Riccati equations are a recurrent and important feature in many theoretical control design results and have been the subject of study for a long time Reid [1972].

When the positive semidefinite solution of the Riccati equation is time invariant we are then reduced to the particular, albeit very useful, subset of equations known as Algebraic Riccati equations (ARE) Lancaster and Rodman [1995], Goodwin et al. [2001]. AREs in particular play a central role in many control design synthesis procedures, Delchamps [1984], Doyle et al. [1989], including $H_2$ optimal control Zhou et al. [1996] and $H_\infty$ optimal control Petersen [1987], Zhou and Khargonekar [1988] (and more recently Petersen [2009]).

Notwithstanding the many theoretical implications that relate to solving an ARE, the solution to the ARE itself is commonly obtained through efficient numerical algorithms, Arnold and Laub [1984], Golberg et al. [1986], Ionescu et al. [1997].

Here we focus on the discrete-time ARE (DTARE), [Goodwin et al., 2001, §22.7], whilst we discuss the continuous-time ARE (CTARE), [Goodwin et al., 2001, §22.5.2], in the companion submission to the present paper.

The main contribution of the present work is a closed-form solution for the class of DTAREs with non repeated eigenvalues. To the best knowledge of the author such closed-form solution is novel.

We compare the closed-form DTARE solution, both numerically and theoretically, to the standard Matlab solution based on the DTARE algorithmic solution developed in Arnold and Laub [1984]. We then show that, as the state weight $Q$ tends to zero, the present paper main contribution retrieves the closed-form solution proposed in [Rojas, 2009, Proposition 1].

We then challenge, for a simple $2 \times 2$ case, our standing assumption of non repeated eigenvalues. We thus aim to illustrate two things: First that the non repeated eigenvalues assumption is not a “hard” limitation when trying to find a closed-form solution. Second that, although possible to indeed find a closed-form solution, the repeated eigenvalues case is far more complex.

As an application of the DTARE the closed-form solution we introduce, in a series of straightforward lemmas, extensions to the class of DTAREs that accept such a closed-form solution. We also present as well related closed-form results that stems from the DTARE closed-form solution (such as for example the optimal regulator gain matrix $K$ involved in a state feedback control law of the type $u(k) = -Kz(k)$, see for example Kwakernaak and Sivan [1972]).

The paper is organised as follows: In Section 2 we introduce the standing assumptions for the present paper, discuss the induced closed loop pole locations by present the DTARE closed-form solution. In Section 3 we discuss the connection to known results and where possible verify numerically the exactness of the proposed DTARE closed-form solution. We also introduce a series of simple extensions of the main result which include a closed-form for the regulator gain as well as a DTARE closed-form solution subject to a non singular transformation of the state. In Section 4 we conclude by presenting our final remarks and possible future research.

**Terminology:** Let $\mathbb{C}$ denote the complex plane. Let $\mathbb{D}^-$, $\mathbb{D}^+$, $\mathbb{D}^*$ and $\mathbb{D}^+$ denote respectively the open unit-circle, closed unit-circle, open and closed unit circle complements in the complex plane $\mathbb{C}$, with $\partial \mathbb{D}$ the unit-circle itself. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}^>_+$ the set of non-negative real numbers and $\mathbb{R}^-$ the set of real negative numbers. Let $\mathbb{Z}^+$ denote the set of positive integers. A discrete-time signal is denoted by $x(k)$, $k = 0, 1, 2, \cdots$, and its $\mathcal{Z}$-transform by $X(z)$, $z \in \mathbb{C}$. The expectation operator is denoted by $\mathbb{E}$. A rational transfer function of a discrete-time system is minimum phase if all
its zeros lie in $\mathbb{D}^-$, and is non minimum phase if it has zeros in $\mathbb{D}^+$. We use bold notation to represent a generic matrix $\mathbf{A}$. Similarly, $\mathbf{0}$ stands for a matrix, of suitable dimensions, with all its entries set to zero and $\mathbf{I}$ for the identity matrix. Denote the element in the $i$th-row, $j$th-column of a matrix $\mathbf{A}$ as $a_{ij}$, and equivalently the overall matrix as $\mathbf{A} = [a_{ij}]$. If $a$ in $\mathbf{A}$, $\bar{a}$ represents its complex conjugate.

2. DTARE EXPLICIT SOLUTION

In the present section we present a technical result which states in closed-form the solution of a class of minimum energy Riccati equations.

