The Stabilization of One-Dimensional Wave Equation by Delayed Output Feedback ⋆

Jun-Min Wang ∗ Bao-Zhu Guo ∗∗ Miroslav Krstic ∗∗∗

Abstract: In this paper, we study the stability of a string equation with time delays in the output feedback loop. When the delay is equal to the even multiples of the wave propagation time, we develop the necessary and sufficient conditions for the feedback gain and time delay which guarantee the exponential stability of the closed-loop system. We also show that when the delay is an odd multiple of the wave propagation time, the closed-loop system is unstable. In the particular case of delay equal to two, the lack of robustness to a small perturbation in time delay is discussed. The semigroup theory and Riesz basis approach are adopted in investigation. Finally, some numerical simulation for the case of the delay equal to two is presented to illustrate the convergence.

Keywords: Wave equation, time delay, boundary control, distributed parameter system.

1. INTRODUCTION

In general, a time delay in actuation or sensing brings difficulty for the design of a control system [12]. For distributed parameter systems, the stabilization with time delay in observation and control creates particularly difficult mathematical challenges ([2, 8]). On the other hand, intentionally introduced time delay has been used successfully in feedback control design [7, 16] and filter design [13]. A control design called “proportional minus delay controller” (PMD) was introduced in [14, 15] with the objective of improving the performance of the system.

Very recently, it was found in [4] that a specific time delay in boundary observation can be used to stabilize the string vibration, which can be described by the following one-dimensional wave equation under boundary control and collocated observation with time delay:

\[
\begin{aligned}
\begin{cases}
    w_t(x,t) - w_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\
    w(0,t) = 0, & w_t(1,t) = kw_t(1,t-\tau), & t \geq 0, \tau > 0, \\
    w(x,0) = w_0(x), & w_t(x,0) = w_1(x), & 0 \leq x \leq 1,
\end{cases}
\end{aligned}
\]

where \((w_0, w_1)\) is the initial state. The main result of [4] is that when \(\tau = 2\) and \(k \in [3 - 2\sqrt{2}, 1)\) is sufficient but not necessary. In fact, for this case, a much simpler approach by spectral analysis (Section 6 of this paper) establishes that the system is exponentially stable if and only if \(k \in (0, 1)\).

In this paper, motivated by [4], we study the delays \(\tau\) equal to even multiples of the wave propagation time rather than the specific \(\tau = 2\). When \(\tau = 2, 4, 6\) and 8, we present the sufficient and necessary condition for \(k\) so that the closed-loop system (1) is exponentially stable. When \(\tau = 1, 3, 5\) and \(\tau = \frac{1}{2}\), we show that there exists no \(k\) to make the system (1) stable. The lack of robustness to delay perturbation for \(\tau = 2\) is also analyzed. Following the method presented in this paper, the necessary and sufficient condition for the feedback gain \(k\) to be stabilizing for a rational delay \(\tau = \frac{p}{q} > 0\) can be obtained by checking whether two Jury matrices [9, 10] of order \((2p + q - 1) \times (2p + q - 1)\) are positive innerwise.

Instead of the characteristic line method used in [4], for the general \(\tau > 0\) in this paper, we use the semigroup approach to explain the well-posedness of the system (1) and the Riesz basis approach to get the spectrum determined growth condition. The detailed spectral analysis gives the necessary condition of the stabilizing feedback gain \(k\) for the delay equal to the general even multiples of the wave propagation time to make the system exponentially stable.

We proceed as follows. In Section 2 we formulate the problem as a time delay free problem by considering the time delay as dynamics. The \(C_0\)-semigroup approach is used to prove the well-posedness of the system. Section 3 is devoted to the spectral analysis. The Riesz basis property and spectrum-determined growth condition are
presented in Section 4. In Section 5, the necessary and sufficient conditions are given for the exponential stability with some integer time delays. The specific case of \( \tau = 2 \) is presented in Section 6 to explain the lack of robustness to small time delay perturbation for this delay-dependent stabilizing controller. Finally, some numerical simulations are presented in Section 7 to illustrate graphically the stability for \( \tau = 2 \).

