Coherent Quantum $H^\infty$ Control via a Strict Bounded Real Quantum Controller

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Abstract: We present a systematic method to synthesize a coherent quantum robust $H^\infty$ controller for a class of linear complex quantum stochastic systems, which is defined only in terms of annihilation operators, with norm-bounded structured uncertainties. This controller is required to be stable and strict bounded real in order to be physically realizable. The main idea of our approach is to introduce an additional uncertainty to form an artificial uncertain quantum system used to design the desired quantum controller. The $H^\infty$ control objective is to achieve a strict bounded real closed loop quantum system with a specified disturbance attenuation level. The solution to this control problem is given in terms of stabilizing solutions to the parameterized complex Riccati equations.

Keywords: coherent quantum control, robust $H^\infty$, strict bounded real.

1. INTRODUCTION

Coherent quantum feedback control is a class of quantum control techniques where the controller itself is a quantum system; e.g., see James et al. (2008); Nurdin et al. (2009); Maalouf and Petersen (2009); Harno and Petersen (2010). Importantly, a coherent quantum feedback controller has to be physically realizable; e.g., see Petersen (2009a); Shaiju and Petersen (2009). In this paper, we consider coherent quantum robust $H^\infty$ control for a class of linear complex quantum stochastic systems with norm-bounded structured uncertainties. The dynamics of an uncertain quantum system in this class is determined only by annihilation operators and described in terms of linear quantum stochastic differential equations (QSDEs); see Maalouf and Petersen (2009). Robustness against perturbations is also a central issue in quantum control studies; e.g., see D’Helen and James (2006). The aim of applying a coherent quantum robust $H^\infty$ controller is thus to achieve a strict bounded real closed loop quantum system with a specified disturbance attenuation level. It is possible to solve this quantum control problem based on the quantum $H^\infty$ control methods presented in James et al. (2008) and Maalouf and Petersen (2009). However, their methods do not always yield a stable and strict bounded real quantum $H^\infty$ controller, which may not be physically realizable.

We thus propose a new systematic method to construct a stable and strict bounded real coherent quantum $H^\infty$ controller, which is guaranteed to be physically realizable. The main idea of our approach is to introduce an additional uncertainty to form an artificial uncertain quantum system used to design the desired quantum controller. The $H^\infty$ control objective is to achieve a strict bounded real closed loop quantum system with a specified disturbance attenuation level. The solution to this control problem is given in terms of stabilizing solutions to the parameterized complex Riccati equations.
if $M$ is an operator matrix, then $M^*$ denotes the operation of taking the adjoint of each entry of $M$.

2. PROBLEM STATEMENT

2.1 Linear complex uncertain quantum system

We consider a class of linear complex quantum stochastic systems with structured uncertainties: (see James et al. (2008); Nurdin et al. (2009); Maalouf and Petersen (2009))

\[
da(t) = F_a(t) dt + G_a dv(t) + G_1 d\xi_1(t) + \sum_{j=1}^k G_{3,j} d\xi_j(t);
\]

\[
dz(t) = H_1 a(t) dt + J_{12} du(t);
\]

\[
d\xi_1(t) = L_1 a(t) dt + M_1 du(t);
\]

\[
d\xi_k(t) = L_k a(t) dt + M_k du(t);
\]

\[
dy(t) = H_2 a(t) dt + J_{20} dv(t) + J_{21} du(t)\]  \hspace{1cm} (1)

where $a(0) = \alpha_0$; $\alpha$ is an $n \times 1$ vector of the plant annihilator operators; $\nu$ is an $n_{\nu} \times 1$ vector of quantum noises; $w$ is an $n_w \times 1$ vector of disturbance inputs; $u$ is an $n_u \times 1$ vector of control inputs; $\xi_j$ is an $n_{\xi_j} \times 1$ vector of uncertainty inputs; $\zeta_j$ is an $n_{\zeta_j} \times 1$ vector of uncertainty outputs; $\zeta$ is an $n_{\zeta} \times 1$ vector of controlled outputs; and $y$ is an $n_y \times 1$ vector of 'measurement' outputs. All the coefficient matrices in (1) are complex matrices, which have compatible dimensions with those of the operators and signals in (1). Quantum systems of this form, defined only in terms of annihilation operators, can be used to represent interconnections of linear passive optical components such as optical cavities, beam-splitters and phase-shifters; e.g., see Maalouf and Petersen (2009); Petersen (2009a).