2.1 Assumptions

The assumptions under consideration are

1) A minimal realisation ($\mathbf{A}, \mathbf{B}, \mathbf{C}, 0$) of a plant model $G(z)$.

2) The eigenvalues of $\mathbf{A}$ are in $\mathbb{C}$ and are all distinct.

3) $\mathbf{A}$ is diagonal and $\mathbf{B} = [1 \cdots 1]^T$.

Other assumptions might be stated when required for some specific results, but the above are standing assumptions for the present paper.

2.2 Closed Loop Poles Location

A DTARE, see for example [Goodwin et al., 2001, §22.7], is defined as

$$\mathbf{X} - \mathbf{A}^T \mathbf{X} \mathbf{A} + \mathbf{A}^T \mathbf{X} \mathbf{B} (\mathbf{R} + \mathbf{B}^T \mathbf{X} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{X} \mathbf{A} - \mathbf{Q} = 0,$$

where both the solution $\mathbf{X}$ and the state weight $\mathbf{Q}$ are symmetric matrices. A well known fact that stems from the solution of the above DTARE is the definition of the regulator gain $\mathbf{K}$ (see for example Kwakernaak and Sivan [1972]) which is given by

$$\mathbf{K} = (\mathbf{R} + \mathbf{B}^T \mathbf{X} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{X} \mathbf{A}.$$

We also define, by means of the state feedback control law $u(k) = -\mathbf{K} x(k)$ the open loop regulator transfer function $L_{reg}(z)$ as

$$L_{reg}(z) = \mathbf{K} (z \mathbf{I} - \mathbf{A})^{-1} \mathbf{B},$$

as well as the sensitivity function $S_{reg}(z)$ as

$$S_{reg}(z) = (1 + \mathbf{K} (z \mathbf{I} - \mathbf{A})^{-1} \mathbf{B})^{-1}.$$

We note that an alternative definition for $S_{reg}(z)$, useful in the context of the present paper, is also

$$S_{reg}(z) = \prod_{i=1}^{n} \frac{z - \rho_i}{z - z_i},$$

where $\rho_i, \forall i = 1, \cdots, n$, are the eigenvalues of $\mathbf{A}$ and $z_i, \forall i = 1, \cdots, n$, are the closed loop pole locations induced by the DTARE solution through the state feedback control law. From (3) we also obtain another definition for $L_{reg}(z)$, alternative to (2), as

$$L_{reg}(z) = \frac{\prod_{i=1}^{n} (z - z_i) - \prod_{i=1}^{n} (z - \rho_i)}{\prod_{i=1}^{n} (z - \rho_i)}.$$  

By means of a spectral factorisation argument, see for example [Aström, 1970, pp. 99–103], induced by the DTARE in (1) we have that $S_{reg}(z)$ satisfies

$$S_{reg}(z)(\mathbf{R} + \mathbf{B}^T \mathbf{X} \mathbf{B})S_{reg}(z)^{-1} =$$

$$R + \mathbf{B}^T (z I - \bar{\mathbf{A}})^{-1} \mathbf{Q} (z^{-1} I - \mathbf{A}^{-1} \mathbf{B}).$$

(6)

Therefore, if for example $\mathbf{Q} = \mathbf{C}^T \mathbf{C}$, the closed loop poles $z_i$ can be seen to be the stable roots of the polynomial

$$p(z) = q(z)q(z^{-1}),$$

where $G(z) = q(z)/p(z)$.

Remark 1. We observe, from comparing the limit as $z \to \infty$ of the transfer functions involved on both sides of the equality in (6), that

$$\mathbf{B}^T \mathbf{X} \mathbf{B} = \left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) - \mathbf{R}.$$

Notice that as $\mathbf{Q} \to \mathbf{0}$, then each $z_i \to 1/\rho_i$ and the above expression converges to $(\prod_{k=1}^{n} |\rho_k|^2)^{-1} - \mathbf{R}$.

Remark 2. It is well known that the closed loop pole locations $z_i$ are defined also as the eigenvalues of $\mathbf{A} - \mathbf{B} \mathbf{K}$. However we use the spectral factorisation argument in (6) and the polynomial (7) because, albeit numerical for most of the cases, they do not explicitly require a priori the solution $\mathbf{X}$ of the DTARE. That is, by means of the proposed spectral factorisation argument, we do not need to solve the DTARE in order to know the closed loop poles.