2. WELL-POSEDNESS OF SYSTEM

Since the time delay itself is a dynamical system, we have to specify its initial condition. To do this, we introduce a new variable

\[
z(x, t) = w_t(1, t - \tau x).
\]

Then the system (1) becomes

\[
\begin{align*}
  w_t(x, t) - w_{xx}(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
  w(0, t) &= w_t(1, t), \quad t > 0, \\
  z_t(x, t) + z_{xx}(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
  z(0, t) &= w_t(1, t), \quad t > 0, \\
  w(x, 0) &= w_0(x), \quad w_t(x, 0) = w_1(x), \quad 0 \leq x < 1, \\
  z(x, 0) &= z_0(x), \quad 0 \leq x \leq 1,
\end{align*}
\]

where \( z_0 \) is the initial value of the variable \( z \).

We consider the system (2) in the energy space

\[
\mathcal{H} = H^1_0(0, 1) \times L^2(0, 1) \times L^2(0, 1)
\]

where \( H^1_0(0, 1) = \{ f \in H^1(0, 1) | f(0) = 0 \} \) and the inner product in \( \mathcal{H} \) is given, for all \( X_t = (f_t, g_t, h_t) \in H, i = 1, 2, \) by

\[
\langle X_1, X_2 \rangle = \int_0^1 \left[ f_1^2 + g_1^2 + \tau h_1^2 \right] dx.
\]

Then (2) can be written as an evolution equation in \( \mathcal{H} \):

\[
\dot{X}(t) = AX(t), \quad t > 0,
\]

where \( X(t) = (w_t(t), w_{xx}(t), z(t), t) \). The inner product induced norm of \( X(t) \) is the energy of the system (2).

Lemma 1. Let \( A \) be given by (4). Then \( A \) admits solutions and is compact. Hence, \( \sigma(A) \), the spectrum of \( A \), consists of isolated eigenvalues of finite algebraic multiplicity only.

Theorem 2. Let \( A \) be given by (4), and \( \alpha > 0, \beta < 0, \gamma > 0 \). For any \( X_t = (f_t, g_t, h_t) \in H, i = 1, 2 \), define

\[
\langle X_1, X_2 \rangle_1 = \int_0^1 e^{\alpha x}(f'_1 - g_1)(f'_2 - g_2) + e^{\beta x}(f'_1 + g_1)(f'_2 + g_2) + \tau \int_0^1 e^{\gamma x}h_1 h_2 dx.
\]

Then \( \cdot, \cdot \_ \) is an inner product in \( H \) whose induced norm is equivalent to the one induced by (3). Moreover, for any given \( \beta < 0 \), by taking \( \alpha > \ln(2 + 4e^\beta) \) and \( \gamma > \max\{0, \ln(k^2(e^\alpha + 2e^\beta))\} \), there exists a constant \( M > 0 \) such that

\[
\text{Re}(AX, X)_1 \leq M(X, X)_1, \quad \forall X \in D(A).
\]

This together with Lemma 1 implies that \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( \mathcal{H} \).

3. SPECTRAL ANALYSIS

Let us now consider the eigenvalue problem of \( A \). \( \mathcal{A} X = \lambda X \), where \( X = (f, g, h) \in D(\mathcal{A}), \) if and only if \( g(x) = \lambda f(x), h(x) = \lambda f(1)e^{-\tau \lambda x} \) with \( f \) satisfying the eigenvalue problem:

\[
\begin{align*}
  f''(x) &= \lambda^2 f(x), \\
  f(0) &= 0, f'(1) &= k \lambda e^{-\tau \lambda} f(1).
\end{align*}
\]