The disturbance input $w(t)$ and the control input $u(t)$ in (1) are represented respectively as

\[
da(t) = \beta_w(t) dt + dw(t); \hspace{1cm} (2)
\]

\[
du(t) = \beta_u(t) dt + du(t)\]  \hspace{1cm} (3)

where $\beta_w(t)$ and $\beta_u(t)$ are adapted processes and $dw(t)$ and $du(t)$ are the noise parts of (2) and (3). Meanwhile, $dt(t)$ represents an additional quantum noise in the plant. The quantum noises $dv(t)$, $dz(t)$ and $du(t)$ have corresponding Ito matrices $F_v$, $F_z$ and commutation matrices $C_v$, $C_z$ and $C_u$. Here, we assume that the Ito matrices are $F_v = F_z = F_u = I$ and the commutation matrices are $C_v = C_z = C_u = I$.

The $j$-th structured uncertainty in (1) is modeled as an additional unknown linear time-invariant complex quantum stochastic system:

\[
\dot{a}_j(t) = A_j \dot{a}_j(t) dt + B_j d\xi_j(t); \hspace{1cm} \dot{a}_j(0) = \hat{a}_j(0);
\]

\[
d\xi_j(t) = C_j \dot{a}_j(t) dt + D_j d\xi_j(t)\]  \hspace{1cm} (4)

with $A_j$ Hurwitz and transfer function matrix

\[
\Delta_j(s) = C_j(sI-A_j)^{-1}B_j + D_j\]  \hspace{1cm} (5)

which is required to satisfy

\[
\|\Delta_j(s)\|_\infty \leq 1, \hspace{1cm} \forall j = 1,2,\ldots,k;\]  \hspace{1cm} (6)

see also James et al. (2008).

2.2 Coherent quantum controller

We aim to control the uncertain quantum system (1), (4), (5), (6) using a dynamic coherent quantum $H^\infty$ controller, which is assumed to be a non-commutative stochastic quantum system. A general form of this controller can be written as

\[
der(t) = F_c e(t) dt + G_{c0} dw_{c0}(t) + G_{c1} dw_{c1}(t) + G_c dy(t); \hspace{1cm} (7)
\]

\[
du(t) = H_c e(t) dt + dw_{c}(t)\]

where $e(0) = c_0$; $c$ is an $n \times 1$ vector of the controller annihilator operators; $w_{c0}$ and $w_{c1}$ are respectively $n_{c0} \times 1$ and $n_{c1} \times 1$ vectors of non-commutative quantum Wiener processes with the Ito matrices $F_{w_{c0}} = F_{w_{c1}} = I$ and the commutation matrices $C_{w_{c0}} = C_{w_{c1}} = I$. At time $t = 0$, it is assumed that $a(0)$ and $\hat{a}(0)$ commute with $c(0)$. The quantum $H^\infty$ controller (7) is required to be stable and strictly bounded real, which implies that it is physically realizable. Referring to see Maalouf and Petersen (2009), we define a physical realizability condition for the quantum $H^\infty$ controller (7) in terms of its bounded real property.

Definition 1. (see Definition 7.1 in Maalouf and Petersen (2009)) The matrices $F_c$, $G_c$ and $H_c$ are said to define a physically realizable controller of the form (7) if there exist matrices $G_{c0}$, $G_{c1}$, $H_{c0}$ and $H_{c1}$ such that

\[
der(t) = F_c e(t) dt + G_{c0} dw_{c0}(t) + G_{c1} dw_{c1}(t) + G_c dy(t);\]

\[
du(t) = H_{c0} e(t) dt + dw_{c0}(t) + dw_{c1}(t) + dy(t)\]  \hspace{1cm} (8)

with $c(0) = c_0$, is physically realizable when

\[
C_{c0} := J_{20} C_v J_{20} + J_{21} C_v J_{21} = I.\]  \hspace{1cm} (9)

Lemma 1. (see Theorem 7.2 in Maalouf and Petersen (2009)) Suppose that the matrices $F_c$, $G_c$ and $H_c$ in (7) are such that (7) is a minimal realization. Then, the coherent quantum controller (7) is physically realizable if and only if $F_c$ is Hurwitz and

\[
\|H_c(sI-F_c)^{-1}G_c\|_\infty \leq 1.\]  \hspace{1cm} (10)

Thus, the quantum controller (7) is bounded real.

