{r+1}(t) \).

Rewrite the system as a continuous-time one:

\[
\dot{s}_j = s_{j+1}, \quad j = 0, \ldots, r - 1,
\]

\[
\dot{s}_{r-1} = s_r - g(t,x)\varphi_{r+1}(\hat{e}_{r+1}(t_{j+1}))), \quad t \in [t_{j+1}, t_j),
\]

\[
\dot{s}_r = h(t,x,u) + g(t,x)u(t) + g(t,x)\varphi_{r+1}(\hat{e}_{r+1}(t))
\]

\[
\dot{u}_j = \varphi_{r+1}(\dot{z}(t)), \quad t \in [t_{j+1}, t_j),
\]

\[
\dot{z} = \varphi_{r+1}(\dot{z}(t), s_j(t)).
\]

Taking into account the noise, the system can be now rewritten in the form of the inclusion

\[
\dot{s}_j = s_{j+1}, \quad j = 0, \ldots, r - 1,
\]

\[
\dot{s}_{r-1} = s_r - \tau[K_m + K_M] \sup \varphi_{r+1},
\]

\[
\dot{s}_r = [-C_1 + [K_m + K_M] \sup \varphi_{r+1}, [y_1, y_{r-1}] \tau^{r-1}],
\]

\[
\dot{z} \in \nu(z(t) \in [0,1], s_j(t) \in [0,1], [y_1, y_{r-1}] \tau^{r-1})
\]

Here \( C_1 \) is a new constant satisfying \( C_1 \geq g_1 + g_1 u_1 \) in some vicinity of the \((r+1)-sliding \) mode. Now, similarly to the proof of Theorem 1, a disturbed finite-time stable \((r+1)-sliding \) homogeneous differential inclusion is obtained. Under the homogeneity transformation

\[
(t, \xi_r, \zeta_1, \ldots, \zeta_r) \mapsto (\kappa t, \kappa \xi_r, \kappa^2 \zeta_1, \ldots, \kappa^r \zeta_r)
\]

with the parameter \( \kappa > 0 \) its solutions with sampling parameter \( \tau \) transfer into solutions with the sampling parameter \( \kappa \tau \). The rest of the proof follows (Levant, 2005). ■

4. SIMULATION EXAMPLE

Practical application of HOSM control is presented in a lot of papers, only to mention here (Bartolini et al., 2003, Massey et al., 2005). Consider the kinematic car control problem

\[
\dot{x} = v \cos \varphi, \quad \dot{y} = v \sin \varphi,
\]

\[
\dot{\varphi} = v/l \tan 0, \quad \dot{\theta} = \mu,
\]

where \( x \) and \( y \) are Cartesian coordinates of the rear-axle middle point, \( \varphi \) is the orientation velocity, \( l \) is the length between the two axles and \( \theta \) is the steering angle (i.e. the real input). The task is to steer the car from a given initial position to the trajectory \( y = g(x) \), where \( g(x) \) and \( y \) are assumed to be available in real time.

Define \( \sigma = y - g(x) \). Let \( v = const = 10 \) m/s, \( l = 5 \) m, \( x = y = \varphi = 0 \) at \( t = 0 \), \( g(x) = 10 \sin(0.05x) + 5 \). The relative degree of the system is 3. Thus, the listed quasi-continuous 3-sliding controller and 2nd-order differentiator solve the problem and are discretized as in Section 3. It was taken \( \alpha = 2, L = 400 \).

The integration was carried out according to the Euler method with the integration step \( 10^{-3} \) (the only reliable integration method with discontinuous dynamics). With the sampling step \( \tau(t) = \max(0.005+0.05 \sin \tau, 1) \) and noise \( \epsilon = 0.08 \) the tracking accuracies \( |\sigma| \leq 0.91, \ |\sigma| \leq 3.2, \ |\sigma| \leq 13.3 \) were obtained. According to Theorems 1, 2 the sampling step \( \tau(t) = \max(0.025+0.025 \sin \tau, 1) \) and noise \( \epsilon = 0.01 \) are taken, resulting in the accuracies \( |\sigma| \leq 0.10, \ |\sigma| \leq 0.67, \ |\sigma| \leq 5.0, \) which mainly corresponds to the stated asymptotics (Fig. 1).

In the absence of noises the accuracy \( |\sigma| \leq 4.3 \times 10^{-5}, \ |\sigma| \leq 2.7 \times 10^{-3}, \ |\sigma| \leq 0.28 \) was obtained with the sampling step \( \tau = 10^{-3}, \) which changed to \( |\sigma| \leq 9.3 \times 10^{-5}, \ |\sigma| \leq 4.9 \times 10^{-5}, \ |\sigma| \leq 0.03 \) with the sampling step \( \tau = 10^{-3}, \) Scheme (11) does not converge with \( \tau \) randomly taking values \( 10^{-3} \) and \( 3 \times 10^{-3}. \)

The 2nd-order differentiator effect \( \varphi_0 = \dot{\varphi}_0 - f(t) \) with \( f(t) = t - 2t^2 + t - 1 \) and \( L = 6 \) is shown in Fig. 2. Accuracies \( |\varphi_0| \leq 0.023, |\varphi_0| \leq 0.0076, |\varphi_0| \leq 0.0076, |\varphi_0| \leq 0.0076, \) were obtained with \( \tau = 10^{-3}. \) They change to \( |\varphi_0| \leq 0.23, |\varphi_0| \leq 0.076, \) \( |\varphi_0| \leq 0.076, \) \( |\varphi_0| \leq 0.0071, \) \( |\varphi_0| \leq 0.0071, \) \( |\varphi_0| \leq 0.06 \) when differentiating \( \sin 0.5t + \cos t \) with \( \tau = 0.001. \)
Fig. 1: Quasi-continuous 3-sliding car control with variable sampling interval \( t(t) \) and sampling noise magnitude \( \varepsilon \).

Fig. 2: Tracking error of the differentiator (8) with \( k = 2, \lambda_0 = 1.1, \lambda_1 = 1.5, \lambda_2 = 2, L = 6, f(t) = t^3 - 2t^2 + t - 1, \tau = 10^{-3} \).

5. CONCLUSIONS

The digital implementation of the closed-loop homogeneous sliding-mode control, based on the simple zero-order-hold control input and internal one-step Euler integration, provides for the same asymptotic accuracy as the (theoretical) continuous-time implementation with discrete time sampling.

REFERENCES


