Optimal Control of Impulsive Hybrid Systems*

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Abstract: We consider an optimal control problem for an impulsive hybrid dynamic system, where jumps of a trajectory may occur only at the moments of hitting a given closed set. A time reparameterization technique is applied to reduce the original problem to the one with bounded controls. We show that the reparameterized problem is equivalent to optimization in a class of generalized solutions to the impulsive hybrid system. Necessary conditions for optimality of generalized control processes are obtained by interpreting the Maximum Principle in the reduced problem.

Keywords: Optimal control, Maximum Principle.

1. INTRODUCTION

The term “hybrid systems” typically comprises a large variety of systems with discontinuous trajectories. The various mathematical models with jumps of trajectories and dynamics switchings naturally arise when studying demographic and epidemiologic processes (Jacquez (1985)), ecologic and economic problems (Berman and Plimmons (1979)) as well as in telecommunication (Foster (1989)), power engineering (Verriest (2003)), and problems in robotics (Arkin (1998), Brogliato (2000), Boccadoro et al. (2002), Sanfelice et al. (2006), and problems in robotics (Arkin (1998), Boccadoro et al. (2002), Sanfelice et al. (2006), Kurzhanski and Tychinin (2009)).

The wide range of applications has been motivated a vast amount of literature on modelling and studying qualitative properties of hybrid systems of various classes, see, for example, Branicky et al. (1998), Van der Schaft and Schumacher (2000), Matveev and Savkin (2000), Haddad et al. (2002), Sanfelice et al. (2006), and optimal control processes for hybrid systems (Boccadoro et al. (2005), Miller and Rubinovich (2005), Kurzhanski and Tychinin (2009)).

In Miller and Rubinovich (2005) a time reparameterization method has been suggested for optimal control of hybrid systems under unilateral constraints. Such models arise, for instance, in problems of control of impacts in mechanical systems (Brogliato (2000), Miller and Bentsman (2006)), where trajectories may have a discontinuity only on the boundary of a given forbidden domain. A jump of a trajectory is defined by a solution of a certain differential equation. The fast motions are assumed to be admissible only in the forbidden domain, so that they move up the system from a state on the boundary of this domain into another boundary point.

In our paper we also consider such discrete-continuous systems that jumps of a trajectory may happen just at the moments of its hitting a certain subset of an extended phase space. Following Branicky et al. (1998) we call such systems by impulsive hybrid ones. However, we assume that the fast motions are controlled, and a constraint on the total impulse of control is imposed. An optimization problem for this kind of systems can be treated as an impulsive optimal control problem under mixed constraints. In § 2 we give the problem statement and introduce a notion of a generalized solution of the impulsive hybrid system. In the third section, to investigate problems of this class, we suggest a method of reduction based on a discontinuous time reparameterization and obtain a series of results which are inspired by and conceptually close to Miller and Rubinovich (2005). By applying the time reparameterization technique we transform the original optimal control problem into a problem in the class of bounded controls under phase and functional constraints. A representation of a generalized solution is given in the form of a measure-driven differential equation. Subsequently, we succeed to establish the existence of a solution to the optimal control problem in the sense of generalized processes of an impulse hybrid system. Thereby, the reduction proposed yields an efficient way to the problem extension. The set of admissible trajectories is completed by means of its closure in the weak-* topology of the space of functions of bounded variations. The concluding section is devoted to necessary conditions of optimality in the class of generalized control processes. To derive the conditions we interpret the Maximum Principle in the reduced problem by means of the time change. Finally, we give an example illustrating the main result.

2. OPTIMAL CONTROL PROBLEM FOR A HYBRID SYSTEM

Consider a problem \((P)\) of minimization of a functional 
\[ I = F(x(T)) \]
under constraints
\[ \dot{x}(t) = f(t, x(t), V(t), u(t)), \quad \dot{V}(t) = 0, \]
\[ [x(\tau)] = \Psi(\tau, x(\tau^-), \nu_{\tau}), \quad [V(\tau)] = [\nu], \]