So \((f, g, h) \neq 0\) if and only if (8) has nonzero solution. Since the general solutions of the first equation of (8) are of the form

\[
f(x) = ae^{\lambda x} + be^{-\lambda x},
\]

where \( a, b \) are two constants, substitute (9) into the boundary conditions of (8) to get

\[
\begin{align*}
  a + b &= 0, \\
  a e^\lambda - b e^{-\lambda} &= ke^{-\tau \lambda}(ae^\lambda + be^{-\lambda}).
\end{align*}
\]

Thus (10) has nontrivial solution if and only if the characteristic equation \( \Delta(\lambda) = 0 \), where

\[
\Delta(\lambda) = \frac{1}{k} \left| e^\lambda - ke^{-\tau(1+\lambda)} - e^{-\lambda} - ke^{-\tau(1+\lambda)} \right| = ke^{-\tau(1+\lambda)} - e^{-\lambda} - ke^{-\tau(1+\lambda)} - e^\lambda.
\]

Hence, we get the following lemma immediately.

Lemma 3. Let \( A \) be given by (4) and let

\[
\Delta(\lambda) = ke^{-\tau(1+\lambda)} - e^{-\lambda} - ke^{-\tau(1+\lambda)} - e^\lambda = 2ke^{-\tau\lambda} \sinh \lambda - 2e^\lambda.
\]

Then \( \sigma(A) = \sigma_p(A) = \{ 0 \in C \} \) (11) for \( \lambda \in \sigma(A) \) is geometrically simple and the corresponding eigenfunction \( X_\lambda = (f_\lambda, g_\lambda, h_\lambda) \) can be given by

\[
\begin{align*}
  f_\lambda(x) &= \sinh(\lambda x), \\
  g_\lambda(x) &= \lambda \sinh(\lambda x), \\
  h_\lambda(x) &= \lambda \sinh(\lambda e^{-\tau \lambda} x).
\end{align*}
\]

For any \( \lambda \in \rho(A) \), we have the following lemma on the expression of the resolvent operator \( R(\lambda, A) = (\lambda - A)^{-1} \).

Lemma 4. Let \( A \) be given by (4) and let \( \Delta(\lambda) \) be given by (11). Then for any \( \lambda \in \rho(A) \) and \( Y = (f_t, g_t, h_t) \in H \), \( X = R(\lambda, A) Y \), where \( X = (f, g, h) \in D(A) \) is given by the following expressions:

\[
\begin{align*}
  f(x, \lambda) &= \frac{2 \sinh \lambda x}{\Delta(\lambda)} G(\lambda, \tau) - \lambda^{-1} G_1(\lambda, x), \\
  f'(x, \lambda) &= \frac{2 \cosh \lambda x}{\Delta(\lambda)} G(\lambda, \tau) - G_2(\lambda, x), \\
  g(x, \lambda) &= \frac{2 \sinh \lambda x}{\Delta(\lambda)} G(\lambda, \tau) - G_1(\lambda, x) - f_1(x), \\
  h(x, \lambda) &= e^{-\tau \lambda x} [g(1, \lambda) + f_1(1)] - G_3(\lambda, x),
\end{align*}
\]

and

\[
\begin{align*}
  G(\lambda, \tau) &= ke^{-\tau \lambda} G_2(\lambda, 1) - G_2(\lambda, 1) + G_3(\lambda, 1), \\
  G_1(\lambda, x) &= \int_0^x \sinh \lambda(x-s)[f_1(s) + g_1(s)] ds, \\
  G_2(\lambda, x) &= \int_0^x \cosh \lambda(x-s)[f_1(s) + g_1(s)] ds, \\
  G_3(\lambda, x) &= f_1(1)e^{-\tau \lambda x} - \tau e^{-\tau \lambda x} \int_0^1 e^{\tau \lambda s} h_1(s) ds.
\end{align*}
\]
Now we characterize the spectrum of $A$.

**Proposition 5.** Let $A$ be given by (4) and let $\Delta(\lambda)$ be given by (11). The following assertions hold for the spectrum of $A$:

(i). There is an $M_0 > 0$, such that for all $\lambda \in \sigma(A)$, $|\text{Re}\lambda| < M_0$, that is, all the eigenvalues of $A$ lie in some vertical strip parallel to the imaginary axis in the complex plane.

