The Method of Characteristics for the Lyapunov Function Design in the Finite-Time Stability Analysis of Sliding Mode Controlled Systems with Uncertainties

Alex Poznyak

Automatic Control Department, CINVESTAV-IPN, Mexico D.F.
(e-mail: apoznyak@ctrl.cinvestav.mx)

Abstract: This presentation deals with designing of Lyapunov functions for the analysis of a finite-time stabilization of the class of nonlinear systems containing the, so-called, second order type sliding-mode controllers. The design of the corresponding Lyapunov functions is based on the solution of a specific Partial Differential Equation (PDE) which can be resolved by the application of "Method of Characteristics" transforming the considered PDE into a system of Ordinary Differential Equations (ODE). The collection of the first integrals of this ODE-system permits to find an analytical expression for the desired Lyapunov Function, in particular, for "twisting", "super-twisting" and "nested" (imbedded) controllers. This paper presents the survey of the works fulfilled in CINVESTAV-IPN during three last years.

1. INTRODUCTION

The method of Sliding Mode Control (SMC) actually is one of the most powerful tools for the stabilization of complex nonlinear systems in the presence of uncertainties or external disturbances Utkin [1992], Emelyanov & Korovin [1997], Edwards & Spergeon [1998] and Utkin, Guldner & Shi [2009]. This method is based on the maintaining of desired properties of a nonlinear system by the application of discontinuous control, making high-frequency switching and realizing the, so-called, Sliding Mode Regime. It provides the complete suppression of external "matched" perturbations (acting in the same subspace as a designed control) and significantly decreases undesired effect of "unmatched" perturbations (uncertainties) of a bounded amplitude. One of the principle disadvantages of SMC, appearing in its practical application, is related with the, so-called, chattering effect Utkin, Guldner & Shi [2009].

High-frequency oscillations take place during the sliding mode regime. This effect may provoke undesired behavior of a controlled system and even its complete destruction. Overcoming this undesired effects constitute one of the main research direction in SMC theory Bartolini et al. [2000], Bartolini et al. [2003] and Boiko & Fridman [2005]. Actually, there exist two effective approach providing a partial suppressing of high-frequency chattering oscillations Levant [1993, 2007], Bartolini et al. [2003] which do not permit any extensions for the state dimensions more than three. Here we survey the corresponding results from Polyakov & Poznyak [2009], Polyakov & Poznyak [2009], which are based on some analytical (non-geometrical) constructions and discuss their possible extensions and perspectives.

2. ON DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDE

2.1 Solution in the Filippov’s sense

Consider a vector model given by

\[ \dot{x} = g(t, x, u(t, x)), \quad t \in T \]  

(1)

where

- \( x \in \mathbb{R}^n \) state-vector at time \( t \geq 0 \) (\( T \) may be unbounded interval),
- \( u(t, x) = (u_1(t, x), u_2(t, x), ..., u_m(t, x))^T \) control-vector function defined in \( \mathbb{R}^{n+m+1} \) and may be discontinuous,
- \( g(t, x, u) \) is continuous in \( \mathbb{R}^{n+m+1} \).
Following to Filippov [1988] denote by $U_i(t; x)$ the set of all limit points of $u_i(\tau, y)$ when $(\tau, y) \to (t, x)$ and $K_g(t; x) = \mathbb{O} \{ g(t, x, u) : u = (u_1, \ldots, u_m)^T, u_i \in U_i(t; x) \}$ a convex closed set.

Definition 1. An absolutely continuous vector-function $x(t)$ defined for all $t \in T$ for which almost everywhere the differential inclusion

$$\dot{x} \in K_g(t; x) \quad (2)$$

holds is referred to as a solution of the differential equation (1) in the Filippov’s sense.

