Sliding Mode Controller and Flatness Based Set-Point Generator for a Three Wheeled Narrow Vehicle

Nestor Roqueiro∗ Marcelo G. de Faria** Enric Fossas Colet***

∗ Universidade Federal de Santa Catarina, Florianópolis, BR (e-mail: nestor@das.ufsc.br).
** Universidade Federal de Santa Catarina, Florianópolis, BR (e-mail: fara@das.ufsc.br).
*** Universitat Politècnica de Catalunya, Barcelona, ES (e-mail: enric.fossas@upc.edu).

Abstract: This paper presents a sliding mode controller for a narrow tilting three wheeled vehicle. A dynamic model of eighteenth order is presented for simulation purposes only. Design and analysis are based on a simpler third order model of a bicycle. The validity of this simplification is verified through simulations. Because of the flatness of the bicycle, trajectories given by a presumed driver are redefined to take into account the limits of the vehicle stability. Finally, tilting and speed controllers are designed in the frame of sliding modes. Analytical and simulation results performed in the simpler and complex model respectively validate these designs.

Keywords: Sliding Mode Control, Flatness, Set-point generator, Tilting Vehicle, Nonlinear Models

1. INTRODUCTION

Nowadays, automobile companies are involved in the design of more efficient vehicles improving the energetic efficiency and making them smaller for the best use of the existing roads and streets. For this end and in order to lighten the traffic in the cities, the design of smaller vehicles with a better weight/load ratio are reported in Ashmore [2006], Johannsen and et al. [2003], Gehre et al. [2001], Brink and Kroonen [2004], Gohl et al. [2006]. There is also a project in development at the “Universidade Federal de Santa Catarina” (Brazil). Specifically the aim of the project is the design of a three wheeled narrow vehicle for two passengers (Vieira et al. [2009]), However, because of their narrowness, this kind of vehicles have stability problems. A stable behavior can be achieved by allowing the vehicle to lean in curves, like a motorcycle. This strategy has been used in several concept vehicles, for instance Mercedes Benz F 300 Life-Jet, General Motors Lean Machine, BMW Clever and Simple and one production vehicle, Carver by VanderBrink.

Recent publications address different topics related to control of such vehicles. In Kidane et al. [2008], the authors applied two different types of control schemes known as steering tilt control (STC) and direct tilt control (DTC). The stability control algorithm for tilting vehicles needs to be developed in such a way that no special operating skills are required by the driver. It means the driver will use the steering input to follow a certain desired trajectory without regard to the vehicle’s tilt stability. That work uses a feedforward plus PID controllers for tilt stabilization and a driver model that defines a look-ahead error of the trajectory.

In Defoort and Murakami [2009] and Nenner et al. [2008], the authors deal with the robust stabilization and trajectory-tracking problems of a riderless bicycle. In the last article, a dynamic model which takes into account geometric-stabilization mechanisms due to bicycle trail is presented. A posture controller combining second-order sliding mode control and disturbance observer is derived. Then, a tracking controller based on the proposed posture controller and the dynamic-inversion framework is designed. The dynamical model of a bicycle presented will be adopted in this work as a simplified model of the tricycle.

Several publications have presented the idea of a virtual driver who has the ability to follow a path without falling. Frezza and Beghi [2003] take the roll angle as control input instead of steering angle avoiding to deal directly with lean instability. Path tracking is defined as an optimization problem. In Saccon et al. [2008] a dynamic inversion of a simplified motorcycle model is used, allowing the computation of state and input trajectories corresponding to a desired output trajectory. A stabilizing feedback is obtained by standard Linear Quadratic Regulator (LQR).

The central issue in S. Fuchshumer and Rittenschober [2005] is the discussion of the differential flatness of the
planar holonomic bicycle model of a car. The bicycle model is obtained from the four wheeled vehicle by merging together the front and the rear wheels to a single (massless) front and rear wheel, respectively, located at the longitudinal axis of the car. The vehicle dynamics control design is accomplished following the flatness based control theory.

Our goal is to design a controller able to tilt the vehicle through steering angle, taking into account the expected limits to avoid losses of tire grip. Furthermore, it should track the trajectory given by the driver through the steering wheel as close as possible. Because it is a light vehicle, its mass changes significantly with full or partial load, shifting the center of gravity. Thus, a robust controller should be used. The controller should be transparent to the driver and reach the optimum tilt angle, improving the vehicle safety and its performance. The aim of this paper is to show how these objectives can be reached using sliding mode control for low level controllers (longitudinal speed and tilt angle).