Remark 1. The matrices $G_{c1}$ and $H_{c1}$ can be set to zero as the exogenous quantum noise $dw_{c1}$ term is not needed in the realization of the quantum controller (7). A consequence of Lemma 1 is that a strict bounded real quantum $H^\infty$ controller of the form (7) must always be physically realizable with $F_c$ Hurwitz and $\|H_c(sI-F_c)^{-1}G_c\|_\infty < 1$. Also, the (strict) bounded real and physical realizability conditions of linear complex quantum systems can be found in Maalouf and Petersen (2009).

2.3 $H^\infty$ control objective

Interconnecting the quantum controller (7) with the uncertain quantum system (1), (4), (5), (6), we obtain a closed loop uncertain quantum system that satisfies the following $H^\infty$ control objective:

\[
\int_0^t \langle \langle z(s)^1 z(s) + z(s)^T z(s)^* + \varepsilon (\eta(s)^1 \eta(s) + \eta(s)^T \eta(s)^* \rangle \rangle ds \leq (\gamma^2 - \varepsilon^2) \times \int_0^t \langle \beta_{w_{c0}}(s)^1 \beta_{w_{c0}}(s) + \beta_{w_{c1}}(s)^T \beta_{w_{c1}}(s)^* \rangle ds + \pi_1 + \pi_2 t\]  \hspace{1cm} (11)

where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants and

\[
der(t) = [da(t)^T \quad dc(t)^T \quad du(t)^T]^T.\]  \hspace{1cm} (12)

This objective is attained if the closed loop uncertain quantum system is strict bounded real with a specified disturbance attenuation level $\gamma > 0$.  

11351
3. MAIN RESULTS

An algorithm to construct a coherent quantum controller of the form (7), which leads to the satisfaction of the $H^\infty$ control objective (11), has been provided in James et al. (2008) and Maalouf and Petersen (2009). However, their algorithms do not always yield a stable and strictly bounded real coherent quantum $H^\infty$ controller, which may not be physically realizable. Thus, we are motivated to provide a new systematic method to synthesize a stable and strictly bounded real quantum $H^\infty$ controller based on the approach in Petersen (2009b). In this case, we force the quantum $H^\infty$ controller to be physically realizable.

The main idea of our approach is to introduce an additional uncertainty to form of an artificial uncertain quantum system used to design a physically realizable coherent quantum $H^\infty$ controller. This approach only provides a sufficient condition such that any suitable quantum controller of the form (7) will lead to a strictly bounded real closed loop uncertain quantum system with disturbance attenuation $\gamma > 0$ when applied to the original uncertain quantum system (1), (4), (5), (6). Moreover, the same quantum controller must be stable and strictly bounded real when applied to a particular open loop uncertain quantum system, while achieving the closed loop $H^\infty$ control objective. These properties hold even when the quantum controller is disconnected from the open loop uncertain quantum system; see Petersen (2009b).

In order to apply this idea, we first consider the following uncertain quantum system:

$$
d\alpha(t) = F\alpha(t)dt + G_0d\xi(t) + G_1d\omega(t) + \sum_{j=1}^{k} G_{2,j}d\xi_j(t);
$$

$$
d\xi(t) = H_1\alpha(t)dt + J_1d\omega(t) + \sum_{j=1}^{k} J_{2,j}d\xi_j(t);
$$

$$
d\zeta_1(t) = H_{2,1}\alpha(t)dt + K_1d\omega(t) + \sum_{j=1}^{k} L_{1,j}d\xi_j(t); \ \vdots \\
$$

$$
d\zeta_k(t) = H_{2,k}\alpha(t)dt + K_kd\omega(t) + \sum_{j=1}^{k} L_{1,j}d\xi_j(t). \quad (13)
$$

with $a(0) = a_0$. Here, the $j$-th structured uncertainty in (13) is modeled as an unknown quantum system:

$$
\begin{align*}
da_j(t) &= A_j\alpha_j(t)dt + B_jd\zeta_j(t); \quad \alpha_j(0) = \alpha_{0,j}; \\
d\xi_j(t) &= C_j\alpha_j(t)dt + D_jd\zeta_j(t)
\end{align*} \quad (14)
$$

with $A_j$ Hurwitz and transfer function matrix

$$
\Delta_j(s) = C_j(sI - A_j)^{-1}B_j + D_j
$$

which is required to satisfy

$$
\|\Delta_j(s)\|_{\infty} \leq 1. \quad (15)
$$

for all $j = 1, 2, \ldots, k$. We now present the following lemma required in subsequent sub-sections.