(ii). The multiplicity of each root of $\Delta(\lambda) = 0$ is at most two.

(iii). If $\tau$ is rational, then the eigenvalues of $A$ are located on finitely many lines parallel to the imaginary axis.

(iv). If $\tau$ is irrational, then all roots of $\Delta(\lambda) = 0$ are simple.

(v). The eigenvalues of $A$ are separated, that is, $\inf_{\lambda_m, \lambda_n \in \sigma(A), \lambda_m \neq \lambda_n} |\lambda_m - \lambda_n| > 0$.

(vi). The algebraic multiplicity of each eigenvalue of $A$ is at most two.

**Remark** It is proved in Section 6 that when $\tau = 2$, $\Delta(\lambda)$ does have root with multiplicity 2.

4. **RIEZS BASIS PROPERTY AND SPECTRUM-DETERMINED GROWTH CONDITION**

Now we are in a position to consider the Riesz basis property and spectrum-determined growth condition for the system (5).

**Theorem 6.** Let $A$ be given by (4). Then the root subspace of $A$ is complete in $\mathcal{H}$, that is, $\text{Sp}(A) = \mathcal{H}$.

Now we are ready to establish the Riesz basis property of the system (5).

**Theorem 7.** Let $A$ be given by (4) and let $\sigma(A) = \{\lambda_i, i \in \mathbb{N}\}$. Then the following assertions hold.

(1) There is a set of generalized eigenfunctions of $A$, which forms a Riesz basis with parentheses for $\mathcal{H}$. More precisely,

$$W = \sum_{i \in \mathbb{N}} \mathcal{P}_\lambda W, \quad \forall W \in \mathcal{H},$$

and there are constants $M_1, M_2 > 0$ such that

$$M_1 \sum_{i \in \mathbb{N}} \|\mathcal{P}_\lambda W\|^2 \leq \|W\|^2 \leq M_2 \sum_{i \in \mathbb{N}} \|\mathcal{P}_\lambda W\|^2,$$

$$\forall W \in \mathcal{H},$$

where $\mathcal{P}_\lambda$ is the spectral projection operator corresponding to eigenvalue $\lambda_i$ of $A$.

(2). The spectrum-determined growth condition holds ([11]), namely, $S(A) = \omega(A)$, where

$$S(A) := \sup_{\lambda \in \sigma(A)} \text{Re}\lambda$$

is the spectral bound of $A$, and

$$\omega(A) := \inf \{\omega \mid \exists M > 0 \text{ such that } \|e^{\omega t}\| \leq Me^{\omega t}\}$$

is the growth order of $e^{\omega t}$.

**Proof.** Let $\Delta(\lambda)$ be defined by (11), which is obviously an entire function of exponential type. Moreover, by Proposition 5 and the fact $|\Delta(\lambda)| \to \infty$ as $\text{Re}\lambda \to \pm \infty$, it follows that $\Delta(\lambda)$ is a sine-type function.

By Proposition 5, the eigenvalues of $A$ can be decomposed into two separable sets (a multiple eigenvalue is repeated in a number of times equal to its algebraic multiplicity), i.e.,

$$\text{eigenvalues of } A = \Lambda = \bigcup_{n=1}^2 \Lambda_n$$

and

$$\inf_{i \neq j, \lambda_i, \lambda_j \in \Lambda_n} |\lambda_i - \lambda_j| > 0, \quad \forall 1 \leq n \leq 2.$$

Since $\{\lambda_i, i \in \mathbb{N}\}$ is located on the strip parallel to the imaginary axis and has no finite accumulation point, we may assume without loss of generality that $\{\text{Im}\lambda_i\}$ is arranged to form a nondecreasing sequence. By Equation (3.3) of [6], the generalized divided difference (GDD) produced by $\lambda_i$ is just $([1, 6])$

$$E(\lambda_i) = \{e^{\lambda_i t}, e^{\lambda_i \tau t}\}.$$

Then, $D^+(\Lambda) < \infty$. According to Proposition 3.2 of [6], for any $T > 2\pi D^+(\Lambda)$, the family of GDD $\{E(\lambda_i)\}_{i \in \mathbb{N}}$ forms a Riesz basis for the closed subspace spanned by itself in $L^2(0, T)$. Therefore, all conditions of Theorem 3.1 of [6] are satisfied. This together with $\text{Sp}(A) = \mathcal{H}$ as established by Theorem 6 deduces the assertions. The proof is complete.

