Robust synchronization of multiple agents with uncertain dynamics

Kiyotsugu Takaba∗
∗ Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University Kyoto 606-8501, Japan (e-mail: takaba@amp.i.kyoto-u.ac.jp).

Abstract: This paper is concerned with robust synchronization of a system of uncertain agents whose mutual communications are constrained by the topology of the associated communication graph. The uncertainty of each agent is modeled as a gain-bounded multiplicative uncertainty. A sufficient condition for achieving the robust synchronization is derived based on the input-output stability argument under the assumption that the nominal dynamics of each agent is relaxed co-coercive. Based on this robust synchronization condition, we investigate the stability margin and the worst-case $L_2$ performance of the closed-loop system for various network topologies of the communication graph.

Keywords: multi-agent systems, cooperative systems, robustness analysis, distributed control, communication graph, incremental passivity

1. INTRODUCTION

Multi-agent coordinations such as synchronization, consensus, and formation have recently been attracting a great attention in the area of systems and control, since such phenomena can be encountered in many applications in physics, biology, robotics, computer science, and so forth (see e.g. Olfati-Saber, Fax and Murray (2007) and the references therein). Although there have been reported a number of researches on synchronization of multi-agent systems (e.g. Stan and Sepulchre (2007), Scardovi et al. (2010)), many of them assume that all agents in the system have the same dynamics. In practice, however, such homogeneity may not be satisfied due to unmodeled dynamics and/or parameter variations in the individual agents.

Some related works (Li, Chen and Aihara (2006), Hill and Zhao (2008), Hamadeh et al. (2008), Hirche and Hara (2008)) considered the synchronization of heterogeneous multi-agent systems based on Lyapunov-type methods using dissipation inequalities.

In this paper, we will consider the output synchronization of heterogeneous multi-agent systems, in which the heterogeneity is modeled as gain-bounded uncertainties in the individual agents. We will take the input-output stability approach to derive a condition for achieving the robust synchronization under the assumption that each nominal agent satisfies a weak version of incremental passivity called relaxed co-coercivity (Scardovi and Leonard (2009), Scardovi et al. (2010)). Based on the obtained robust synchronization condition, we will also discuss the stability margin and the worst-case $L_2$-gain performance of the closed-loop system.

Notations:

\begin{align*}
\langle x, y \rangle_T & := \int_0^T x(t)^T y(t) \, dt, \\
\|x\|_{2,T} & := \langle x, x \rangle_T^{1/2}, \\
\mathcal{L}_{2e}^m & := \{ x : [0, \infty) \to \mathbb{R}^m \mid \|x\|_{2,T} < \infty \ \forall T \in [0, \infty) \} \\
1_n & := (1, 1, \ldots, 1)^T \in \mathbb{R}^n \\
\lambda_{\text{min}}, \lambda_{\text{max}} & \text{ denote the smallest eigenvalue, largest eigenvalue, respectively.}
\end{align*}

$\| \cdot \|$ denotes the Euclidean norm for a vector, and the maximal singular value for a matrix, respectively. Moreover, if an operator $F : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$ is finite gain $L_2$ stable, then its gain is denoted by $\|F\|$.

2. PROBLEM FORMULATION

2.1 Agents with Uncertain Dynamics

Throughout this paper, we consider a multi-agent system consisting of $N$ agents. Each agent is described by the input-output equations

\begin{align}
\dot{y}_i &= P(u_i + v_i + w_i), \\
\dot{w}_i &= \Delta_i(u_i + v_i), \\
i &= 1, 2, \ldots, N,
\end{align}

where $y_i$, $u_i$, and $v_i$ are the output to be synchronized, the control input, and the disturbance, respectively. The input-output maps $P : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$ and $\Delta_i : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$ are causal nonlinear operators corresponding to the nominal system and uncertainty, respectively. Note that the uncertainties $\Delta_i, i = 1, \ldots, N$ come from the heterogeneity and/or unmodeled dynamics in the individual agent.

The notion of relaxed co-coercivity, or incremental feedback passivity, plays an important role in the analysis of synchronization phenomena (Stan and Sepulchre (2007),
Fig. 1. Agent with multiplicative uncertainty

Stan et al. (2007), Scardovi and Leonard (2009), Scardovi et al. (2010).