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are such that \((\tau, x(\tau), V(\tau)) \in \mathcal{Z}, \) \(^{(3)}\)
\[ x(0) = x_0, \quad V(0) = 0, \quad V(T) \leq M, \] \(^{(4)}\)
\[ u(t) \in U, \quad t \in [0, T]; \quad \nu_\tau \in W. \] \(^{(5)}\)
Here, \(x() \in BV([0, T], \mathbb{R}^n), \) \(V() \in BV([0, T], \mathbb{R})\) are right continuous functions of bounded variation, \([x(t)]\) stands for a jump of a trajectory \(x()\) at \(t,\) controls \(u()\) are Borel measurable bounded functions, \(\nu_\tau \in \mathbb{R}^m, |\nu_\tau| = \sum_{j=1}^m |\nu_{\tau_j}|,\) and \(M > 0.\) We assume that \(U \subset \mathbb{R}^r\) is a compact set, \(W\) is a closed convex set in the nonnegative orthonth \(\mathbb{R}^m_+\) such that \(0 \in W,\) functions \(f\) and \(\Psi\) are continuous with respect to all variables, Lipschitz continuous in \(x\) and \(V,\) and satisfy the linear growth condition in \(x\) and \(V,\) \(\Psi(t, x, 0) = 0\) for any \(t, x.\) We suppose also that system \((1), (2), (4), (5)\) is robust \(\) \((\text{the correctness assumption}),\) see, e.g., Miller and Rubinovich (2005). These assumptions ensure the existence and uniqueness of a solution to discrete continuous system \((1), (2), (4), (5).\) The set \(\mathcal{Z} \subset [0, T] \times \mathbb{R}^m_+ \times \mathbb{R}\) is supposed to be closed. The last inequality in \((4)\) is a bound on the total “resource” of dynamics switching, while the value \(V(t)\) indicates a current amount of this resource, which is spent up to the moment \(t.\)

A tuple
\[ \sigma = (x(), V(), u(), \{\nu_\tau\}_{\tau \in \tau}), \]
satisfying conditions \((1)-(5)\) is called an admissible control process for problem \((P).\) Here, \(\tau\) is a finite or countable sequence of instants \(\tau\) of dynamics switching \(\) \((i.e.\) those time moments when the system is subject to nonzero controls \(\nu_\tau).\)

Note that problem \((P)\) can be regarded as an optimization problem in the class of purely impulsive controls, where vectors \(\nu\) specify direction and intensity of impulse impacts. As is well known, a solution of the problem may not exist. This is implied by the fact that the set of trajectories corresponding to purely impulsive controls is not closed in the weak-* topology of the space of functions of bounded variation. It has to be enriched with the motions through the set \(\mathcal{Z},\) and differential equations with measures are proved to be a adequate way of a dynamics description for such systems.

**Definition 1.** A couple \((x(), V())\) of functions, which are defined on the interval \([0, T],\) right continuous and have bounded variation, is called a generalized solution to hybrid system \((1)-(5)\), if

\begin{enumerate}
  \item \((1)\) the inclusion
    \[ \text{supp}(dV) \subseteq \{t \in [0, T] : (t, x(t), V(t)) \in \mathcal{Z}\} \]
  holds, where \(dV\) is the measure induced by the function \(V,\) and \(\text{supp}(dV)\) stands for the support of the measure \(dV,\)
  \item \((2)\) there exists a sequence
    \[ \sigma^k = (x^k(), V^k(), u^k(), \{\nu^k_\tau\}_{\tau \in \tau}) \]
  of processes satisfying \((1)-(5),\) except possibly the condition \((3),\) and such that the sequence of trajectories \((x^k, V^k)\) converges to \((x(), V())\) in the weak-* topology of the space of functions of bounded variation.
\end{enumerate}

Note that the weak-* convergence of trajectories \((x^k, V^k)\) is equivalent to convergence of this sequence at all points of continuity of the limit function \((x, V)\) and at the instants \(t = 0, t = T.\)

In what follows we expound a constructive method for defining the desired extension by means of a reduction of problem \((P)\) based on the discontinuous time change. In impulsive control theory \(\) \((\text{see, for instance, Miller and Rubinovich (2005)})\) the reaction of the system to instantaneous impulsive effects is usually regarded as results of motions in a fast time scale. The main idea of the time reparameterization is to make such fast motions comparable in duration with motions in the natural time scale by extending the instants of impulses into intervals. If a hybrid system is robust, the possibility of application of such an approach is implied by the representation \((\text{Miller and Rubinovich (2005)}))\)
\[ \Psi(\tau, x(\tau), \nu) = x(\tau) + \kappa(1), \]
where \(\kappa(\theta)\) is a solution of a system
\[ \dot{\kappa}(\theta) = G(\tau, \kappa(\theta))\nu, \quad \kappa(0) = x(\tau^-), \]
and \(\text{on the interval } [0, 1], \) and \(G(t, x) = \Psi_q(t, x, \nu))_{\nu = 0.}\) System \((7)\) is called a limit one, it determines a path connecting in a fast time scale the left and right limits of a trajectory at a discontinuity point.