First, we briefly present a model of a tricycle with nine degrees of freedom that will represent the real system. Then, we propose a simple model of a bike that leans as an approximation of three-wheeled vehicle for the project and analysis of the controllers. Simulations allow us to ensure that both models are similar enough to design controllers using the simpler one. A flatness based approach for trajectory tracking is presented and used to define the set points for low level controllers. Next section shows briefly the project of sliding mode low level controllers. We present some simulations of closed loop system using the nine degrees of freedom model with and without perturbations. Finally, conclusions and some topics on future developments close the paper.

2. THE THREE WHEELED TILTABLE VEHICLE

This work starts from the tilting tricycle model of six degrees of freedom reported in Vieira et al. [2009]. Three extra degrees of freedom were added to allow modelling independent vertical movements of each wheel. In addition, lateral wind and road roughness were modelled as external disturbances.

The vehicle consists of a central mass (body 2) and three wheels, two located in the front (bodies 3 and 4) and one in the rear (body 1). The origin of the coordinate system is the contact point of the rear wheel with the road and the clockwise convention is adopted (Figure 1). The three wheels are assumed to be always in contact with the ground.

In this frame, the velocity of each of the masses is defined. It results in a model of nine degrees of freedom with positions and velocities described as (see Figure 2):

- Longitudinal vehicle movement \( (x) \), \( \dot{x} = u \).
- Transverse vehicle movement \( (y) \), \( \dot{y} = v \).
- Rear wheel vertical movement \( (z_1) \), \( \dot{z}_1 = w_1 \).
- Main body vertical movement \( (z_2) \), \( \dot{z}_2 = w_2 \).
- Right side front wheel vertical movement \( (z_3) \), \( \dot{z}_3 = w_3 \).
- Left side front wheel vertical movement \( (z_4) \), \( \dot{z}_4 = w_4 \).
- Rotation with respect to Z axis \( (\psi) \).
- Rotation with respect to X axis \( (\phi) \).

\[ \ddot{q} + \Lambda(q) \dot{q} + \sum_{i=1}^{4} M_i(q) \ddot{z}_i + \sum_{i=1}^{4} K_i(q) \dot{z}_i = F(q, \dot{q}, t) \]  

where \( M(q) \), \( C(q) \) and \( K(q) \) are respectively the inertia, damping and stiffness matrices, \( q \) is the freedom degrees vector defined by \( q = \{x, y, z_1, z_2, z_3, z_4, \psi, \phi, \theta \} \) and

In this way the vertical movements of all of the bodies are taken into account independently, allowing to simulate suspensions. Vehicle rotation with respect to Y axis \( (\alpha) \) will be modelled by a variable parameter in velocities formulation. The variable \( \alpha \) represents the inclination of the road.

![Inertial coordinate system](image)

**Fig. 1. Inertial coordinate system**

- Rotation of the body 2 with respect to an axis parallel to Y axis \( (\theta) \). This is due to a movement of body 2 (not punctual) with respect to its own rotation center which will be considered to be placed in its mass center. The rotation around its transverse axis causes a movement that interferes with the behaviour of the front and rear suspensions.

In this way the vertical movements of all of the bodies are taken into account independently, allowing to simulate suspensions. Vehicle rotation with respect to Y axis \( (\alpha) \) will be modelled by a variable parameter in velocities formulation. The variable \( \alpha \) represents the inclination of the road.

To create the dynamical model, a multi-body analysis approach was chosen, following two steps to achieve the final model. They are:

- **Velocity equations**: They represent the way that velocities works on bodies in 9 degree of freedom. These equations are based on vehicle velocity and represent the different components for each body.
- **Equation of motion**: Building equations to kinetic and potential energy based on the previously definition, we can define a Lagrangian Equation of Motion, that describes a differential equation set.

The basic geometry of body 2 is depicted in Figure 3 and geometric definitions of bodies 3 and 4 are depicted in Figure 4.

The full model of the vehicle was presented in Roqueiro and Fossas [2010]. As in Leal et al. [2008], using the Lagrangian formulation, a second order vector dynamic model can be written as:

![Velocities definition for mass 1](image)

**Fig. 2. Velocities definition for mass 1**
F(q, \dot{q}, t) the vector of external forces and momenta acting on the vehicle.