**Lemma 2.** Consider the uncertain quantum system (13), (14), (15), (16) and let $\tau_1 > 0, \tau_2, \ldots, \tau_k > 0$ be given constants. Suppose that $F$ in (13) is Hurwitz and the scaled quantum system:

$$
\begin{align*}
d\hat{\alpha}(t) &= F\hat{\alpha}(t)dt + G_0d\xi(t) + \hat{G}_1d\hat{\omega}(t) \quad (17) \\
d\hat{\xi}(t) &= B\hat{\alpha}(t)dt + J\hat{\omega}(t)
\end{align*}
$$

where

$$
\hat{G}_1 = \left[ \begin{array}{c} \gamma^{-1}G_1 \sqrt{\tau_1}^{-1}G_{2,1} \sqrt{\tau_2}^{-1}G_{2,2} \cdots \sqrt{\tau_k}^{-1}G_{2,k} \end{array} \right];
$$

$$
J = \left[ \begin{array}{c} \gamma^{-1}J_1 \sqrt{\tau_1}^{-1}J_{2,1} \sqrt{\tau_2}^{-1}J_{2,2} \cdots \sqrt{\tau_k}^{-1}J_{2,k} \end{array} \right];
$$

Then, there exists a matrix $K$ such that the uncertain quantum system (20), (4), (5), (6) is strictly bounded real with disturbance attenuation $\gamma > 0$. The uncertain quantum system (13), (14), (15), (16) is strictly bounded real with disturbance attenuation $\gamma > 0$. The uncertain quantum system (13), (14), (15), (16) is strictly bounded real with disturbance attenuation $\gamma > 0$.

3.1 Artificial uncertain quantum system

Prior to defining an artificial uncertain quantum system based on the original uncertain quantum system (1), (4), (5), (6), we need to construct a matrix $K$ such that $(F + G_2K)$ is Hurwitz and the following uncertain quantum system:

$$
d\alpha(t) = (F + G_2K)\alpha(t)dt + G_0d\xi(t) + G_1d\omega(t) + \sum_{j=1}^{k} G_{2,j}d\xi_j(t);
$$

$$
d\xi(t) = (H_1 + J_2K)\alpha(t)dt; \quad d\zeta_1(t) = (L_1 + M_1K)\alpha(t)dt; \quad \vdots \quad d\zeta_k(t) = (L_k + M_kK)\alpha(t)dt; \quad (20)
$$

with $a(0) = a_0$ and (4), (5), (6) is strictly bounded real with disturbance attenuation $\gamma > 0$. This requirement is satisfied under a condition which is dependent on the existence of a solution to a parameterized complex Riccati equation defined as follows: Let $\kappa_1 > 0, \kappa_2 > 0$ be given constants and consider a complex Riccati equation

$$
(F - G_2E_1^{-1}J_{12}H_1)X + X(F - G_2E_1^{-1}J_{12}H_1)^* + X(G_1G_1^* - G_2E_1^{-1}G_2^*)X + H_1^*(I - J_{12}E_1^{-1}J_{12}H_1)H_1 = 0 \quad (21)
$$

where

$$
G_1 = \left[ \begin{array}{c} \gamma^{-1}G_1 \sqrt{\tau_1}^{-1}G_{3,1} \cdots \sqrt{\tau_k}^{-1}G_{3,k} \end{array} \right];
$$

$$
H_1 = \left[ \begin{array}{c} H_1 \sqrt{\tau_1}^{-1}L_{1,1} \cdots \sqrt{\tau_k}^{-1}L_{1,k} \end{array} \right]; \quad J_{12} = \left[ \begin{array}{c} J_{12} \sqrt{\tau_1}^{-1}M_{1,1} \cdots \sqrt{\tau_k}^{-1}M_{1,k} \end{array} \right]; \quad E_1 = J_{12}^*J_{12}. \quad (22)
$$

**Assumption 1.** Given constants $\kappa_1 > 0, \kappa_2 > 0$, the uncertain quantum system (1), (4), (5), (6) is assumed to be such that $E_1 > 0$.