### 3. PROBLEM TRANSFORMATION

For further purposes we assume that the function \(f\) satisfies the Lipschitz and linear growth conditions in \(t\), and the function \(G\) given by \(G(t, x) = \Psi_q(t, x, \nu))_{\nu = 0.}\) satisfies the Lipschitz and linear growth conditions with respect to \(t\) and \(x,\) and the partial derivative \(G_x(t, x)\) is locally Lipschitz continuous in \(x.\) On an unifized time interval \([0, S], T \leq S \leq T + 2M,\) we consider an optimal control problem \((RP)\)
\[ J = F(z(S)) \to \inf, \]
\[ \dot{\xi} = \alpha, \quad \xi = (1 - \alpha)\beta e, \quad \dot{y} = (1 - \alpha)(1 - \beta)e, \]
\[ \dot{\zeta} = (1 - \alpha)\beta |e|, \quad \dot{\eta} = (1 - \alpha)(1 - \beta)|e|, \]
\[ \dot{y} = \alpha f(\xi, \zeta, \eta, \nu) + (1 - \alpha)\beta G(\xi, \nu) e, \]
\[ \zeta = \alpha f(\xi, \zeta, \eta, \nu) + (1 - \alpha)(1 - \beta)G(\xi, \nu) e, \]
\[ y(0) = z(0) = x_0, \quad \zeta(0) = \eta(0) = 0, \quad \zeta(0) = \pi(0) = 0, \quad \xi(0) = 0, \]
\[ \xi(S) = T, \quad \zeta \leq \eta, \]
\[ v \in U, \quad e \in \text{cone}(W) \cap B, \quad \alpha, \beta \in [0, 1], \]
\[ J_1 = \int_0^S (1 - \alpha)(1 - \beta)e|Q(\xi, \nu, \zeta)| ds = 0, \]
\[ J_2 = \int_0^S \alpha \rho(\pi - \zeta) ds = 0. \]

Here, trajectories \(\xi, y, z, \zeta, \eta, \zeta,\) and \(\pi\) are absolutely continuous, controls \(v,\) \(\alpha, \beta,\) and \(e\) are Borel measurable bounded functions, the sets \(U\) and \(W\) are the same as in \((5),\) \(\text{cone}(W)\) stands for the conical hull of the set \(W,\) \(B\) is the unit ball in \(\mathbb{R}^m\) centered at zero, \(\rho(\mu)\)
and $Q(t, x, V)$ are nonnegative continuous scalar functions vanishing only at zero and on the set $Z$, respectively. Note that a function $Q$ with the required properties does always exist, furthermore, there is an infinitely smooth function characterizing (in the referred respect) any closed subset of a finite dimensional space.

By $\sigma = (\gamma(.), \omega(.); S)$ we denote an admissible control process of problem (RP), where $\gamma(.), \omega = (\xi, y, z, \xi, \eta, s, \pi)$, and $\omega(.), \omega = (x, \alpha, \beta, e)$, satisfy (8)–(17).

**Theorem 1.** Let $\sigma = (x(.), V(.), u(.), \nu, \nu \in \mathcal{T})$ be an admissible process in problem (P). Then in problem (RP) there exists an admissible control $\omega(\cdot)$ defined on the interval $[0, S]$, where $S = T + 2 \sum_{\tau \in T} |\nu|$, and such that the corresponding solution $\gamma(\cdot)$ of system (8)–(17) is related with $(x(\cdot), V(\cdot))$ by:

$$x(t) = y(\Gamma(t)) = z(\Gamma(t)), \quad V(t) = \zeta(\Gamma(t)) = \eta(\Gamma(t)), \quad (18)$$

$$\Gamma(t) = \inf \{s \in [0, S] : \xi(s) > t\}, \quad t \in [0, T], \quad \Gamma(T) = S. \quad (19)$$

Our proof for the most part follows arguments for the similar result (Miller and Rubinovich (2005)) on the reduction for discrete-continuous systems. We just adduce explicitly the form of the sought control $\omega(\cdot)$. Suppose that a control $\nu$ is applied at a moment $t \in \mathcal{T}$, when a trajectory reaches the set $Z$. We define the sequence $s_\tau = t + 2 \sum_{\theta \in \mathcal{T}, \theta < t} |\nu|$

The time reparameterization consists in stretching of each point $\tau$ into the interval $I_\tau = s_\tau + [0, 2|\nu|]$. The interval $I_\tau$ is split into two intervals of equal length: $I_{\tau}^0 = s_\tau + [0, |\nu|]$, and $I_{\tau}^1 = s_\tau + [|\nu|, 2|\nu|]$. Denote $I = \bigcup_{\tau \in \mathcal{T}} I_{\tau}$, and $I' = \bigcup_{\tau \in \mathcal{T}} I_{\tau}^1$, where $i = 0, 1$. Then the control $\omega(\cdot)$ takes the form:

$$v(s) \begin{cases} \in U, & s \in I, \\ u(\xi(s)), & s \in Z, \\ \text{otherwise}. \end{cases}$$

$$\alpha(s) = \begin{cases} 0, & s \in I, \\ 1, & s \in \mathcal{T}, \\ \text{otherwise}, \end{cases}$$

$$\beta(s) = \begin{cases} 0, & s \in I_\tau^0, \\ 1, & s \in I_\tau^1, \\ \text{otherwise}, \end{cases}$$

$$\epsilon(s) = \begin{cases} \nu / |\nu|, & s \in I_{\tau}, \tau \in \mathcal{T}, \\ \text{otherwise}. \end{cases}$$