2.2 Global attractivity or "dichotomy"

Definition 2. The invariant set $D$ of the differential inclusion (2) is called a set of all trajectory points never leaving this set, i.e.,

$$\forall x(0) = x_0 \in D \implies x(t) \in D \forall t \geq 0$$

The invariant set $D$ is said to be globally attractive (or, dichotomic) for the differential inclusion (2) if

$$d(x(t), D) := \inf_{z \in D} \| x(t) - z \| \to 0$$

whenever $t \to \infty \forall x_0 : \| x_0 \| < \infty$

A globally attractive set $D$ we will call a reachable in a finite-time if for any $x_0 : \| x_0 \| < \infty$ there exists a finite time $t_{reach} = t_{reach}(x_0)$ such that

$$x(t) \in D \forall t \geq t_{reach}(x_0)$$

2.3 The Lyapunov-type theorem on a global finite-time convergence

Theorem 1. Let for a closed convex set $D \subset \mathbb{R}^n$ there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

1) $V(x) = 0$ if $x \in D$ and $V(x) > 0$ if $x \in \mathbb{R}^n \setminus D$;
2) $V(x)$ is a local Lipschitz in an neighborhood of $x \in \mathbb{R}^n$;
3) for any solution $x(t)$, $t > 0$ of the differential inclusion (1) with an arbitrary initial condition $x_0 : \| x_0 \| < \infty$ the function $V(x(t))$ fulfills for almost all $t > 0$ and $k > 0, \rho \in [0, 1)$ the differential inequality

$$\dot{V}(x(t)) \leq -kV^\rho(x(t)) \quad (3)$$

Then the set $D$ is globally attractive with a finite reaching time estimated as

$$t_{reach}(x_0) \leq [k(1 - \rho)]^{-1} V^{1-\rho}(x_0) \quad (4)$$

Proof. Since a solution of the differential equation (1) is an absolutely continuous function $x(t)$, then under the assumptions of this theorem the function $\dot{V}(x(t)) := V(x(t))$ is also absolutely continuous. Then (3) follows

$$d\dot{V}(t) / \dot{V}^\rho(t) \leq -k dt \implies 0 \leq (1 - \rho)^{-1} \dot{V}^{1-\rho}(t) \leq -k t + (1 - \rho)^{-1} \dot{V}^{1-\rho}(0)$$

so that $V(x(t))$ does not increase and reach the value $V(x(t)) = 0$ in finite time $t_{reach}(x_0)$ estimated from above by (4). Then statement of the theorem results from a non-local extension of the solution.

Corollary 2. Let $0 \in K_g(t; x)$ and the conditions of Theorem 1 are fulfilled for $D = \{ 0 \}$. Then the differential equation (1) has the trivial solution $x(t) \equiv 0$ which is globally attractive and a reachable in a finite-time.

3. HIGH-ORDER SLIDING MODES

Below in this paper we will to design a control action $u(t, x)$ which stabilizes in the origin the scalar "output" $\sigma = \sigma(t, x(t)) \in \mathbb{R}$ (defined on the solutions of (2)) in a finite-time.

3.1 The general definition

Definition 3. If

- the smooth output $\sigma(t, x)$ has the full-time derivatives $\sigma(t, x), \dot{\sigma}(t, x), \ldots, \sigma^{(r-1)}(t, x)$ which are continuous on $x$;
- for the system (1) the set

$$\Sigma := \{ x \in \mathbb{R}^n : \sigma(t, x) = \sigma(t, x) = \ldots = \sigma^{(r-1)}(t, x) = 0 \}$$

is globally attractive and a reachable in a finite-time, then the movements of this system on the set $\Sigma$ is called the $r$-order sliding mode regime and the set $\Sigma$ is referred to as the $r$-order sliding manifold.

3.2 Second-order sliding modes for affine controlled systems

Below we will consider the system (1) of the relative degree 2 and the affine scalar control $u$ which is a function of $\sigma$ and $\dot{\sigma}$, i.e.,

$$\ddot{\sigma} = a(t, x) + b(t, x) u(\sigma, \dot{\sigma}) \quad (5)$$

Main assumption (MA): the functions $a(t, x)$ and $b(t, x)$ are assumed to be unknown but bounded

$$|a(t, x)| \leq C, 0 < b_{\min} \leq b(t, x) \leq b_{\max} \quad (6)$$

Then any solution of (5) satisfies the differential inclusion

$$\ddot{\sigma} \in [-C, C] + [b_{\min}, b_{\max}] u(\sigma, \dot{\sigma}) \quad (7)$$

which is autonomous (stationary) depending only on $u$ and, therefore, all definitions above concerning the global convergence in a finite-time can be directly applied to this differential inclusion.