**Lemma 3.** Let $\kappa_1 > 0, \kappa_2 > 0$ be given constants. Suppose that the uncertain quantum system (1), (4), (5), (6) is such that Assumption 1 is satisfied and the complex Riccati equation (21) has a stabilizing solution $X \geq 0$. Then, there exists a matrix $K$ such that the uncertain quantum system (20), (4), (5), (6) is strictly bounded real
with disturbance attenuation $\gamma > 0$. That is, $(F + G_2K)$ is Hurwitz and \[
\| (H_1 + J_12K)(sI - (F + G_2K))^{-1}G_1 \|_\infty < \gamma \tag{23}
\]
where \[
K = -E^{-1}_c (G_2^r X + J_2^r H_1). \tag{24}
\]

Using the matrix $K$ as described in (24) and introducing additional uncertainty input $d_{\xi}(t)$ and uncertainty output $d_{\xi+1}(t)$, we form an artificial uncertain quantum system as follows: (see Petersen (2009b))

\[
da(t) = \dot{F} a(t) dt + G_0 dv(t) + \dot{G}_1 dv(t) + \dot{G}_2 du(t) + \sum_{j=1}^{k+1} G_{3,j} d_{j}(t); \tag{25}
\]
\[
d\bar{z}(t) = \dot{H}_1 a(t) dt + \dot{J}_1 du(t) + N_0 d_{\xi+1}(t); \tag{26}
\]
\[
d_{\xi}(t) = \dot{L}_1 a(t) dt + \dot{M}_1 du(t) + N_1 d_{\xi+1}(t); \tag{27}
\]
\[
d_{\xi+1}(t) = \dot{L}_{k+1} a(t) dt + \dot{M}_{k+1} du(t) + P dv(t); \tag{28}
\]
\[
dy(t) = \dot{H}_2 a(t) dt + J_20 dv(t) + \dot{J}_{21} du(t) + N_{k+1} d_{\xi+k+1}(t). \tag{29}
\]

where $a(0) = a_0$; $\dot{d}_{\xi}(t) = \dot{\beta}_0(t) dt + \dot{d}v(t)$;
\[
d\bar{z}(t) = \frac{dw_1(t)}{dw_2(t)}; \quad d\bar{z}(t) = \frac{d\bar{z}_1(t)}{d\bar{z}_2(t)}; \quad \ddot{\bar{z}} = \ddot{\bar{z}}_1 + \ddot{\bar{z}}_2 = \ddot{\beta}_0(t) dt + \ddot{d}v(t);
\]
\[
\bar{G}_1 = \begin{bmatrix} G_1 & 0 \end{bmatrix}; \quad \bar{G}_2 = \begin{bmatrix} G_2 & {G}_3 \end{bmatrix}; \quad \bar{G}_3 = \begin{bmatrix} G_2 & 0 & 0 \end{bmatrix} R^{-1};
\]
\[
\bar{H}_1 = \begin{bmatrix} H_1 & 0 \end{bmatrix}; \quad \bar{J}_1 = \begin{bmatrix} J_1 & 0 \end{bmatrix} \gamma I; \quad N_0 = \begin{bmatrix} 0 & -I & 0 \end{bmatrix} R^{-1};
\]
\[
\bar{L}_1 = L_1 + \begin{bmatrix} M_1 & 0 \end{bmatrix} K; \quad \bar{M}_1 = \begin{bmatrix} M_1 & 0 \end{bmatrix} R^{-1};
\]
\[
\bar{L}_k = L_k + \begin{bmatrix} M_k & 0 \end{bmatrix} K; \quad \bar{M}_k = \begin{bmatrix} M_k & 0 \end{bmatrix} R^{-1};
\]
\[
\bar{L}_{k+1} = \frac{1}{2} R \begin{bmatrix} K & 0 \\ H_1 & 0 \end{bmatrix}; \quad \bar{M}_{k+1} = \frac{1}{2} R \begin{bmatrix} J_2 & 0 \\ 0 & I \end{bmatrix}; \quad P = \frac{1}{2} R \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} R^{-1};
\]
\[
\bar{H}_2 = \frac{1}{2} H_2; \quad \bar{J}_2 = \frac{1}{2} \begin{bmatrix} J_2 & I \end{bmatrix}; \quad N_{k+1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} R^{-1}. \tag{26}
\]

Note that $R$ is any $n_r \times n_r$ non-singular scalar matrix, where $n_r = 2n_u + n_u + n_d$; $w_0$ and $z_2$ have the same dimensions as those of $y$ and $u$, respectively.