5. **EXPONENTIAL STABILITY**

In this section, we discuss the exponential stability of the system (5). Since the spectrum determined growth condition holds for system (5) as established by Theorem 7, we only need to check the location of the eigenvalues of $A$ on the complex plane. Let $\Delta(\lambda)$ be given by (11). Then by $\Delta(\lambda) = 0$, we have

$$e^{-\lambda} + ke^{-(1+\tau)\lambda} + e^{\lambda} - ke^{(1-\tau)\lambda} = 0,$$

which is equivalent to

$$e^{(2+\tau)\lambda} - ke^{2\lambda} + e^{\tau\lambda} + k = 0.$$

In what follows, we assume that $\tau = n \in \mathbb{N}^+$. Then (18) becomes

$$e^{(2+n)\lambda} - ke^{\lambda} + e^{\lambda} + k = 0.$$

Set $y = e^{\lambda}$. The Equation (19) can be further written as

$$D(y) = y^{(2+n)} - ky^2 + y + k = 0.$$

So finding the roots of $\Delta(\lambda) = 0$ on the left half complex plane is equivalent to finding the roots of $D(y) = 0$ inside the unit circle.

Since the left-hand side of (20) is a polynomial, there are only finitely many roots inside the unit circle. The following lemma is obvious.

**Lemma 8.** Let $\tau = n \in \mathbb{N}^+$. Then $\text{Re}\lambda < -\omega$ for all $\lambda \in \sigma(A)$ and some $\omega > 0$ if and only if all roots of $D(y) = 0$ lie inside the unit circle.

**Theorem 9.** Let $A$ be given by (4) and let $\tau = n \in \mathbb{N}^+$. If all roots of $D(y) = 0$ lie inside the unit circle, then the system (5) is exponentially stable.

**Proof.** If all roots of $D(y) = 0$ lie inside the unit circle, by Lemma 8, there is a positive constant $\omega > 0$ such that all eigenvalues $\lambda \in \sigma(A)$ satisfy $\text{Re}\lambda < -\omega$. Moreover,
Theorem 7 says that the spectrum determined growth condition of the system (5) is valid. Hence, the system (5) is exponentially stable. The proof is complete. □

For a given \( n \in \mathbb{N}^+ \), by using the Schur-Cohn Criterion (See e.g., [9] on page 34-36 or Proposition 5.3 of [10] on page 27), we can get the necessary and sufficient condition on \( k \) such that all the roots of \( D(y) \) lie inside the unit circle.

First we have the following necessary condition for even numbers \( n \).

\[ \text{Theorem 10. Suppose that } n \in \mathbb{N}^+ \text{ is an even number, and all the roots of } D(y) \text{ lie inside the unit circle. Then}
\]
\[ \begin{align*}
&\text{(i) if } n = 4m, m \in \mathbb{N}^+, \text{ then } k \in (-1, 0); \\
&\text{(ii) if } n = 4m - 2, m \in \mathbb{N}^+, \text{ then } k \in (0, 1).
\end{align*} \]

\[ \text{Proof. When } n = 4m \text{ and } n = 4m - 2, m \in \mathbb{N}^+, D(y) = y^{4m+2} + y^{4m} - ky^2 + k \text{ and } D(y) = y^{4m} + y^{4m-2} - ky^2 + k, \text{ respectively. By the relationship between the coefficients and the roots of a polynomial, if all the roots of } D(y) \text{ lie inside the unit circle, it must be that } |k| < 1.
\]