Graphs of the corresponding trajectories $y$ and $z$ are obtained by completing the graph of the discontinuous trajectory $y$ in the way presented in Fig. 1. On the first subinterval $I_{\tau}^0$ the trajectory $y$ is defined by inserting the graph of a trajectory of limit system (7), and $y$ keeps the constant value $y(s) = y(s_\tau)$. Thus, while $z$ displays the fast dynamics, the auxiliary trajectory $y$ is introduced to give a signal that $x$ hits the resetting set $Z$. On the second subinterval $z$ and $y$ are swapped around such that both the trajectories coincide since the instant $s_\tau + 2|\nu|$

Like in Miller and Rubinovich (2005), we obtain the result on the reduction in terms of generalized solutions to hybrid system (1)–(5):

**Theorem 2.** For any generalized solution $(x(\cdot), V(\cdot))$ of hybrid system (1)–(5), there exists an admissible in (RP) control $\omega(\cdot)$ defined on an interval $[0, S], T \leq S \leq T + 2M$, and such that the corresponding solution $\gamma(\cdot)$ of system $Q(\cdot)$ is related with $(x(\cdot), V(\cdot))$ by formulas (18), (19).

Now, we give a representation of a generalized solution to the hybrid system by means of a differential equation with a measure.

**Theorem 3.** Assume that, for an admissible process $\sigma = (\gamma(.), \omega(.); S)$ in problem (RP), the functions $x(\cdot), V(\cdot)$ are defined by the relations $x(t) = z(\Gamma(t)), \quad V(t) = \eta(\Gamma(t))$, where $t \in [0, T]$, and $\Gamma$ is given in (19). Then there exist:

1) a Borel measurable bounded control $u(\cdot)$ subject to (5);
2) a regular cone($W$)–valued vector measure $d\mu$ induced by a right continuous function $\mu(\cdot) \in BV([0, T], \mathbb{R}^m)$, $\mu(0) = 0$, such that:

$$\int_0^T |d\mu(\theta)| \leq M, \quad (20)$$

$$\text{supp}\{d\mu\} \subseteq \{t \in [0, T] : (t, x(t), V(t)) \in Z\}, \quad (21)$$

and, for all $t \in [0, T]$, the functions $x(\cdot), V(\cdot)$ satisfy the integral equalities:

$$x(t) = x_0 + \int_0^t f(\theta, x(\theta), V(\theta), u(\theta))d\theta +$$

$$+ \int_0^t G(\theta, x(\theta))d\mu(\theta) + \sum_{\kappa(\tau) < x(\tau)}(\kappa(\tau) - x(\tau)), \quad (22)$$

$$V(t) = \int_0^t |d\mu(\theta)|. \quad (23)$$

Here, $d\mu_{ac}, d\mu_{sc},$ and $d\mu_{dis}$ are absolutely continuous, singular continuous, and discrete components of the measure $d\mu$, respectively, and $d\mu = d\mu_{ac} + d\mu_{sc} + |d\mu|$ is a measure induced by the total variation of the function $\mu$. For each $t$ the function $\kappa(\tau)$ is a solution to limit system (7) with $\nu = |\mu(\tau)|$, and the sum is taken over all $\tau \in \text{supp}\{d\mu\}$ such that $\tau \leq t$. 

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The proof is based on a change of variable under the sign of Lebesgue-Stieltjes integral and uses the properties of the time reparameterization. Note that the function $\Gamma$ is right continuous, strictly monotone increasing on $[0,T]$, and the relation $\Gamma(\xi(s)) = t$ holds for all $t \in [0,T]$ as well as $\Gamma(\xi(s)) = s$, if $\xi = \xi(s)$ is the point of continuity of $\Gamma$. Given an admissible control $\omega(\cdot)$ in problem (RP), we determine a control $u(\cdot)$ in the original problem as follows: $u(t) = v(\Gamma(t)) \mathcal{L}$-a.e. (almost everywhere with respect to the Lebesgue measure) on $[0,T]$. The desired control measure $du$ is defined by its distribution function

$$
\mu(t) = \int_0^t (1 - \alpha(s))(1 - \beta(s)) e(s) ds. \quad (24)
$$

If we apply the substitution $s = \Gamma(t)$ in (8)–(11) and take into account the relations $x(t) = z(\Gamma(t))$, $V(t) = \eta(\Gamma(t))$, we obtain that $x$ and $V$ satisfy (22), (23) with the measure $du$. In view of the constraint $S \leq T + 2M$ it is easy to check estimation (20). From conditions (14) and (16) it follows the constraint (21) on the support of the measure $du$.

Notice that, for any generalized solution $(x(\cdot), V(\cdot))$ and the corresponding trajectory $\gamma(\cdot)$ of the reduced system, we have also the relations $x(t) = y(\Gamma(t))$, $V(t) = \zeta(\Gamma(t))$, $t \in [0,T]$.