In (25), the uncertainty input $d_{\xi}(t)$ is related to the uncertainty output $d_{\xi}(t)$ according to (4) for $j = 1, 2, \ldots, k$. Also, the additional uncertainty input $d_{\xi+1}(t)$ is related to the additional uncertainty output $d_{\xi+1}(t)$ according to
\[
d_{\xi+1}(t) = \Delta_{k+1} d_{\xi+1}(t) \tag{27}
\]
where $\Delta_{k+1} \in \mathbb{R}$ is an unknown real scalar uncertain parameter satisfying $|\Delta_{k+1}| \leq 1$. Moreover, the $H^\infty$ control objective for the artificial uncertain quantum system (25), (4), (5), (6), (27) is as follows:
\[
\int_0^t \left\langle \bar{z}(s)^T \bar{z}(s) + \bar{z}(s)^T \bar{y}(s)^* + \varepsilon \left( \bar{\eta}(s)^T \bar{\eta}(s) + \bar{\eta}(s)^T \bar{\eta}(s)^* \right) \right\rangle ds \leq \left( 1 - \varepsilon^2 \right) \int_0^t \left\langle \bar{\beta}_0^T \bar{\beta}_0 + \bar{\beta}_0^T \bar{\beta}_0^* \right\rangle ds + \pi_1 + \pi_2 t \tag{28}
\]
where $\varepsilon, \pi_1, \pi_2 > 0$ are real constants.

We now consider two special cases for $\Delta_{k+1}$ in order to verify that any suitable coherent quantum controller of the form (7) for the artificial uncertain quantum system (25), (4), (5), (6), (27) is indeed stable and strictly bounded real, and solves the original quantum control problem.
for the artificial uncertain quantum system (25), (4), (5), (6), (27). To proceed with this approach, we first introduce scaling constants \( \tau_1 > 0, \ldots, \tau_{k+1} > 0 \) so that we can rewrite the QSDEs (25) of the artificial uncertain quantum system as follows:

\[
\begin{align*}
\dot{a}(t) &= \dot{F} a(t) dt + G_0 a(t) dt + G_1 \dot{a}(t) + G_2 a(t) dt; \\
\dot{z}(t) &= \dot{H}_1 a(t) dt + N \dot{a}(t) + J_{12} a(t) dt; \\
\dot{y}(t) &= \dot{H}_2 a(t) dt + J_{20} a(t) dt + J_{21} \dot{a}(t) \\
\end{align*}
\]

where \( a(0) = a_0; \dot{a}(t) = \dot{\beta}_0 a(t) dt + \dot{d}(t); \)

\[
\begin{align*}
\dot{d}(t) &= \left[ \begin{array}{c}
\dot{a}(t) \\
\dot{z}(t) \\
\dot{y}(t)
\end{array} \right] = \left[ \begin{array}{c}
\sqrt{\tau_1 N_1} d_1(t) \\
\sqrt{\tau_1 N_2} d_2(t) \\
\sqrt{\tau_1 N_3} d_3(t)
\end{array} \right]; \\
\end{align*}
\]

\[
\begin{align*}
\dot{G}_1 &= \left[ \begin{array}{c}
\gamma^{-1} \hat{G}_1 \\
\gamma^{-1} \hat{J}_{12} \\
0
\end{array} \right]; \\
\dot{J}_{21} &= \left[ \begin{array}{c}
\gamma^{-1} \hat{J}_{21} \\
0 \\
0
\end{array} \right]; \\
\end{align*}
\]

\[
\begin{align*}
\hat{H}_1(t) &= \left[ \begin{array}{c}
\hat{H}_1(t) \\
\hat{J}_{12}(t) = \left[ \begin{array}{c}
0 \\
\sqrt{\tau_1 N_2} d_1(t)
\end{array} \right]; \\
\hat{N} &= \left[ \begin{array}{c}
0 \\
0 \\
0
\end{array} \right]; \\
\end{align*}
\]

The \( H^\infty \) control objective corresponding to the quantum system (30) is stated as follows:

\[
\int_0^T \left\{ \left( \hat{\beta}_0 a(t) \right)^\dagger \hat{\beta}_0 a(t) + \left( \hat{\beta}_0 a(t) \right)^T \hat{\beta}_0 a(t)^* \right\} ds + \pi_1 + \pi_2 t
\]

where \( \varepsilon, \pi_1, \pi_2 > 0 \) are real constants. Here, we notice that the \( \hat{N} \)-term, which appears in the QSDEs (30), results in a non-standard \( H^\infty \) control problem so that we can apply a loop shifting transformation to eliminate this term; e.g., see Section 17.2 in Zhou et al. (1996). To do so, we first need to satisfy the following assumption:

**Assumption 2.** Given a non-singular scaling matrix \( R \) and constants \( \tau_1 > 0, \ldots, \tau_{k+1} > 0 \), the uncertain quantum system (25), (4), (5), (6), (27) is assumed to be such that \( \hat{N}^\dagger \hat{N} < I \).