3. TILTABLE BICYCLE

The bicycle models presented in Getz [1995] and Frezza and Beghi [2003] begin with a description of a simplified kinematic model of a car considering the wheels in its medium plane. Second, it incorporates the dynamic characteristics of the inverted pendulum. Thirdly, the trajectory problem is solved with kinematic model. Then, a viable solution to avoid the bicycle falling is obtained through an optimization process. In this article, dynamics model is defined instead of the kinematics one prior to solve the reference's tracking problem.

When riding a non-autonomous tilting tricycle, which is our bicycle-like vehicle case study, the desired trajectory given by the driver should be followed through the steering wheel and the accelerator pedal. The stable tilting dynamic behavior of the vehicle have priority over trajectory and speed. This means that if some reference should be relaxed, first it must be the speed in order to meet the trajectory and angle of inclination. Failure to meet these two specifications means that the reference trajectory should be relaxed in order to prevent vehicle falling. Through the steering wheel the driver sets the desired values of front wheels angle \( \delta_d(t) \) and with the accelerator pedal the driver sets the vehicle velocity \( u_d(t) \). So, we model the tilting behavior and then we will project a controller in order to drive as close as possible to the desired trajectory. We use a simple model for a bike that uses Newton’s second law to model the tilting dynamics as:

\[
\ddot{\phi}(t) = g \sin(\phi(t)) + \frac{\cos(\phi(t))}{a_4} u^2(t) \delta(t) \tag{2}
\]

It is just a force balance between gravitational momentum and fictitious inertial momentum, with \( g \) the gravitational acceleration, \( u \) the longitudinal speed of the vehicle \( a_4 \) the vehicle’s length and \( \delta \) the steering angle. The angle \( \phi \) is restricted by the road grip to a maximum value \(|\phi(t)| < \phi_{\text{max}}|\).

For \( x - a \) longitudinal movement, the model can be written as:

\[
m \ddot{u}(t) = \frac{2n T_m(t) \eta_T}{d} - \frac{1}{2} C_x u^2(t) A \rho \tag{3}
\]

It is a force balance which considers the thrust and the drag force of wind, where \( n \) is the transmission reduction, \( T_m \) the traction torque, \( d \) the wheel diameter, \( \eta_T \) the transmission efficiency, \( C_x \) the aerodynamic drag coefficient for longitudinal flow, \( A \) the vehicle’s frontal area and \( \rho \) the air density.

A comparison between the two models used in this work is presented below. The controller project, including analysis of closed loop stability, is facilitated using the simplified model. Obviously, some dynamic behaviours can not be described. In this case we are interested in controlling two variables directly, the leaning angle \( \phi \) and the speed \( u \). Then, a simple model considering the dynamics of this two variables could be enough. Equations (2) and (3) are taken as a simplified model. The other objective is to follow a trajectory given by the driver. The trajectory is defined by the velocity \( u_d \) and the steering angle \( \delta_d \) as a first approximation. A comparison between those two models gives a qualitative and perhaps quantitative information on the behaviour of the models adopted. The following figures show some comparative results.

In the simulation shown in the Figure 5 we used the same driver inputs for \( \delta_d(0.001\text{rad}) \) and velocity \( u_d(26\text{m/s}) \). The speed control is done with a Sliding Mode controller (Roqueiro et al. [2010]), and tilt control is done with PID with gain schedule. We note that the dynamic behaviour is similar with the same steady states values. The oscillations seen in the dynamic response of the bicycle and that do not appear in tricycle’s response are due to interactions between the front wheels and the central body.

A simulation with system in open loop for speed and PID Gain Schedule controller for tilt angle is shown in Figure 6. Simultaneous changes were implemented in the speed and steering angle in order to test the effects of coupling.

The conclusion is that this model can be used successfully for project control and analysis of the dynamic properties of the closed loop system.

Any controller based on the bicycle model ought to deal with strong oscillations in transient. So, if it succeed, it
will show better performance in the three-wheeled vehicle. A similar approach to the design of controllers through the use of simplified models of two wheeled vehicles is used in Kidane et al. [2007], Kidane et al. [2008].