4. LYAPUNOV FUNCTIONS DESIGNING FOR SYSTEMS WITH UNCERTAINTIES

4.1 An affine system with uncertainties

Consider the following affine controlled system

$$\begin{cases}
\dot{x} = g(x, y) \\
\dot{y} = a(x, y) + b(t, x, y) u(x, y) + \xi(t, x, y)
\end{cases} \quad (8)$$

where
- $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$, $a(x, y)$ is a known vector-function,
- $u(t, x) \in \mathbb{R}^m$ is a given (may be, discontinuous) control and
- $b(t, x, y)$ and $\xi(t, x, y)$ are bounded uncertainties satisfying

$$b(t, x, y) = \{ b_{i,j}(t, x, y) \}_{i,j=1}^{k,m}$$

$$0 \leq b_{\min} ^{i,j} \leq b_{i,j}(t, x, y) \leq b_{\max} ^{i,j}$$

$$|\xi_i(t, x, y)| \leq C_i, \quad i = 1, 2, \ldots, k$$

769
4.2 An upper estimate of the Lyapunov function full-time derivative

If a function $V = V(x, y)$ is differentiable on its variables then along the trajectories of (8) we have

$$
\dot{V} = \sum_{s=1}^{n} g_s(x, y) \frac{\partial V}{\partial x_s} + \sum_{s=1}^{k} \left( a_s(x, y) + \sum_{j=1}^{m} b_{s,j}(t, x, y) u_j(x, y) + \xi_i(t, x, y) \right) \frac{\partial V}{\partial y_i} 
$$

$$
\leq \sum_{s=1}^{n} g_s(x, y) \frac{\partial V}{\partial x_s} + \sum_{s=1}^{k} h_i(x, y, \mu_{i1}, ..., \mu_{im}, \gamma_i) \frac{\partial V}{\partial y_i}
$$

with

$$
h_i(x, y, \mu_{i1}, ..., \mu_{im}, \gamma_i) := a_i(x, y) + \sum_{j=1}^{m} \mu_{ij} u_j(x, y) + \gamma_i,
$$

$$
\gamma_i = C_i \text{sign} \left( \frac{\partial V}{\partial y_i} \right) \in \{-C_i, C_i\}
$$

\[ \mu_{ij} = \frac{1}{2} \max \left( -\text{sign} \left( \frac{\partial V}{\partial y_i} u_i(x, y) \right), \frac{\partial V}{\partial y_i} \right) \]

\[ \mu_{ij} = \frac{1}{2} \max \left( +\text{sign} \left( \frac{\partial V}{\partial y_i} u_i(x, y) \right), \frac{\partial V}{\partial y_i} \right) \]

where

$$
\text{sign}[\rho] := \begin{cases} 
1 & \text{if } \rho > 0 \\
-1 & \text{if } \rho < 0 \\
0 & \text{if } \rho = 0
\end{cases}
$$

So, instead of (8) we may consider the mixed "ODE-Differential Inclusion"

\[ \dot{x} = g(x, y) \]

\[ \dot{y} \in a(x, y) + [b_{ij}^{\min}, b_{ij}^{\max}] u + [-C_i, C_i] \]

where

\[ b_{ij}^{\min} = \{b_{ij}^{\min}\}, \ b_{ij}^{\max} = \{b_{ij}^{\max}\}, \ C = (C_1, ..., C_k)^T \]

4.3 The method of characteristics

Theorem 3. Any function $V = V(x, y)$ satisfying the following system of the characteristic ODE

\[ \frac{dx_1}{g_1(x, y)} = ... = \frac{dx_n}{g_n(x, y)} = \frac{dy_1}{h_1(x, y, \mu_{i1}, ..., \mu_{im}, \gamma_1)} = ... = \frac{dy_k}{h_k(x, y, \mu_{k1}, ..., \mu_{km}, \gamma_k)} = \frac{dV}{-qV^p}, \ q > 0, \ \rho \in [0, 1] \]

automatically fulfills

\[ \dot{V} \leq \sum_{s=1}^{n} g_s(x, y) \frac{\partial V}{\partial x_s} + \sum_{s=1}^{k} h_j(x, y, \mu_{j1}, ..., \mu_{jm}, \gamma_j) \frac{\partial V}{\partial y_j} = -qV^p \]

Proof. Following Poznyak [2008] (see Lemma 19.13), the relations (10) imply

\[ dx_i = \frac{g_i(x, y)}{-qV^p} dV, \ i = 1, ..., n \]

\[ dy_j = \frac{h_j(x, y, \mu_{j1}, ..., \mu_{jm}, \gamma_j)}{-qV^p} dV, \ j = 1, ..., k \]

So, for $V = V(x, y)$ we have

\[ dV = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} dx_i + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} dy_j = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} (-qV^p) dV + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} (-qV^p) dV \]

\[ \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} g_i(x, y) dV + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} h_j(x, y, \mu_{j1}, ..., \mu_{jm}, \gamma_j) dV = \]

which leads to the right-hand side equality in (11).