Denote $\tilde{\sigma} = (x(\cdot), V(\cdot), u(\cdot), du)$. According to Theorems 2 and 3 we call a tuple $\tilde{\sigma}$ by a generalized control process in problem (P).

Now we can state the existence of an optimal generalized control process in the original problem.

**Theorem 4.** Suppose that the set \{$(f(t,x,V,u))_{u \in U}$\} is convex for all $(t,x,V)$. Then, in problem (RP) there exists an optimal process $\tilde{\sigma}^* = (\gamma^*(\cdot), \omega^*(\cdot), S^*)$, and in (P) there is an optimal generalized process $\tilde{\sigma}^* = (x^*(\cdot), V^*(\cdot), u^*(\cdot), du^*)$ satisfying (5), (20)–(23). Moreover, the processes $\sigma^*$ and $\tilde{\sigma}^*$ are related by (18), (19), and

$$
I(\tilde{\sigma}^*) = \inf_{\tilde{\sigma} \in \tilde{D}} I = \inf_{\sigma \in D} J(\sigma^*),
$$

where $\tilde{D}$ stands for the set of all generalized processes of problem (P), and $\tilde{D}$ is the set of admissible processes in (RP).

4. NECESSARY CONDITIONS OF OPTIMALITY

In the Section we obtain necessary conditions for optimality in the class of generalized control processes of problem (P). The conditions will take the form of a Maximum Principle for impulsive controls and will be presented as a result of an interpretation of the Maximum Principle [Ioffe and Tikhomirov (1979)] for problem (RP). Note that such an interpretation becomes possible because we have furnished ourselves with the time reparameterization technique (see Theorems 2–4).

Consider again problem (RP). It contains phase and terminal constraints (14) as well as functional ones (16), (17). Moreover, the problem is considered on an unfixed time interval. However, since the velocity set of the reduced system contains zero, then, any admissible process $\tilde{\sigma}$ of (RP) can be extended to the maximal interval $[0, \tilde{S}]$, $\tilde{S} = T + 2M$, by putting $\alpha(s) = 0$ and $e(s) = 0$ $\mathcal{L}$-a.e. on $[S, \tilde{S}]$. Thus, (RP) can be reduced to a problem on the fixed time interval $[0, \tilde{S}]$. Solutions of such an extended problem ($\tilde{R}P$) and (RP) are related in an apparent way. An optimal time duration in ($\tilde{R}P$) is given by $S = T + \tilde{\eta}(\tilde{S})$, where $\eta$ (i.e., $\tilde{\eta}$ extended) is a component of an optimal trajectory in problem ($\tilde{R}P$). To restore the respective optimal control in (RP) one should redefine the extended control by left shifting over the intervals, where $\alpha(s) = 0$ and $e(s) = 0$ $\mathcal{L}$-a.e. In what follows we will deal with problem ($\tilde{R}P$) and drop the tilde sign.

Assume that the set \{$(f(t,x,V,u))_{u \in U}$\} is convex for all $(t,x,V)$, the function $Q$ is continuous in all variables together with its partial derivatives $Q_t$, $Q_x$, and $Q_V$. For the sake of simplicity we take $\rho(\mu) = (\mu, \mu)$. We introduce the following constructions:

- the terminal Lagrange function $L(\gamma(\tilde{S})) = \lambda_0 F(z(\tilde{S})) + \lambda_1 (\xi(\tilde{S}) - T)$,
- the Pontryagin function

$$
h(\gamma, \psi, \omega) = ah^1 + (1 - \alpha) \{\beta h^2 + (1 - \beta)h^3\}, \quad (25)
$$

where

$$
h^1(\gamma, \psi, \omega) = \psi \psi - \lambda_2 \rho(\pi - \xi),
$$

$$
h^2(\gamma, \psi, \omega) = (\xi)^2 \psi^2 + \psi^2 = \lambda_3 \rho(\pi - \xi),
$$

$$
h^3(\gamma, \psi, \omega) = (\xi)^2 \psi^2 + \psi^2 e + \psi^2 \psi = \lambda_4 \rho(\pi - \xi).
$$

Here, $\psi = (\psi^2, \psi^2, \psi^2, \psi^2, \psi^2, \psi^2)$ is a dual phase vector, and $\lambda_i$, $i = 0, 1, 2, 3$, are scalars.

Now we formulate the necessary conditions of optimality in problem (RP) (Ioffe and Tikhomirov (1979)).