We now use the two alternative definitions for $S_{reg}(z)$, the one in (4) and the one in (3), to state the following proposition that we require for the proof of the present paper main result.

Proposition 3. The following equality holds

$$\left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) r_i = \rho_i \sum_{j} x_{ij}.$$  

(8)

Proof. Observe from (5) that a partial fraction description of $L_{reg}(z)$ results in

$$L_{reg}(z) = \sum_{i=1}^{n} \frac{r_i}{z - \rho_i},$$

with the residue factor $r_i$’s defined as

$$r_i = \prod_{j=1 \atop j \neq i}^{n} (\rho_j - z_i), \forall i = 1, \cdots, n.$$

On the other hand, from the equivalent definition of $S_{reg}(z)$ in (3) we have that a partial fraction description of $L_{reg}(z)$ results in

$$L_{reg}(z) = \sum_{i=1}^{n} \frac{k_{i1}}{z - \rho_i}.$$

From the definition of

$$\mathbf{K} = (\mathbf{R} + \mathbf{B}^T \mathbf{X} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{X} \mathbf{A} = [k_{i1}],$$

we have then that

$$k_{i1} = \left( \prod_{k=1}^{n} \frac{z_k}{\rho_k} \right) \sum_{j} \rho_j x_{ij},$$

and from the fact that both partial fraction description are equivalent, we obtain the equality proposed in (8), which concludes the proof.

Remark 4. We observe that Proposition 3 can also be restated as

$$\left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) r_j = \rho_j \sum_{i} x_{ij}.$$
The above expression can be easily verified by invoking the symmetric condition of X from which we have that \( x_{ij} = x_{ji} \).

2.3 Main Result

In this subsection we present the closed-form solution to the general DTARE in (1) under the assumptions stated in Subsection 2.1.

**Proposition 5. (Closed-Form Solution for R = 1)** The closed-form solution \( \hat{X} = [x_{ij}] \) to the DTARE in (1) with \( R = 1 \) is given by

\[
x_{ij} = \frac{\left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) \bar{r}_i r_j - q_{ij}}{\bar{\rho}_i \rho_j - 1},
\]

(9)

with \( r_i \) defined as

\[
r_i = \frac{\prod_{k=1}^{n} (\rho_k - z_k)}{\prod_{j \neq i}^{n} (\rho_k - \rho_j)}, \quad \forall i = 1, \cdots, n,
\]

(10)

and \( \rho_k \), \( \forall i = 1, \cdots, n \), the distinct eigenvalues of \( A \).

**Proof.** As a first step let us recognise that matrices \( A \) and \( B \) that satisfy assumptions 2) and 3) are given by

\[
A = \begin{bmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.
\]

Rewrite now (1) as

\[
A^T X A - X + Q = A^T X B \left( R + B^T X B \right)^{-1} B^T X A. \quad (11)
\]

The LHS and RHS of (11), replacing \( X \) as in (9), are then given by

\[
LHS = \left[ \bar{\rho}_i \rho_j \right] \left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) \bar{r}_i r_j - q_{ij} \frac{\rho_k}{\bar{\rho}_i \rho_j - 1},
\]

and

\[
RHS = \left[ \left( \prod_{k=1}^{n} \frac{\rho_k}{z_k} \right) \bar{r}_i r_j \right].
\]

Now from Proposition 3 we then have that the LHS of (11), after substituting \( \hat{X} \) as in (9), is equal to the RHS of (11) which concludes the proof.

**Remark 6.** Note from the definition of the residue factor \( r_i \) in (10) that a loss of controllability is evident when any \( \rho_k \) matches any \( \rho_j \).

**Remark 7.** Note that the complex conjugate case, arising from possible second order factors in the plant model, is included in the main result of Proposition 5. This is so since complex conjugate eigenvalues do not violate the assumption of distinct eigenvalues and it is the reason why we have maintained the complex conjugate notation in (1).

The result of Proposition (5) can now serve as a figurative stepping stone to obtain further results in closed-form which we investigate in the next section.

3. DISCUSSION AND EXTENSIONS

In the present section we compare Proposition 5 to the algorithmic solution obtained through the command care in Matlab, both numerically and theoretically. We then show how, as \( Q \to 0 \), we recover the result presented as [Rojas, 2009, Proposition 1]. We further investigate, in a simple \( 2 \times 2 \) setting, the DTARE closed-form solution for the case of repeated eigenvalues in \( A \). We then conclude the section with a series of straightforward extensions of Proposition 5, including the closed-form solution for \( R \neq 1 \), subject to a state transformation and for multivariable systems with a block-diagonal \( Q \).