Claim 1. If we are able to resolve the characteristic ODE-system (10) and find its first integrals

\[ \varphi_r(V, x, y | \mu, \gamma, q, p) = \text{const} := c_r, \ r = 1, 2, ..., n + k, \mu = \{\mu_{ij}\}, \gamma = \{\gamma_1, ..., \gamma_k\} \]

then the function $V(x, y | \mu, \gamma, q, p)$ can be found as the solution of the following algebraic equation

\[ \Phi(\varphi_1(V, x, y, \mu, \gamma, q, p), ..., \varphi_{n+k}(V, x, y, \mu, \gamma, q, p)) = 0 \]

(13)

where $\Phi$ is any function (since any function of constant is a constant). The parameters $q, p$ for each $\mu, \gamma$ should be selected in such a way that

1) $V(x, y | \mu, \gamma, q, p)$ is continuous (in fact, absolutely continuous for any fixed parameters $\mu, \gamma, q, p$),

2) and

\[ V(0, 0 | \mu, \gamma, q, p) = 0 \]

\[ V(x, y | \mu, \gamma, q, p) > 0 \text{ if } \|x\| + \|y\| > 0 \]

5. THE LYAPUNOV FUNCTION CANDIDATE FOR THE SECOND-ORDER SLIDING MODE

5.1 The general form

Denote $x = \sigma$ and $y = \dot{\sigma}$. Then (8) can be represented as

\[ \dot{x} = y \]

\[ \dot{y} \in a(x, y) + [b_{ij}^{\min}, b_{ij}^{\max}] u(x, y) \]

(14)

where $u(x, y)$ is a partially constant function taking a finite number of values (relay control). Then for $\rho = \frac{1}{2}$ the partial differential equation (11) becomes

\[ \frac{\partial V}{\partial x} + \gamma \frac{\partial V}{\partial y} = -qV^p \]

with

\[ \gamma = \gamma + \mu u, \ \mu = C \text{sign} \left( \frac{\partial V}{\partial y} u \right) \]

\[ \mu = \frac{1}{2} b_{ij}^{\min} \left( 1 - \text{sign} \left( \frac{\partial V}{\partial y} u \right) \right) + \frac{1}{2} b_{ij}^{\max} \left( 1 + \text{sign} \left( \frac{\partial V}{\partial y} u \right) \right) \]

The system (10) of the characteristic equations (using the relay property of $u$) is

\[ \frac{dx}{y} = \frac{dy}{\dot{y}} = \frac{dV}{-qV^p} \]

and two first integrals are as follows

770
\[ \varphi_1(x, y) = x - \frac{y^2}{2\gamma} \]
\[ \varphi_2(V, y) = -\frac{y}{\gamma} - \frac{2\sqrt{V}}{q} \]

Selecting then
\[ \Phi(\varphi_1, \varphi_2) = \varphi_2 + p\sqrt{\varphi_1} = 0 \]

we get
\[ \frac{2\sqrt{V}}{q} = -\frac{y}{\gamma} + p\sqrt{\left| x - \frac{y^2}{2\gamma} \right|} \geq 0 \]

which finally leads to

\[ V(x, y) = \frac{q^2}{4} \left( p\sqrt{\left| x - \frac{y^2}{2\gamma} \right|} - \frac{y}{\gamma} \right)^2 \]  \hspace{1cm} (15)

where the functional parameter \( p \) should fulfill the inequality

\[ p\sqrt{\left| x - \frac{y^2}{2\gamma} \right|} = \frac{y}{\gamma} (u(x, y)) \geq 0 \] \hspace{1cm} (16)

for all admisible \( x \) and \( y \). Remember that here \( u = u(x, y) \) is a relay control.