4. THE TRAJECTORY PLANNING PROBLEM

From current and targeted steering angle and linear velocity values provided by the driver, a continuous reference trajectory is planned. The system given by equations (2) and (3) is flat and the tilting angle and the linear velocity are flat outputs. Thus, from the driver’s goal values, a tilting target is computed and reference trajectories for the tilting angle and the linear velocity are planned.

We say that a tilting target $\phi_{k+1}$ is feasible if $|\phi_{k+1}| \leq \phi_{max}$. Hence, using equation (2), we define the tilting target as:

$$\phi_{k+1} = \text{sign}(\delta_{k+1}) \max \left\{ \left| \frac{\arctan\left(\frac{u_{k+1}^2 \delta_{k+1}}{a_4 g}\right)}{\phi_{max}} \right|, \phi_{max} \right\}$$

If $\phi_{k+1} = \phi_{max}$, $u_{k+1}$ is calculated again from equation (2) so that $\delta_{k+1}$ can be reached. Thus, from $\phi_k$, $\phi_{k+1}$, $u_k$ and $u_{k+1}$, the reference trajectories are defined by:

$$\phi_{k+1}(t) = \phi_k + (\phi_{k+1} - \phi_k) \frac{(t-t_k)}{(t_{k+1}-t_k)}$$

$$u_{k+1}(t) = u_k + (u_{k+1} - u_k) \frac{(t-t_k)}{(t_{k+1}-t_k)}$$

Remark that these trajectories yield uniform rectilinear movements (second derivatives are zero). Furthermore, feedforward controller can be obtained from equations (2) and (5) together with the actuator dynamics.

There are some pre-established conditions that agree to the solution of the trajectory problem. Namely, if the target steering angle and linear velocity yield a non feasible target tilting angle, the linear velocity must be reduced so that the target steering angle is kept.

5. STATEMENT OF THE CONTROL PROBLEM

In four wheeled vehicles, the rotation with respect to the X axis (angle $\phi$, also called rolling) is not desired. However, two wheeled vehicles (i.e. bikes and motorcycles) take benefit of rolling angle to compensate fictitious inertial force effects when cornering, improving the stability and the performance. The bearing angle $\phi$ can be dynamically adjusted (controlled) through a tilting mechanism. One of the aims of this work is to carry out a controller such that, acting on the steering wheels, it corrects the tilting angle $\phi$ in a way that forces to the wheel’s plane are cancelled. The desired tilting angle $\phi_d$ is given in equation (4). Furthermore, there is a second control objective: to track the forward velocity.

5.1 Sliding mode control

In Roqueiro et al. [2010] and Moreira et al. [2009], single input controllers were designed for the tricycle. The selected system input was the front wheels steering angle, and the selected output was the bodywork tilting angle. In this paper, a two-input two-output sliding mode controller based in the model described in section 3 is reported. The input variables are the front wheels steering angle $\delta$ and the motor torque $T_m$. The output variables are $\phi$, the vehicle tilting angle and $u$, the vehicle speed.

5.2 Sliding surfaces

The outputs to be regulated to zero are the errors in the output variables. Namely, zero tilting error and zero speed error in the X axis with relative degrees two and one, respectively. Then, the following sliding functions are selected:

$$S_1 = \frac{d}{dt} (\phi_d - \phi) + (\phi_d - \phi)$$

$$S_2 = u_d - u$$

are selected. The closed loop system is input-output decoupled. Input $\delta$ manages $S_1$ while the input $T_m$ manages $S_2$. Indeed, the Jacobian of the switching function is a $2 \times 2$-matrix with zeroes in the diagonal. Then, the control action defined by:

$$\delta = \begin{cases} -k_1 & \text{if } S_1 > 0 \\ k_1 & \text{if } S_1 < 0 \end{cases}$$

$$T_m = \begin{cases} -k_2 & \text{if } S_2 > 0 \\ k_2 & \text{if } S_2 < 0 \end{cases}$$

yields ($S_1$, $S_2$) = (0, 0), provided that $|T_{eq}| < k_2$ and $|\delta_{eq}| < k_1$. Hence, $u = u_d$ and $u_d = \phi_d - \phi$. In the subset of the intersection $\{x \mid S_1 = 0 \land S_2 = 0\}$ defined by the preceding inequalities.