3.1 Numerical Comparison with the Matlab Solution

Here we compare of the closed-form solution \( \hat{X} \) and the one obtained with Matlab.

**Example 8.** Consider for this example the following selection

\[
A = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad Q = I, \quad R = 1.
\]

The first eigenvalue \( \rho_1 \) of \( A \) is a variable pole defined in the range \( ] - 2, 2 [ \). Define \( E = [e_{ij}] = \hat{X} - X_m \) where \( X_m \) is the DTARE solution obtained from Matlab. We then propose an error function the expression

\[
error = \sum_i \sum_j [e_{ij}^2]. \quad (12)
\]

In Figure 1 we observe the plot of the resulting error function for the present example. As the value of \( \rho_1 \) approaches the value of \( \rho_2 \) and \( \rho_3 \) the error spikes out due to the loss of controllability. Otherwise, we can appreciate, from Figure 1 that indeed \( \hat{X} \) is equivalent to \( X_m \).

To clarify that the observed spikes in Figure 1 are indeed due to the loss of controllability and not to a sudden di-
vergence between the $\hat{X}$ and $X_m$ solutions let us introduce the following alternative error function
\[ F = \begin{bmatrix} f_{ij} \end{bmatrix} = X - \hat{X} \]

We evaluated $F$ alternatively for $\hat{X}$ and for $X_m$ and use a similar error index as before
\[ \text{error} = \sum_i \sum_j [f_{ij}^2] . \]

From Figure 2 we have then that the spikes observed in

\[ \text{Fig. 2. Numerical error upon replacing the solution obtained from Matlab and the proposed closed-form solution into the DTARE definition in (1).} \]

Figure 1 are also present, independently for $\hat{X}$ and $X_m$, thus reinforcing the idea that these are indeed due to loss of controllability whenever $\rho_1 \to \rho_2$ or $\rho_1 \to \rho_3$.

We conclude the present subsection by remarking that the numerical comparison presented in the above example is not to renge of the closed-form solution in Proposition 5. The objective of the example was to clearly show that the closed-form DTARE solution is numerically comparable, when implemented through Matlab, to the available standard algorithmic DTARE solution.

3.2 Theoretical Comparison with the Matlab Solution

From Arnold and Laub [1984] we observe that the Matlab command dare, for the proposed class of AREs, solves an eigenproblem defined by the Hamiltonian matrix
\[ M = \begin{bmatrix} A + BR^{-1}B^TA^{-T}Q & -BR^{-1}B^TA^{-T} \\ -A^{-T} & A^T \end{bmatrix} . \]

The solution $W$ to the eigenproblem is such that
\[ W^{-1}MW = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} , \]

with all the eigenvalues of $S_{11}$ strictly inside the unit circle. Thus, as intuition would have it, from Proposition 5 we can also obtain a closed-form solution to the eigenproblem solved by the algorithm proposed in Arnold and Laub [1984], namely
\[ \hat{W} = \begin{bmatrix} \hat{X}^{-1}Y \\ Y \end{bmatrix} , \]

where
\[ Y = (A + A^{-T} - BR^{-1}B^TA^{-T} + BR^{-1}B^TQ + A^{-T}Q)^{-1}(\hat{X}^{-1}S_{12} - S_{12}) . \]

Note that for the above expression, as well as in (13), we assume that $S$ is known (or in its default that just $S_{12}$ is known), see Arnold and Laub [1984] for more details.

3.3 Convergence to the ARE with Vanishing State Weight Result

Here we discuss the convergence of the DTARE solution in closed-form from Proposition 5 to known results. We refer in particular to the closed-form solution for the class of AREs with vanishing state weight, that is $Q \to 0$, presented in [Rojas, 2009, Proposition 1]. Notice that as $Q \to 0$ the induced closed loop poles $z_i$ tend to the set of stable eigenvalues and the mirrored (with respect to the unit circle) unstable eigenvalues of $A$. As a result for each stable eigenvalue we will have that the associated $r_i$ will tend to zero, whilst for the mirrored unstable eigenvalues we will have
\[ r_i \to (1 - |\rho_i|^2) \prod_{k=1}^{m} \frac{(1 - \rho_k\rho_i^*)}{(1 - \rho_k\rho_i^*)} , \quad \forall i = 1, \ldots, m , \quad (14) \]