5.2 The Lyapunov function candidate for the "twisting controller"

For the "twisting controller" (Levant [1993])
\[ u = -r_1 \text{sign}(x) - r_2 \text{sign}(y), \quad r_1 > 0, \quad r_2 > 0 \] \hspace{1cm} (17)

Taking into account the continuity property:
\[ V_{tw}(x, y) \rightarrow \left( p\sqrt{\frac{2}{\gamma}} - \text{sign}(y)/\gamma \right)^2 q^2 y^2/4 \quad \text{for} \quad x \rightarrow \pm 0 \]
\[ V_{tw}(x, y) \rightarrow q^2 p^2 |x|/4 \quad \text{for} \quad y \rightarrow \pm 0 \]

we obtain that the parameters \( p \) and \( q \) have to satisfy the relations
\[ q^2 \left( p\sqrt{\frac{2}{\gamma}} - \text{sign}(y)/\gamma \right)^2 = k^2, \quad q^2 p^2 = 1 \]

for some \( k > 0 \) which fulfills, for example, for
\[ p := \frac{\sqrt{2/\gamma} \text{sign}(xy)}{\sqrt{2/\gamma}|k|}, \quad q := \frac{1}{|p|} \] \hspace{1cm} (18)

Lemma 1. The condition (16) for \( p \) is fulfilled if
\[ r_2 + C/b_{\text{min}} < r_1 < \left( (b_{\text{min}} + b_{\text{max}}) r_2 - 2 C \right) / (b_{\text{max}} - b_{\text{min}}) \] \hspace{1cm} (19)

and
\[ \left( 2(b_{\text{min}} (r_1 + r_2) - C) \right)^{-1/2} < k < \left( 2(b_{\text{max}} (r_1 - r_2) + C) \right)^{-1/2} \] \hspace{1cm} (20)

Notice that the condition \( r_2 + C/b_{\text{min}} < r_1 \) in (19) coincides with the analogous condition in Levant [1993]. The Lyapunov function for the "twisting controller" with
\[ b_{\text{min}} = 3/4, \quad b_{\text{max}} = 1, \quad r_1 = 2, \quad r_2 = 1, \quad C = 1/2 \]

is (see Fig.1-Fig.3)
\[ V_{tw}(x, y) = \begin{cases} q^2 \left( p\sqrt{\left| xy^2/(2\gamma) - y/\gamma \right|} \right)^2/4 & \text{for} \quad xy \neq 0 \\ k^2 y^2/4 & \text{for} \quad x = 0 \\ |x|/4 & \text{for} \quad y = 0 \end{cases} \] \hspace{1cm} (21)

5.3 The Lyapunov function candidate for the "Nested (embedded) controller"

Let the relay control in the system (8) is designed as
\[ u(x, y) = -\alpha \text{sign}(y + \beta \sqrt{|x|} \text{sign}(x)) \] \hspace{1cm} (22)

where \( x, y \in \mathbb{R} \) - scalars and \( \alpha, \beta \) - positive constants. For the corresponding closed system we have
\[
\dot{x} = y
\]
\[
\dot{y} \in [-C, C] - \alpha [b_{\text{min}}, b_{\text{max}}] \text{sign} \left( y + \beta \sqrt{|x| \text{sign}(x)} \right)
\]  

(23)

The attractive set \(D\) includes the origin \(\{0,0\}\) as well as the curve

\[
z(x, y) := y + \beta \sqrt{|x| |\text{sign}(x)|} = 0
\]

The Lyapunov function (15) becomes

\[
V_{\text{nest}}(x, y) = \begin{cases} 
\frac{q^2}{4} \left( p \sqrt{\frac{x^2 - 2(\gamma + \mu u)}{y}} - \frac{y}{\gamma + \mu u} \right)^2 & \text{for } yz(x, y) \neq 0 \\
\frac{k^2 y^2 / 4}{|x| / 4} & \text{for } z(x, y) = 0 \\
\end{cases}
\]

where

\[
\gamma := C \text{sign}(z(x, y)), \mu := b_{\text{min}}, \quad q := 1/|p|
\]

\[
p := \frac{\text{sign}(y)/(\gamma + \mu u)}{\sqrt{1/\beta^2 + 0.5 \text{sign}(y)/(\gamma + \mu u)}}
\]

and

\[
\sqrt{\frac{0.5}{|\gamma + \mu u|}} < \beta^2 < \sqrt{\frac{0.5}{|\gamma + \mu u|} + \frac{1}{\beta^2 \cdot 2|\gamma + \mu u|}}
\]