Naturally, nobody expects a bang-bang action for inputs $T_m$ and $\delta$ that are continuous functions. However, discontinuous control actions can be designed for the corresponding actuators (two DC-motors with torque and position as outputs). In the Laplace domain, they can be modelled as:

$$G_{T_m}(s) = \frac{b}{s + 1}$$

$$G_{\delta}(s) = \frac{a}{s(a s + 1)}$$

Taking the actuators into account, two new switching functions $\hat{S}_1$ and $\hat{S}_2$ have to be considered.

$$\hat{S}_1 = p_1 \frac{d}{dt} (\phi_d - \phi)$$

$$\hat{S}_2 = p_2 \frac{d}{dt} (u_d - u)$$
where \( p_i \left( \frac{d}{dt} \right) \) for \( i = 1, 2 \) are polynomials in the variable \( \frac{d}{dt} \) of degrees 2 and 3 respectively. Furthermore, the couple \((\hat{S}_1, \hat{S}_2)\) has a well defined vector relative degree \((4,2)\), but the new input-output system \((v_1, v_2; \hat{S}_1, \hat{S}_2)\) is no longer decoupled. Hence, the discontinuous control is designed for the new input variables

\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \nabla \hat{S} G \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
\]

where \( G \) is the input matrix. See details in the appendix.

Thus, the controllers

\[
v_i = \frac{V_{i,m} + V_{i,M}}{2} + \frac{V_{i,m} - V_{i,M}}{2} \text{sign}(\hat{S}_i), \quad \text{for } i = 1, 2
\]

locally force \( \hat{S}_1 = 0 \) and \( \hat{S}_2 = 0 \) in finite time. These equalities, in turn, yield \( z_1 = z_{1,d} \) and \( z_5 = z_{5,d} \) asymptotically.

In the original input variables the controllers result in:

\[
\begin{align*}
    w_1 &= - \left( \frac{V_{1,m} + V_{1,M}}{2} + \frac{V_{1,m} - V_{1,M}}{2} \text{sign}(\hat{S}_1) \right) \\
    w_2 &= \frac{V_{2,m} + V_{2,M}}{2} + \frac{V_{2,m} - V_{2,M}}{2} \text{sign}(\hat{S}_2)
\end{align*}
\]

As for the discontinuous gains \( V_{1,m}, V_{1,M}, V_{2,m}, V_{2,M} \), they will define the sliding domain which must include the target dynamics. There is a constrain on these values given by the power of the traction motor. However, the gain corresponding to the steering angle can be adjusted by appropriate gears.

5.3 Tracking a reference

First the model is simulated in extreme conditions without perturbations in order to observe the ideal sliding dynamics. The selected velocities are 4 \( m/s \), 8.3 \( m/s \) and 30 \( m/s \) to reflect maximum and minimum velocities and an intermediate cruising speed in town. The simulation consists in turning the steering wheel to the driver’s desired angle \((-0.3 rad, \pm 0.07 rad \) and \( \pm 0.003 rad \)\) for the three selected velocities and then going back to the rest position.

Table 1 shows the Root Mean Square (RMS) error and the max absolute error achieved in this simulation.

<table>
<thead>
<tr>
<th>Speed</th>
<th>RMS Error</th>
<th>Max. Abs. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 ( m/s )</td>
<td>0.0024</td>
<td>0.0047</td>
</tr>
<tr>
<td>8.3 ( m/s )</td>
<td>0.0024</td>
<td>0.0047</td>
</tr>
<tr>
<td>4 ( m/s )</td>
<td>0.0024</td>
<td>0.0047</td>
</tr>
</tbody>
</table>

5.4 Tracking a reference in the perturbed model

In the second simulation, wind perturbation and road roughness will be considered. Specifically, a lateral wind of 15 \( m/s \) in both directions for high speed (at simulation time 30 and 50, ceasing at 70 seconds), 10 \( m/s \) for cruising speed (at simulation time 130, 150 and ceasing at 70 seconds) and 2.5 \( m/s \) (at simulation time 230, 250 and ceasing at 270 seconds) for low speed. The speed changes occurs at time 90 \( seconds \) (from 30 to 8.3 \( m/s \) and time 190 (from 8.3 to 4 \( m/s \)). The road roughness perturbation is modelled by means of a random signal of frequency 0.2 Hz and width 0.5 cm, generating a vertical force in each wheel.

Figure 7 shows the system’s performance under tilting control. It is worth to remark that the velocity control system is not significantly affected by this perturbation.

Figure 8 shows the steering (control action) performed by the vehicle to achieve the tracking of the reference tilting angle.