where all the $\rho_i$ in the expression above are part of the set of $m$ unstable eigenvalues of $A$. Assume, for the sake of the exposition, that the set of $m$ unstable eigenvalues of $A$ are also the first $m$ eigenvalues of $A$. We can then observe that the expression for $\hat{X}$ from Proposition 5, as $Q \to 0$, will tend to
\[ \hat{X} \to 0 \begin{bmatrix} \frac{r_1r_1}{\rho_1\rho_1} & \cdots & \frac{r_mr_m}{\rho_m\rho_m} \\ \vdots & \ddots & \vdots \\ \frac{r_mr_1}{\rho_m\rho_1} & \cdots & 0 \end{bmatrix} , \]

where each $r_i$ is given now as in (14), recovering the result from [Rojas, 2009, Proposition 1].

3.4 Repeated Poles Case

The study of repeated poles is also of interest, but it is at the same time more complex. As an example we present next the case of a pair of repeated eigenvalues. We treat in the present example the case that $A$ contains a pair of repeated eigenvalues, that is we have
\[ A = \begin{bmatrix} \rho & 1 \\ 0 & \rho \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \]

The closed-form solution of the DTARE equation (1), with $A$ and $B$ as above, $R = 1$ and any symmetric choice of $Q$ is given by
\[ \hat{X} = \begin{bmatrix} \rho^2(\rho - z_1^2)z_1^2(\rho - z_2^2)z_2^2Q(1,1) & \rho(\rho - z_1)(\rho - z_2) \\ \rho(\rho - z_1^2)(\rho - z_2^2) & \rho^2(\rho - z_1^2)(\rho - z_2^2) \end{bmatrix} . \]

Although tedious, the above $\hat{X}$ solution can be verified by direct replacement in (1), invoking the link between $Q$ and the closed loop poles imposed by the spectral factorisation presented in (6).
Example 9. Consider for this example the following selection: \( \rho \) in \([-2, 2] \) and \( A \) and \( B \) as in (15), \( R = 1 \) and \( Q = \begin{bmatrix} 1 & 1 \end{bmatrix} \). We use the same error function as in (12) with \( \text{hat}X \) as in (16) and \( X_m \) the DTARE solution obtained from Matlab. In Figure 3 we can observe that the error function is very small. Although it seems to increase in the unstable range of \( \rho \) it still remains quite negligible. We then have that the present example shows that the unstable range of eigenvalues, to gain sufficient insight as to find \( X \) in closed-form.

3.5 Extensions

We now introduce a series of straightforward extensions to showcase the possible uses of the closed-form DTARE solution. Due to their simplicity we omit the proof of most such corollaries.

Lemma 10. (Closed-Form Solution for \( R = \lambda \)) The closed-form solution to the DTARE in (1) with weights \( \lambda \) and \( Q \) is given by

\[
\hat{X}^\lambda = \lambda \hat{X},
\]

where \( \hat{X} \) is the closed-form solution to the DTARE in (1) with weights \( R = 1 \) and \( Q/\lambda \).

After the first and most natural extension of Proposition 5 we study the case in which the original state space representation of the plant model

\[
x(k + 1) = Ax(k) + Bu(k)
\]

\[
y(k) = Cx(k),
\]

is subject to a nonsingular transformation of the state \( T \) such that \( \bar{x} = Tx \). The state representation in the new state coordinates \( \bar{x}(k) \) becomes

\[
\bar{x}(k + 1) = \bar{A}\bar{x}(k) + \bar{B}u(k)
\]

\[
y(k) = \bar{C}\bar{x}(k),
\]

where \( \bar{A} = \text{hat}A \), \( \bar{B} = \text{hat}B \) and \( \bar{C} = \text{hat}C \).

Lemma 11. (Transformed Closed-Form Solution) Consider a nonsingular transformation of the state \( T \). The closed-form solution to the DTARE in (1) with matrices \( A \) replaced by \( \bar{A} = \text{hat}A \), \( B \) replaced by \( \bar{B} = \text{hat}B \) and weights \( \bar{R} = 1 \) and \( \bar{Q} = \text{hat}Q \), is given by

\[
\hat{X} = T^{-T}\hat{X}T^{-1},
\]

where \( \hat{X} \) solves the DTARE in (1) with matrices \( A \) replaced by \( \bar{A} = \text{hat}A \), \( B \) replaced by \( \bar{B} = \text{hat}B \) and weights \( \bar{R} = 1 \) and \( \bar{Q} = \text{hat}Q \).