Definition 1. An input-output map $P : \mathcal{L}_2^m \rightarrow \mathcal{L}_2^m$ is said to be relaxed co-coercive, if there exists a constant $\varepsilon \in \mathbb{R}$ such that

$$
\langle u - v, Pu - Pv \rangle_T \geq \varepsilon \|Pu - Pv\|_{L_2}^2,
$$

(2)

with $u, v \in \mathcal{L}_2^m$, $\forall u, v \in \mathcal{L}_2^m$, $\forall T \geq 0$.

In particular, $P$ is called incrementally passive if $\varepsilon = 0$, and strictly incrementally output passive if $\varepsilon > 0$ (see e.g. Zames (1966), Desoer and Vidyasagar (1975)).

We make the following assumption throughout this paper. 

Assumption 1. The nominal input-output map $P$ is relaxed co-coercive, i.e. there exists $\varepsilon \in \mathbb{R}$ satisfying (2).

The uncertainty $\Delta_i$, $i = 1, 2, \ldots, N$, is unbiased finite gain $\mathcal{L}_2$ stable with $\|\Delta_i\| \leq \delta$, namely

$$
\|u\|_{L_2,T} \leq \delta \|u\|_{L_2,T}, \forall u \in \mathcal{L}_2^m.
$$

Denote by $B_\delta$ the set of such gain-bounded uncertainties:

$$
B_\delta := \{ \Delta : \mathcal{L}_2^m \rightarrow \mathcal{L}_2^m | \|\Delta\| \leq \delta \}
$$

Then, (1) is equivalently rewritten as

$$
\begin{bmatrix}
\Delta_1(u_1) \\
\Delta_2(u_2) \\
\vdots \\
\Delta_N(u_N)
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix}
$$

Then, Assumption 2. The communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is time-invariant, i.e. the network topology of $\mathcal{G}$ is fixed all the time.

We now introduce some useful matrices from the algebraic graph theory (Godsil and Royle (2001)). Denote with $B$ the incidence matrix of $\mathcal{G}$. Then, the Laplacian $L$ of $\mathcal{G}$ is defined by $L = BB^T$. Note that $\text{rank}L = \text{rank}B \leq N - 1$, and that the equality is attained when $\mathcal{G}$ is a connected graph. By definition, $L$ has at least one zero eigenvalue with the eigenvector $1_N$, i.e. $L1_N = 0$. Let $Q \in \mathbb{R}^{(N-1) \times N}$ be such that $Q^T1_N = I_{N-1}1_N = 0$, and $Q^TQ + \frac{1}{N}1_N1_N^T = I_N$. Then, $U = [Q^T \frac{1}{\sqrt{N}}1_N]$ is an orthogonal matrix, and

$$
U^T Lu = \begin{bmatrix}
QLQ^T & 0 \\
0 & 0
\end{bmatrix}
$$

Let

$$
0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N
$$

be the eigenvalues of $L$. It then follows from the above equation that $\lambda_N = \lambda_{\text{max}}(QLQ^T)$, $\lambda_2 = \lambda_{\text{min}}(QLQ^T)$, and $\mathcal{G}$ is connected iff $\lambda_2 > 0$.

For later use, define

$$
B_m = B \otimes I_m, \quad L_m = L \otimes I_m, \quad Q_m = Q \otimes I_m.
$$

Then, $L_m = B_mB_m^T$, and $[Q_m^T \frac{1}{\sqrt{N}}1_{Nm} \otimes I_m]$ is an orthogonal matrix, namely

$$
Q_m^T Q_m + \frac{1}{N} (1_N \otimes I_m)(1_N \otimes I_m)^T = I_{Nm},
$$

(4a)

$$
Q_m(1_N \otimes I_m) = 0,
$$

(4b)

$$
Q_mQ_m^T = I_{(N-1)m}.
$$

(4c)

We also have

$$
\lambda_2 = \lambda_{\text{max}}(Q_mL_mQ_m^T),
$$

$$
\lambda_N = \lambda_{\text{max}}(Q_mL_mQ_m^T).
$$

Lemma 1. For $u_i, v_i \in \mathcal{L}_2^m$, $i = 1, \ldots, N$, we define

$$
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix}
$$

Then,

$$
\langle \tilde{u}, \tilde{v} \rangle = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle u_i - u_j, v_i - v_j \rangle_T, \forall T \geq 0
$$

Proof. The proof is omitted since it can be proved in a similar way to Lemma 1 in Scardovi and Leonard (2009) and Scardovi et al. (2010).

It should be noted that, in Lemma 1, $(\tilde{u}, \tilde{v})_T$ is independent of the choice of $Q$.