(i). When \( n = 4m \), for simplicity, we may assume \( s = y^2 \) and denote \( D_1(s) = s^{2m+1} + s^{2m} - k s + k \) without confusion. Since \( D_1(s) \) is of odd order, by the Schur-Cohn Criterion, it follows that \( D_1(1) = -D_1(-1) = 2k > 0 \), in order for all of its roots to be inside the unit circle. Since
\[
\begin{align*}
D_1(1) &= 1 - k + k + 1 = 2 > 0, \\
-D_1(-1) &= 1 - k - k - 1 = -2k,
\end{align*}
\]
by the Schur-Cohn Criterion, if all the roots of \( D_1(y) \) lie inside the unit circle, it should be that \( D_1(-1) = -2k > 0 \), namely, \( k < 0 \). This together with \( |k| < 1 \) shows that \( k \in (-1, 0) \).

The proof is complete. □

(ii). When \( n = 4m - 2 \), under the assumption, \( D_1(s) = s^{2m} + s^{2m-1} - k s + k \) with \( s = y^2 \). Since \( D_1(s) \) is of even order, then, by the Schur-Cohn Criterion, it follows that \( D_1(1) \) and \( D_1(-1) \) must be positive, in order for all of its roots to be inside the unit circle. Since
\[
D_1(1) = 1 - k + k + 1 = 2 > 0, \quad D_1(-1) = 1 + k + k - 1 = 2k,
\]
by the Schur-Cohn Criterion, if all the roots of \( D_1(y) \) lie inside the unit circle, then it should be that \( D_1(-1) = 2k > 0 \). This together with \( |k| < 1 \) shows that \( k \in (0, 1) \).

We now turn our attention to the case where \( \tau = \frac{2}{p} \) is a rational number, where \( p, q \in \mathbb{N}^+ \) are coprime. Then (18) becomes
\[ e^{(2+q/p)\lambda} - k e^{2\lambda} + e^{(q/p)\lambda} + k = 0. \]

Let \( y = e^{\lambda} \). Then (21) becomes
\[ D(y) = y^{2p+q} - ky^{2p} + y^q + k = 0. \]

We can discuss the roots of \( D(y) \) in the same fashion as in the integer case. The following proposition gives the result for a special case of \( \tau = \frac{2}{p} \).

\[ \text{Proposition 13. For } \tau = \frac{2}{p}, \text{ there is no } k \in \mathbb{R} \text{ such that all the roots of } D(y) \text{ lie inside the unit circle.} \]

We list in Table 1 the region of \( k \) for the first few even numbers for which the exponential stability holds.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>The stability region of the feedback ( k )</th>
<th>The sign of ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(( -\infty, 0 ))</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>(( 0, 1 ))</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>(( 1 - \sqrt{2}, 0 ))</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>(( 0, 2 - \sqrt{3} ))</td>
<td>+</td>
</tr>
<tr>
<td>8</td>
<td>(( 1 - \sqrt{2}, 0 ))</td>
<td>-</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The case of odd delay values is complicated. Some special odd values discussed in Proposition 12 below suggest that there is no \( k \in \mathbb{R} \) such that all the roots of \( D(y) \) lie inside the unit circle, but the general case requires further investigation.

\[ \text{Proposition 12. For } n = 1, 3, 5, \text{ there is no any } k \in \mathbb{R} \text{ such that all the roots of } D(y) = 0 \text{ lie inside the unit circle.} \]

In this section, for a special case of \( \tau = 2 \), we give the exact expressions for the eigenvalues. These eigenvalues ensure the exponential stability of the system considered in [4]. We show, using the method of [3], that the feedback loop is not robust to a small perturbation in time delay, which coincides with the general result of [8].

\[ \text{Theorem 14. Let } \mathcal{A} \text{ be given by (4) and let } \tau = 2. \text{ The system (5) is exponentially stable if and only if } k \in (0, 1). \]