Then, we can define

\[
\Phi := I - \hat{N}^\dagger \hat{N} > 0; \quad \Phi^- := I - \hat{N} \hat{N}^\dagger > 0
\]

and also

\[
\begin{align*}
\dot{d}(t) &= \Phi^- \dot{d}(t) + \Phi^- \dot{N} \left[ \hat{H}_1 a dt + J_{12} a dt \right]; \\
\dot{z}(t) &= \Phi^- \dot{z}(t) + \Phi^- \left[ \hat{H}_1 a dt + J_{12} a dt \right].
\end{align*}
\]

From (34), it is straightforward to verify that

\[
\begin{align*}
\| \dot{d}(t) \|^2 - \| \dot{z}(t) \|^2 &= \| \dot{d}(t) \|^2 - \| \dot{z}(t) \|^2; \\
\end{align*}
\]

Now, we can rewrite the QSDEs (30) as

\[
\begin{align*}
\dot{a}(t) &= \dot{F} a(t) dt + G_0 a(t) dt + G_1 \dot{a}(t) + G_2 a(t) dt; \\
\dot{z}(t) &= \dot{H}_1 a(t) dt + J_{12} a(t) dt; \\
\dot{y}(t) &= \dot{H}_2 a(t) dt + J_{20} a(t) dt + J_{21} \dot{a}(t) \\
\end{align*}
\]

where \( a(0) = a_0; \dot{a}(t) = \dot{\beta}_0 a(t) dt + \dot{d}(t); \)

\[
\begin{align*}
\dot{F} &= \dot{F} + \hat{G}_1 \hat{N} \hat{G}_1^\dagger H_1; \\
\dot{G}_2 &= \dot{G}_2 + \hat{G}_1 \hat{N} \hat{G}_1^\dagger J_{12}; \\
\dot{H}_1 &= \dot{H}_1 + \hat{N} \hat{H}_1; \\
\dot{H}_2 &= \dot{H}_2 + \hat{N} \hat{H}_2; \\
\dot{J}_{12} &= \dot{J}_{12} + \hat{N} \hat{J}_{12}; \\
\end{align*}
\]

(37)

Furthermore, we also define

\[
\dot{y}(t) := \dot{y}(t) - \hat{J}_{21} \dot{a}(t)
\]

and if (38) is substituted into (36), we obtain

\[
\begin{align*}
\dot{a}(t) &= \dot{F} a(t) dt + G_0 a(t) dt + G_1 \dot{a}(t) + G_2 a(t) dt; \\
\dot{z}(t) &= \dot{H}_1 a(t) dt + J_{12} a(t) dt; \\
\dot{y}(t) &= \dot{H}_2 a(t) dt + J_{20} a(t) dt + J_{21} \dot{a}(t) \\
\end{align*}
\]

(39)

(40)

where \( \varepsilon, \pi_1, \pi_2 > 0 \) are real constants.

The solution to the coherent quantum \( H^\infty \) control problem for the linear quantum system (30) is given in terms of solutions of the parameterized complex Riccati equations:

\[
\begin{align*}
&\dot{F} - \hat{G}_2 \hat{N} \hat{G}_2^\dagger \hat{H}_1 = \hat{X} + \hat{X} \left( \hat{G}_1 \hat{G}_2^\dagger \right)^\dagger \hat{H}_1 + \hat{H}_1 \left( I - \hat{J}_{12} \hat{N} \hat{G}_2^\dagger \right) \hat{H}_1 = 0; \\
&\dot{G}_2 = \hat{G}_2 \hat{N} \hat{G}_2^\dagger \hat{H}_1 = \hat{Y} + \hat{Y} \left( \hat{G}_1 \hat{G}_2^\dagger \right)^\dagger \hat{H}_1 + \hat{H}_1 \left( I - \hat{J}_{12} \hat{N} \hat{G}_2^\dagger \right) \hat{H}_1 = 0; \\
&F, G_2, H_1, J_{12} > 0; \hat{F}, \hat{G}_2, \hat{H}_1, \hat{J}_{12} > 0.
\end{align*}
\]

(41)

(42)
to the uncertain quantum system (1), (4), (5), (6) is strict bounded real with disturbance attenuation $\gamma > 0$.