Theorem 4. If

\[
4(b_{\text{min}} - C)/\sqrt{3} > \beta^2 > 2(b_{\text{min}} - C)
\]

then \(V_{\text{nest}}(x, y)\) possesses the following properties:

- \(V_{\text{nest}}(x, y)\) is positive definite and locally Lipschitzian in \(\mathbb{R}^2\), and, moreover, continuously differentiable if \(yz(x, y) \neq 0\);
- along the trajectories of the closed-loop system (23) it satisfies

\[
V_{\text{nest}} \leq \min \{0.5k_2^2, q_{\text{min}}\} \sqrt{V_{\text{nest}}}
\]

(24)

Next figures (see Fig.4 and Fig.5) illustrate this function for \(b_{\text{min}} = 4/5, b_{\text{max}} = 1, \alpha = 1, \beta = 5/4\) and \(C = 10^{-1}\).

Consider the system (see Levant [2007])

\[
\begin{cases}
\dot{x} = -\alpha \sqrt{|x| \text{sign}(x)} + y \\
\dot{y} \in [-L, L] - \beta \text{sign}(x)
\end{cases}
\]

(25)

where \(x, y \in \mathbb{R}\) - scalars and \(\alpha, \beta, C\) - positive constants. In fact, the stability analysis such systems can not be analyzed directly using the Lyapunov function (15) since (25) contains (non directly) in \(x\)-dynamics the term \(\beta \int_0^r \text{sign}(x(t)) \, dt\). But this analysis may be done repeating the same scheme suggested above. For this system the PDE-equation is

\[
\begin{aligned}
\frac{\partial V}{\partial x} &= -kV \\
\frac{\partial V}{\partial y} &= -\gamma V
\end{aligned}
\]

(26)

and the characteristic equations are

\[
\frac{dx}{V} = \frac{dy}{-\gamma V} = \frac{dV}{-kV}
\]

(27)

The first integrals are

\[
\begin{align*}
\varphi_1(x, y) &= \frac{\ln |s(x, y)|}{2} + m(x, y) \\
\varphi_2(y, V) &= \frac{y \text{sign}(x)}{\gamma} - \frac{2\sqrt{V}}{k}
\end{align*}
\]

(28)

with

\[
\begin{align*}
s(x, y) &= 2\gamma |x| - \alpha \sqrt{|x| \text{sign}(x)y + y^2} \\
m(x, y) &= \frac{1}{\sqrt{y - 1}} \arctan \left( \frac{\alpha \sqrt{|x| \text{sign}(x) - 2y}}{2\sqrt{y - 1}y} \right)
\end{align*}
\]

(29)

The function \(V\) can be found from the equations

\[
\begin{align*}
\Phi(\varphi_1, \varphi_2) &= k_0 e^{\varphi_1} + \varphi_2 = 0, \quad k_0 \in R \\
\frac{2V}{k} &= \frac{y \text{sign}(x)}{\gamma} + k_0 e^{m(x, y)} \sqrt{s(x, y)}
\end{align*}
\]

(30)

The last relation leads to

\[
V_{\text{sstw}}(x, y) = \begin{cases} 
\frac{k_0^2}{4} \left( \frac{\text{sign}(x)}{\gamma} + k_0 \sqrt{s(x, y)e^{m(x, y)}} \right)^2 & \text{for } xy \neq 0 \\
2k_0^2 y^2 / \alpha^2 & \text{for } x = 0 \\
|x| / 2 & \text{for } y = 0
\end{cases}
\]

772
with some constraints to $k_0$ and $k$ which are fulfilled if
\[ \beta > 5L, \quad 32L < \alpha^2 < 8(\beta - L) \]
This conditions looks stronger that ones in Levant [2007]. The Lyapunov function for the "super-twisting controller" for $\alpha = \beta = 1$ and $L = 0.2$ is shown at Fig.6-Fig.7.

![Fig. 6. $V_{stw}(x, y)$](image)

![Fig. 7. Level lines for $V_{stw}(x, y)$](image)

7. CONCLUSIONS

This paper suggests the universal approach for the designing of the Lyapunov function for the finite-time stability analysis of the second-order sliding mode control systems such as "twisting","nested" and "super-twisting". The desired Lyapunov function satisfied the special PDE which solution may be found by the method of characteristics.

REFERENCES


773