The eigenvalues of \( \mathcal{A} \) have the following form:
\[
\left\{ \begin{array}{l}
\lambda_n = \frac{1}{2} \ln \frac{k - 1 \pm \sqrt{(k-1)^2 - 4k}}{2} + \left( n + \frac{1}{2} \right) \pi i; \\
\quad \text{when } 0 < k < 3 - 2\sqrt{2}, \\
\lambda_n = \frac{1}{2} \ln \frac{k - 1}{2} + \left( n + \frac{1}{2} \right) \pi i; \\
\quad \text{when } 3 - 2\sqrt{2} < k < 1,
\end{array} \right.
\]
where \( n \in \mathbb{Z} \) and \( \theta \) is given by
\[ \theta = \arctan \frac{\sqrt{4k - (k-1)^2}}{k - 1}. \]
In particular, when \( k = 3 - 2\sqrt{2} \), each eigenvalue \( \lambda_n \) is of algebraic multiplicity two.

**Proof.** The exponential stability was ensured by Theorem 9 and Proposition 11. Now, the characteristic equation \( \Delta(\lambda) = 0 \) given by (11) is reduced to
\[
e^{4\lambda} + (1-k)e^{2\lambda} + k = 0,
\]
which is equivalent to
\[
e^{2\lambda} = k - 1 \pm \sqrt{(k-1)^2 - 4k}.
\]
There are three cases:

**Case 1:** \( 0 < k < 3 - 2\sqrt{2} \). In this case, \((k-1)^2 - 4k > 0\).

\[
-1 < k - 1 - \sqrt{(k-1)^2 - 4k} < 0,
\]
we get the eigenvalues of \( \lambda_n \) as follows:
\[
\lambda_n = \frac{1}{2} \ln \left( \frac{k - 1 \pm \sqrt{(k-1)^2 - 4k}}{2} \right) + \left( n + \frac{1}{2} \right) \pi i, \quad n \in \mathbb{Z}.
\]

**Case 2:** \( k = 3 - 2\sqrt{2} \). In this case, \((k-1)^2 - 4k = 0\).

So \( e^{2\lambda} = \frac{k - 1}{2} \) is a root of the characteristic equation (24) with multiplicity two. Therefore, the eigenvalues of \( \lambda_n \) are found to be
\[
\lambda_n = \frac{1}{2} \ln \left( \frac{k - 1}{2} \right) + \left( n + \frac{1}{2} \right) \pi i, \quad n \in \mathbb{Z},
\]
and by Proposition 5, each eigenvalue \( \lambda_n \) is of algebraic multiplicity two.

**Case 3:** \( 3 - 2\sqrt{2} < k < 1 \). In this case, \((k-1)^2 - 4k < 0\).

\[
\lambda_n = \frac{1}{2} \ln k + \frac{1}{2} \pi i + \left( n + \frac{1}{2} \right) \pi i, \quad n \in \mathbb{Z},
\]
where \( \theta \) is determined by (23). The proof is complete. \( \square \)

The lack of robustness proof follows from the idea of [3].

**Theorem 15.** Let \( \lambda_n \) be given by (4), let \( k \in (0,1) \), and
\[
\delta_m = \frac{1}{m + \frac{1}{2}}, \quad m = 0,1,2,\ldots
\]
Suppose that \( \tau = \tau_m = 2 + \delta_m \) in (1). Then for each \( m \), the system (5) has an eigenvalue given by
\[
\lambda_m = \lambda^1_m + \left( m + \frac{1}{2} \right) \pi i, \quad \lambda^1_m > 0
\]
and
\[
\text{Re}(\lambda_m) = \lambda^1_m \rightarrow \lambda_0 \text{ as } m \rightarrow \infty,
\]
where
\[
\lambda_0 = \frac{1}{2} \ln \left[ k + 1 \pm \sqrt{(k+1)^2 + 4k} \right] > 0
\]
satisfies
\[
\frac{e^{-2\lambda_0} + 1}{e^{2\lambda_0} - 1} = \frac{1}{k}.
\]