**Theorem 5.** Let an admissible process $\tilde{\sigma}^* = (\gamma^*(\cdot), \omega^*(\cdot))$ be optimal in problem (RP). Then, there exists a tuple of Lagrange multipliers $(\lambda, da, \rho(\cdot))$ such that $\lambda = (\lambda_0, \ldots, \lambda_3)$, $da$ is a regular measure induced by a function $a(\cdot) \in BV([0, \tilde{S}], R)$, $a(0-) = 0$, $\rho(\cdot)$ is a vector function with the mentioned components defined on $[0, \til{S}]$, and the conditions (C1)–(C5) hold. The conditions are as follows:

(C1) Nonnegativity and nontriviality

$$
\lambda_0 \geq 0, \quad da \geq 0, \quad |\lambda| + a(S) > 0. \quad (26)
$$

(C2) Complementary slackness condition

$$
supp\{da\} \subseteq \{s : \zeta^*(s) = \eta^*(s)\}. \quad (27)
$$

(C3) Each component $\psi^*(\cdot)$, $q = \xi, y, z, s, \pi$, of the vector function $\psi(\cdot)$ is absolutely continuous and satisfies $\mathcal{L}$-a.e. on $[0, \til{S}]$ the adjoint equation

$$
\til{\psi}^q = -h_q(\gamma^*, \psi^*, \omega^*), \quad (28)
$$

where $h_q = \frac{\partial h}{\partial q}$. The components $\psi^q(\cdot)$ and $\psi^q(\cdot)$ are right continuous functions of bounded variation satisfying $da$-a.e. (almost everywhere with respect to the measure $da$) and $\mathcal{L}$-a.e. on $[0, \til{S}]$ the adjoint differential equations with measures

$$
d\psi^q = -h_q(\gamma^*, \psi^*, \omega^*) ds + da, \quad (29)
$$

$$
d\psi^q = -h_q(\gamma^*, \psi^*, \omega^*) ds - da. \quad (30)
$$
(C\textsubscript{4}) Transversality condition

\[
\psi(\tilde{S}) = -L_\gamma(\gamma^*(\tilde{S})).
\]

(\text{C5}) Maximum condition

\[
h(\gamma^*, \psi, \omega^*) = \max_{\omega \in \mathcal{U}} h(\gamma^*, \psi, \omega),
\]

\(\mathcal{L}\)-a.e. on \([0, \tilde{S}],\) where \(U = U \times [0,1] \times [0, 1] \times (\text{cone}(W) \cap B).\)

In this theorem a measure plays the role of a Lagrange multiplier associated with the phase constraint \(\zeta \leq \eta.\)

By a simple calculation one can obtain that the Hamilton function \(h\) of the reduced problem takes the form

\[
h(\gamma, \psi) = \max \left\{ \tilde{h}_1, \tilde{h}_2, \tilde{h}_3 \right\}.
\]

Therefore, \(\tilde{h}_1(\gamma, \psi) = \max h(\gamma, \psi, \omega),\) \(i = 1, 2, 3,\) and the maximum is taken over \(\omega \in U.\) The structure of extremal controls \(a\) is given by the following explicit formula

\[
\tilde{a}(\gamma, \psi) = \begin{cases} 
1, & g(\gamma, \psi) > 0, \\
0, & g(\gamma, \psi) < 0, \\
\in [0,1], & g(\gamma, \psi) = 0,
\end{cases}
\]

where \(g = \tilde{h}_1 - \max \left\{ \tilde{h}_2, \tilde{h}_3 \right\}.\)

In problem (\(P\)) we introduce the Pontryagin functions

\[
H^1(t, x, V, u, \phi, l) = (\phi^1 + \phi^2 f(t, x, V, u)) + \phi^3,
\]

\[
H^2(t, x, \phi, l) = \phi^1 G(t, x) + \phi^2 V^1,\]

\[
H^3(t, x, V, l) = (\phi^2 G(t, x) + \phi^2 V^2 - \lambda_2 Q(t, x, V),
\]

\[
H^0(t, x, V, \phi, l) = \max \{H^2(t, x, \phi, l), H^3(t, x, V, l)\},
\]

and the Hamiltonians

\[
H^1(t, x, V, \phi, u) = \max_{u \in U} H^1(t, x, V, u, \phi, l),
\]

\[
H^0(t, x, V, \phi) = \max_{l \in \text{cone}(W) \cap B} H^0(t, x, V, \phi, l),
\]

\[
H(t, x, V, \phi) = \max \{H^1(t, x, V, \phi, l), H^0(t, x, V, \phi)\}.
\]

Here, \(\lambda_0, \lambda_1, \lambda_2, \) are scalars, \(\phi^i, \phi^1, \phi^2, \phi^1, \phi^2, \phi^2, \phi^2) = \phi(x, \phi^1, \phi^2, \phi^1, \phi^2),\) where \(\phi^i \in \mathbb{R}\) and \(\phi^V \in \mathbb{R}.\)

The following result provides necessary conditions of optimality in the class of generalized processes of the original problem.