Remark 2. Although the coherent quantum controller (7), (42) is guaranteed to be physically realizable, the additional uncertainty in the artificial uncertain quantum system (25), (4), (5), (6), (27) introduces some additional conservatism to the quantum controller design process.

4. AN ILLUSTRATIVE EXAMPLE

To demonstrate the coherent quantum controller design method presented in Section 3, we consider an example of designing a strict bounded real coherent quantum controller for a first order optical cavity; see James et al. (2008) and Maalouf and Petersen (2009). That is,

\[
\begin{align*}
da(t) &= -\frac{g}{2} a(t) dt - \sqrt{k_1} d_1(t) - \sqrt{k_2} d_2(t) - \sqrt{k_3} d_3(t), \\
dz(t) &= \sqrt{k_3} a(t) dt + d_3(t), \\
dy(t) &= \sqrt{k_2} a(t) dt + d_2(t)
\end{align*}
\]

(43)

where $k_1 = 2.25$, $k_2 = 1.00$, $k_3 = 1.00$ and $g = k_1 + k_2 + k_3$. We assume that the optical cavity (43) does not have an uncertainty term. If we apply the controller design algorithms proposed in James et al. (2008) and Maalouf and Petersen (2009), they do not necessarily yield a stable and strict bounded real coherent quantum controller. For this example, we have that $k_1 > k_2 + k_3$, but $\sqrt{k_1} < \sqrt{k_2} + \sqrt{k_3}$, which implies that the standard quantum $H^\infty$ controller for the quantum system (43) will not be physically realizable; see Maalouf and Petersen (2009).

Applying our method and a differential evolution algorithm (e.g., see Price (2008)), we then obtain $\gamma = 0.9132$ and $\tau_1 = 1.6641$, and the quantum $H^\infty$ controller is

\[
F_c = -11.0014; \quad G_c = -0.0118; \quad H_c = -0.4566
\]

(44)

with $\|H_c(sI - F_c)^{-1}G_c\|_\infty = 0.0005$. Apparently, the quantum $H^\infty$ controller (44) is physically realizable as it is stable and strict bounded real.

From Maalouf and Petersen (2009), we know that there exists $X_0 > 0$ such that

\[
F_c^T X_0 + X_0 F_c + X_0 G_c^T \tilde{X}_0 + H_c^T H_c = 0.
\]

(45)

That is, $X_0 = 0.0095$, which is then used to determine $G_c$ and $H_c$ as follows:

\[
G_c = -X_c^{-1} H^T_c = 48.1908; \quad H_c = -G^T_c X_c = 0.0001
\]

(46)

such that the quantum $H^\infty$ controller (44) is physically realizable. Meanwhile, $G_c = 0$ and $H_c = 0$ as they are not required in the realization of a coherent quantum controller; see Remark 1. Now, using (44) and (46), we can apply the algorithm presented in Petersen (2009a) to physically construct the quantum $H^\infty$ controller (44) as a generalized 2-mirror cavity with two inputs and two outputs as follows:

\[
\tilde{F}_c = -11.0014; \quad \tilde{G}_c = [4.6907 \quad -0.0011]; \quad \tilde{H}_c = [-4.6907 \quad 0.0011]
\]

(47)

using passive optical devices such as optical cavities, beam-splitters and phase shifters. Applying the quantum $H^\infty$ controller (47) to the quantum system (43), we obtain a closed loop quantum system as shown in Fig. 2 with $k_{c_1} = (4.6907)^2$, $k_{c_2} = (0.0011)^2$ and $\tilde{F}_c = -\frac{1}{2}(k_{c_1} + k_{c_2})$.

5. CONCLUSIONS

We have presented a systematic method to solve a coherent quantum robust $H^\infty$ control problem for a class of linear complex quantum stochastic systems with norm-bounded structured uncertainties. This method results in a stable and strict bounded real quantum $H^\infty$ controller, which is thus physically realizable.

REFERENCES


Petersen, I.R. (2009b). Robust $H^\infty$ control of an uncertain system via a strict bounded real output feedback controller. Optimal Control Applications and Methods, 30(3), 247–266.