<table>
<thead>
<tr>
<th>Speed</th>
<th>RMS Error</th>
<th>Max. Abs. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 ( m/s )</td>
<td>0.0086</td>
<td>0.0006</td>
</tr>
<tr>
<td>8.3 ( m/s )</td>
<td>0.0066</td>
<td>0.0054</td>
</tr>
<tr>
<td>4 ( m/s )</td>
<td>0.0053</td>
<td></td>
</tr>
</tbody>
</table>

6. CONCLUSIONS AND FURTHER RESEARCH

Simulation results show that a three dimension flat model of a bicycle is sufficient to design sliding mode controllers for a three wheeled narrow vehicle. The controllers yield a good performance of the eighteenth order system even...
when it is submitted to lateral wind perturbations. When linear actuators are included into the model, the system results in a two-input two-output coupled system with well-defined vector relative degree. It has been decoupled from a change of variables in the inputs for control design purposes. The improvement of the model including tire-road effects is left as a further research.

REFERENCES


N. Roqueiro, R. S. Vieira, and M. Gaudenzi. Tilting control of a three-wheel vehicle through steering. In XVIII Coimbra Brasil de Automatica, Bonito, MS, Brazil, 2010.

APPENDIX

The full model: bicycle plus actuators is a system of ODE in R^6. Let z = (z_1, z_2, z_3, z_4, z_5, z_6) and w = (w_1, w_2) be the state and the input variables respectively, z_i = \phi the tilting angle, z_3 = \delta the steering angle, z_4 = u the forward velocity and z_6 = T_m the traction torque. w_1 and w_2 are the actuator inputs. The vector field that describes the dynamics is

$$\frac{dz}{dt} = f(z) + G \cdot w^T$$

where

$$f(z) = \begin{pmatrix} \frac{z_2}{l} - \cos(z_1) \cdot \frac{z_3 \cdot z_5}{l} \\ \frac{z_4}{a} - \frac{z_4}{a} \cdot z_6 \cdot \alpha - \beta \cdot \frac{z_5^2}{a} \\ -\frac{z_6}{a} \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & b \end{pmatrix}$$

It is straightforward to check that the vector relative degree of \((z_1 - z_1, z_3 - z_3)\) is well defined and it is equal to (4, 2). Thus, the switching functions can be defined by \(\tilde{S}_1 = p_1 \left( \frac{d}{dt} (z_1 - z_1, z_3 - z_3) \right)\), \(\tilde{S}_2 = p_2 \left( \frac{d}{dt} (z_5 - z_5) \right)\), where \(p_1 \left( \frac{d}{dt} \right)\) and \(p_2 \left( \frac{d}{dt} \right)\) are Hurwitz polynomials in the derivative with respect to time of degrees three and one respectively. Since these polynomials are designed by the user, the corresponding poles can be placed appropriately so that higher derivatives could not be heard in mind.

Remember that the system is input-output decoupled when \(\delta\) and \(T_m\) are the inputs. However, when the actuators dynamics are considered this is no longer true. One gets,

$$\frac{\partial \tilde{S}}{\partial z} \cdot G = \begin{pmatrix} \frac{\cos(z_1) \cdot z_3^2}{l} & 2 \cos(z_1) \cdot z_3 \cdot z_5 \cdot \alpha \cdot b \\ \frac{\cos(z_1) \cdot z_3^2}{l} & \frac{\cos(z_1) \cdot z_3 \cdot z_5 \cdot \alpha \cdot b}{l} \end{pmatrix}$$

Thus, one proceed as usual in order to simplify the control design and a new set of inputs \(v = (v_1, v_2)\) is considered; namely, \(v^T = \frac{\partial \tilde{S}}{\partial z} \cdot G \cdot w^T\). Let us assume discontinuous gains \(V_{i,m} < V_{i,M} \) so that \(V_{i,m} \leq v_i \leq V_{i,M}\), then

$$v_i = \frac{V_{i,m} + V_{i,M}}{2} + \frac{V_{i,m} - V_{i,M}}{2} \cdot \text{sign}(\tilde{S}_i), \quad \text{for } i = 1, 2$$

locally force \(\tilde{S}_1 = 0\) and \(\tilde{S}_2 = 0\) in finite time, which, in turn yield \(z_1 = z_{1,d}\) and \(z_5 = z_{5,d}\) asymptotically.