We next proceed to obtain the regulator gain \( K \) in closed-form.

Lemma 12. (Closed-Form Regulator Gain) Given \( \hat{X} \) from Proposition 5, the Kalman gain matrix is then obtained as

\[
K = \left( \prod_{k=1}^{n} \frac{\bar{P}_k}{\bar{P}_k} \right) \begin{bmatrix} \rho_1 \sum_{i=1}^{n} \frac{\bar{P}_{r_1}^{-1} + \rho_{r_1}}{\bar{P}_{r_1}^{-1} - \rho_{r_1}} \bar{P}_{r_1}^{-1} q_{11} \\
\vdots \\
\rho_n \sum_{i=1}^{n} \frac{\bar{P}_{r_n}^{-1} + \rho_{r_n}}{\bar{P}_{r_n}^{-1} - \rho_{r_n}} \bar{P}_{r_n}^{-1} q_{nn} \end{bmatrix},
\]

where \( K = (1 + B^T \bar{X}B)^{-1}B^T \bar{X}A \) and \( R = 1 \).

From the plain regulator gain \( K \) in closed-form for \( R = 1 \), we now move to the regulator gain \( K \) in closed-form case when subject to a nonsingular transformation \( T \) of the state. The closed-form regulator gain \( \bar{K} \) in closed-form case when subject to a nonsingular transformation \( T \) of the state.

Lemma 13. (Transformed Closed-Form Regulator Gain) Given \( \hat{X} \) from Proposition 5 and a nonsingular transformation matrix \( T \), then the Kalman gain matrix \( \bar{K} \) in the transformed state coordinates is obtained as

\[
\bar{K} = KT,
\]

where \( K \) is as in (18) and \( R = 1 \).
and $\sum_{j=1}^{r} n_j = n$. We introduce next the MIMO closed-form characterisation of $\hat{X}$, the non-trivial solution to the DTARE.

Proposition 14. (MIMO Closed-Form Solution for $R = I$) the closed-form solution to the DTARE in (1) with $B$ as in (20), and

$$Q = \begin{bmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_r \end{bmatrix},$$

is given by

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{X}_r \end{bmatrix}.$$  

Each $\hat{X}_j, \forall j = 1, \cdots, r$, in turn is given by

$$\hat{X}_j = \begin{bmatrix} \prod_{k=1}^{n_j} (\rho_{1,j} + z_{k,j})^{-1} r_{1,j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \prod_{k=1}^{n_j} (\rho_{n_j,j} + z_{k,j})^{-1} r_{n_j,j} \end{bmatrix},$$

with $r_{i,j}, \forall j = 1, \cdots, r$, defined as

$$r_{i,j} = \prod_{k=1}^{n_j} (\rho_{i,j} - z_{k,j}) \prod_{l \neq i}^{n_j} (\rho_{i,j} - \rho_{l,j}), \forall i = 1, \cdots, n_j.$$  

Proof. Consider the DTARE in (1) with $R = I$. Due to (20) this is equivalent to a collection of DTAREs, each given by

$$X_j - A_j^T X_j A_j - Q_j + \Lambda_j^T X_j B_j (1 + B_j^T X_j B_j)^{-1} B_j^T X_j A_j = 0,$$

for $\forall j = 1, \cdots, r$. By means of Proposition 5 we have the solution for each of the above minimum energy AREs which is given by (22) with $r_{i,j}$ defined as in (23). Finally, the complete solution $\hat{X}$ can be constructed by collecting each solution as in (21), which concludes the proof.

Notice that the result in Proposition 14 does not include the more challenging case of a full symmetric matrix $Q$.

4. CONCLUSION

In the present paper we have presented a closed-form solution for the DTARE, whenever the eigenvalues of the $A$ matrix are distinct. We have shown how the present closed-form result can be linked to previous work about AREs, Arnold and Laub [1984], Rojas [2009], both numerically and theoretically. We have presented a series of straightforward extension of the main result including a closed-form version of the regulator gain $K$. Future research should consider the more demanding cases of repeated eigenvalues for $A$, MIMO with full $Q$ matrix and the possible application of the DTARE closed-form solution to existing $H_2$ or $H_\infty$ control design results.

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