**Theorem 6.** Suppose that \(\tilde{\sigma}^* = (x^*(\cdot), V^*(\cdot), u^*(\cdot), \mu^*(\cdot))\) is an optimal generalized control process in problem (\(P\)). Then there exist Lagrange multipliers \((\lambda, dw, \phi(\cdot)),\) such that \(\lambda = (\lambda_0, \lambda_1, \lambda_2), dw\) is a scalar nonnegative regular measure, \(\phi(\cdot)\) is a right continuous function of bounded variation on \([0, T],\) and \(\tilde{\sigma}^*\) satisfies the conditions (\text{C}\textsubscript{1})–(\text{C}\textsubscript{4}): (\text{C}\textsubscript{1}) Nonnegativity and non-triviality conditions

\[
\lambda_0 \geq 0, \quad dw \geq 0, \quad |\lambda| + w(T) > 0.
\]

(\text{C}\textsubscript{2}) The vector function \(\phi(\cdot)\) is a solution to the adjoint system of differential equations with measures

\[
\begin{align*}
\phi^x &= -H_1^2 dt - (H_2^2 + H_3^2) dV^*, \\
\phi^{x^1} &= (\lambda_2 Q_x - H_2^1) dV^*, \\
\phi^{x^2} &= -H_1^2 dt - (\lambda_2 Q_x + H_2^1) dV^*, \\
\phi^{V^1} &= -H_1^2 dV^* + dw, \\
\phi^{V^2} &= -H_1^2 dt - dw,
\end{align*}
\]

where the derivatives of the Pontryagin functions are computed along \(\tilde{\sigma}^*\) with \(l^*(\cdot) = \mu^*/|\mu^*|\) in mind. Here, \(\mu^* / |\mu^*|\) stands for the Radon–Nikodym derivative of a measure \(\mu^*\) with respect to the measure \(|\mu^*|\) (the density of \(\mu^*\)).

(\text{C}\textsubscript{3}) Transversality conditions

\[
\phi(0) = -\lambda_1, \quad \phi^{V^1}(0) = \phi^{V^2}(T) = 0,
\]

\[
\phi^{V^1}(T) = 0, \quad \phi^{V^2}(T) = -\lambda_0 F_2(x^*(T)).
\]

(\text{C}\textsubscript{4}) Maximum conditions

\[
\mathcal{H}^{i}, i = 0, 1, \quad \text{L.a.e.} \cup \mathcal{H}^{i*} - \text{a.e. \{0, 1\}},
\]

\[
\mathcal{H} \leq \mathcal{H}^{i}, \quad \lambda_0 \cup \text{a.e. & } \mathcal{H}^{i*} - \text{a.e. \{0, 1\}},
\]

\[
\mathcal{H} \leq \mathcal{H}^{i}, \quad \lambda_0 \cup \text{a.e. & } \mathcal{H}^{i*} - \text{a.e. \{0, 1\}},
\]

along \(x^*(\cdot), V^*(\cdot), \phi(\cdot), u^*(\cdot),\) and \(l^*(\cdot).\) Furthermore, for each \(\tau \in \text{supp}\{\mu^*_0\} \) and \(\vartheta \in [0, 1]\) the inequality

\[
\mathcal{H} \leq \mathcal{H}^{0}
\]

holds along \((\tau, \kappa^{x^*}(\cdot), \kappa^{V^*}(\cdot), \vartheta).\) Here, the function \(\kappa^{x^*}(\cdot)\) is a solution on \(\{0, 1\}\) to the limit system

\[
\dot{\kappa}^x(\vartheta) = G(\tau, \kappa^{x^*}(\vartheta)) |\mu^*(\tau)|, \quad \kappa^x(0) = x^*(\tau) - 1,
\]

\(\kappa^{V^*}(\cdot)\) is defined by

\[
\kappa^{V^*}(\vartheta) = V^*(\tau) - \vartheta |V^*(\tau)|, \quad \vartheta \in [0, 1],
\]

and the function \(p(\cdot, \cdot, p(\cdot), l(\cdot))\) is a solution on \([0, 1]\) to the adjoint limit system

\[
\begin{align*}
p^i &= -H_2^2 - H_3^2, \\
p^{x^1} &= (\lambda_2 Q_x - H_2^1) |V^*(\tau)|, \\
p^{x^2} &= -\lambda_2 Q_x + H_2^1 |V^*(\tau)|, \\
p^{V^1} &= -H_2^1 + |w(\tau)|, \\
p^{V^2} &= -|w(\tau)|,
\end{align*}
\]

with the initial condition

\[
p(1) = \phi(\tau).
\]

In (43) the derivatives of the Pontryagin functions are computed at \((\tau, \kappa^{x^*}(\vartheta), \kappa^{V^*}(\vartheta), p(\vartheta), l(\vartheta))\) with \(l(\vartheta) = |\mu^*(\tau)| |\mu^*(\tau)|, \quad \vartheta \in [0, 1].\)

In this list there is no any complementary slackness condition corresponding to (27). It disappears following the translation of Theorem 5 into terms of generalized processes. Indeed, given a Lagrange multiplier \(da\) in the reduced problem one can define the function

\[
W(t) = \int_0^t da(s).
\]

This function induces the measure \(dw\) from the formulation of Theorem 6. Since the measure \(dw\) is supported on the set \(\{s : \zeta(s) = 0\}\) and in view of relations (18) one can observe that there are no additional constraints on \(\text{supp}\{dw\}\).