**Proof.** Let \( \tau = \tau_m = 2 + \delta_m \). By (11), the characteristic equation \( \Delta(\lambda) = 0 \) becomes
\[
e^{4\lambda_m + 2\lambda} - k \tanh \lambda = 0,
\]
which can be rewritten as
\[
e^{4\lambda_m + 2\lambda} - ke^{2\lambda} e^{-\lambda} = e^{4\lambda_m + 2\lambda} - k e^{2\lambda} - 1 = 0.
\]
This further yields
\[
e^{4\lambda_m} - k e^{-2\lambda} = 0.
\]
Let the imaginary part of \( \lambda \) satisfy \( \text{Im}\lambda = (n + \frac{1}{2}) \pi, \) \( n \in \mathbb{N} \). Then (32) has an eigenvalue \( \lambda_m \) with \( \lambda_m = \lambda^1_m + (m + \frac{1}{2}) \pi i, \lambda^1_m \in \mathbb{R}, \) whose real parts satisfy
\[
e^{4\lambda^1_m} \delta_m = ke^{-4\lambda^1_m} + 1
\]
or
\[
\delta_m = \frac{1}{\lambda^1_m} \left[ \log \left( \frac{e^{-4\lambda^1_m} + 1}{e^{2\lambda^1_m} - 1} \right) + \log k \right].
\]
Since \( \tau_m \) satisfies (26), it follows from (33) or (34) that \( \lambda^1_m \) is positive. Let
\[
f(s) = \frac{e^{-2s} + 1}{e^{2s} - 1}, \quad s \in \mathbb{R}^+,
\]
Since for any \( s \in \mathbb{R}^+ \),
\[
f'(s) = \frac{-2e^{-2s} (e^{2s} - 1) - 2e^{2s} (e^{-2s} + 1)}{(e^{2s} - 1)^2}
\]
\[
= -2 + 2e^{-2s} - 2e^{2s} < 0,
\]
f(\( s \)) is monotonically decreasing on \((0,\infty)\), and hence (33) or (34) has a unique solution. Moreover, solve \( f(\lambda_0) = \frac{1}{k} \) to give
\[
\frac{e^{-2\lambda_0} + 1}{e^{2\lambda_0} - 1} = \frac{1}{k},
\]
which yields
\[
e^{4\lambda_0} - (k + 1)e^{2\lambda_0} - k = 0.
\]
This gives
\[
e^{2\lambda_0} = \frac{k + 1 \pm \sqrt{(k+1)^2 + 4k}}{2},
\]
Due to \( e^{2\lambda_0} > 0 \), we get \( \lambda_0 \) given by (28) directly. Hence the right side of (34) is greater than zero on \((0, \lambda_0)\) and less than zero on \((\lambda_0, \infty)\). Therefore, when \( m \rightarrow \infty \), we have \( \tau_m \rightarrow 2 \) and \( \lambda^1_m \rightarrow \lambda_0 \). The proof is complete. \( \square \)

**Remark** The same argument in the proof of Theorem 15 can be used to get the non-robustness result for negative \( \delta_m \):
\[
\delta_m = -\frac{1}{m + \frac{1}{2}}, \quad m = 0,1,2,\ldots
\]
Due to the page limitation, we omit the proof here.

7. **NUMERICAL SIMULATION FOR THE CASE \( \tau = 2 \)**

In this section, we give some numerical simulation results for the system (5) with \( \tau = 2 \) and \( k \in (0, 3 - 2\sqrt{2}) \), which is beyond the validity of [4]. The finite difference method
In the numerical scheme, the space grid size \( N = 40 \) and \( \Delta t = 10^{-3} \) time step are used. The parameters are taken to be \( \tau = 2, k = 0.1, 0.05 \), respectively. The initial conditions are

\[
 w_0(x) = w_1(x) = x^2 - 2x, \; \forall \; x \in [0, 1]. \tag{36}
\]

It is seen that in these two cases, the state \((w(x, t), w_t(x, t))\) is convergent to zero.

REFERENCES