Given a functional Lagrange multiplier \(\psi(\cdot)\) from Theorem 5 we determine a function \(\phi(\cdot)\) by the relation \(\phi(t) = \psi(\Gamma(t)), \quad t \in [0, T].\) Here, the component \(\phi^x\) corresponds to \(\psi^x,\) while \(\phi^{x^2}\) is obtained from \(\psi^y\). The other components are related similarly. Note that the function \(\phi(\cdot)\) can serve as the sought Lagrange multiplier in the necessary conditions of optimality formulated in Theorem 6.
Conditions (38)–(40) display the choice between two types of dynamics. Indeed, if $H^1 > H^0$, then we proceed with the absolutely continuous dynamics, while, in the case $H^1 < H^0$, the singular dynamics takes effect. If $H^1 = H^0$, then there is no priority between these two modes of the system behavior. Condition (40) convinces us of definite advantages of applying impulses only at the moments $\tau \in \text{supp}(d\nu^*)$. It is easy to see that all of these situations are related to the structure (33) of an extremal control $\alpha$ in the reduced problem.

4.1 Example

We consider the following simple example:

Maximize $x(1)$ under the constraints

$$\dot{x} = u, \ |u| \leq 1, \ x(0-) = 1, \ (45)$$

$$x(\tau) = \nu_{\tau}, \ \text{if} \ x(\tau-) = 0, \ (46)$$

$$\nu_{\tau} \geq 0, \ \sum_{\tau \leq t} \nu_{\tau} \leq M. \ (47)$$

Notice that the reachable set of system (45)–(47) is the interval $[0, \max\{2, M\}]$. It is easy to see that in the case $M > 2$, an optimal control is $u^*(t) \equiv -1$, and $d
\nu^* = M \delta(t-1)$. If $M < 2$, then $u^*(t) \equiv 1$ and no impulses occur as the condition $x = 0$ never holds. So, we have the diametrically opposite conventional control strategies depending on the total resource of impulsive control.

To find a solution we may use the result of Theorem 6 in the following way. Since $x(t) \geq 0$ on $[0, 1]$, we may take $Q(x) = x$. The arguments of maxima of Pontryagin functions and the Hamiltonians take the form:

$$\widehat{u}(\phi) = \text{sign}(\phi x^1 + \phi x^2), \ \widehat{H}^0(\phi) = \arg\max_{\phi \in [0, 1]} H^0,$$

$$H^1(\phi) = |\phi x^1 + \phi x^2| + \phi^l,$$

$$H^0(x, \phi) = \max\{\phi x^1 + \phi V^1 + |\phi x^2| + \phi V^2 - \lambda_2 x^2\}.$$  

By analyzing possible alternatives one can find appropriate Lagrange multipliers

$$\lambda_0 = 1, \ \lambda_1 = -2, \ \lambda_2 = 1, \ dw(t) = V(1) \delta(t-1),$$

and the corresponding solutions to adjoint system (35):

$$\phi^1(t) = 2, \ \phi^2(t) = V(t) - V(1), \ \phi^2(t) = 1, \ \phi^2(t) = -V(1), \ \text{and} \ \nu^2(t) = V(1), \ t \in [0, 1].$$

Here, $V(t) = \sum_{\tau \leq t} \nu_{\tau}.$

Hence, an extremal control takes the form

$$\tilde{u}(t) = \tilde{u}(\phi(t)) = \text{sign}\{1 + V(t) - V(1)\}, \ t \in [0, 1].$$

Suppose that $V(1) > 1$. We initialize the system motion with $\tilde{u}(0) = -1$ and keep this control strategy while $V(t) < V(1)-1$. The corresponding trajectory $x(t) = 1 - t$ does not hit the resetting set $x = 0$ on $[0, 1]$. Hence $V(t) = 0, \ t \in [0, 1]$, and, as a consequence, $\tilde{u} \equiv -1$. For $t \in [0, 1]$ we have

$$1 + V(1) = H^1 \geq H^0 = V(1),$$

and conditions (38) and (39) are met. Further, consider the solutions to the limit systems $\kappa(\theta) = V(1)\theta, \ p^1(\theta) = 2,$

$$p^1(\theta) = (\theta - 1)V(1), \ \ p^2(\theta) = 1, \ \ p^2(\theta) = -V(1), \ \text{and} \ \ p^2(\theta) = V(1), \ \theta \in [0, 1].$$

One has

$$H^1(\kappa(\theta)) \leq H^0(\kappa(\theta), p(\theta))$$

for all $\theta \in [0, 1]$, and condition (40) holds.

In a similar way we find that $\tilde{u}(t) = 1$, and $x(t) = 1$ on $[0, 1]$ for $V(1) < 1$.

Finally, we obtain a solution to the problem by putting $V(1) = M$.

REFERENCES