2.3 Synchronization Problem

The output synchronization problem is to design a distributed control law which makes the outputs of all agents behave in the same way, i.e., $y_i - y_j$, $i, j \in \mathcal{V}$ should converge to zero, or should be sufficiently small in some sense.
Define
\[ \tilde{y} = Q_m y, \quad \tilde{z} = Q_m z, \quad \tilde{u} = Q_m u, \quad \tilde{v} = Q_m v, \quad \tilde{w} = Q_m w. \]
for the underlying signals in the multi-agent systems (3).

Under the communication constraints due to \( G \), a typical control strategy for achieving coordinative tasks such as synchronization, formation, consensus, etc., is to apply the relative output feedback law
\[ u_i = -k \sum_{(i,j) \in E} (y_i - y_j), \quad i \in \mathcal{V}, \quad (5) \]
or equivalently,
\[ u = -kL_m y, \]
where \( k > 0 \) is a constant feedback gain to be designed. Moreover, by using (4a) and \( L_m(1_N \otimes I_m) = 0 \), the above control law can be rewritten as
\[ u = -kL_mQ_m^T Q_m y = -kL_mQ_m^T \tilde{y}. \quad (6) \]

It also follows from Lemma 1 that
\[ \| \tilde{y} \|_{2,T} = \left( \frac{1}{2N} \sum_{i,j \in \mathcal{V}} \| y_i - y_j \|_{2,T}^2 \right)^{1/2}. \quad (7) \]

Thus, in this paper, we adopt the worst-case \( L_2 \)-gain of the closed-loop map from \( v \) to \( \tilde{y} \) as the measure of synchronization performance.

**Definition 2.** For the multi-agent system given by (1) and \( G \), the feedback control law (6) is said to achieve the robust synchronization if there exists a positive constant \( \alpha \) such that
\[ \| \tilde{y} \|_{2,T} \leq \alpha \| v \|_{2,T} \]
\[ \forall v \in \mathcal{L}_{2m} \mathcal{V}, \forall (\Delta_1, \ldots, \Delta_N) \in B^N \mathcal{V}, \forall T \geq 0. \quad (8) \]

**3. ROBUST SYNCHRONIZATION**

In this section, we will derive a sufficient condition for achieving the robust synchronization in the sense of Definition 2.

It follows from Assumption 1 and \( \tilde{u} = Q_m u \) that
\[ \varepsilon \| \tilde{y} \|_{2,T}^2 \leq \langle \tilde{u} + \tilde{v} + \tilde{w}, \tilde{y} \rangle_T \leq \langle -kQ_m L_m Q_m \tilde{y} + \tilde{v} + \tilde{w}, \tilde{y} \rangle_T \]
holds for the multi-agent system (1). Hence, we have
\[ (k\lambda_2 + \varepsilon) \| \tilde{y} \|_{2,T}^2 \leq \varepsilon \| \tilde{y} \|_{2,T}^2 + \langle kQ_m L_m Q_m \tilde{y}, \tilde{y} \rangle_T \leq \langle \tilde{v}, \tilde{y} \rangle_T + \langle \tilde{w}, \tilde{y} \rangle_T \]
Applying the Schwartz inequality to the most right-hand side of the above equation yields
\[ (k\lambda_2 + \varepsilon) \| \tilde{y} \|_{2,T}^2 \leq (\| \tilde{v} \|_{2,T} + \| \tilde{w} \|_{2,T}) \| \tilde{y} \|_{2,T}. \]
Moreover, since \( w = Q_m v, w = \Delta(u + v) \), and \( \| Q_m \| \leq 1 \), we get
\[
\begin{align*}
(k\lambda_2 + \varepsilon) \| \tilde{y} \|_{2,T}^2 & \leq \| \tilde{v} \|_{2,T}^2 + \| \tilde{w} \|_{2,T}^2 \\
& \leq \| v \|_{2,T}^2 + \| w \|_{2,T}^2 \\
& = \| v \|_{2,T}^2 + \| \Delta(u + v) \|_{2,T} \quad (9)
\end{align*}
\]
It also follows from \( \| \Delta \| \leq \delta \) and (6) that
\[
\begin{align*}
\| v \|_{2,T}^2 + \| \Delta(u + v) \|_{2,T} & \leq \| v \|_{2,T}^2 + \delta \| u + v \|_{2,T} \\
& \leq \| v \|_{2,T}^2 + \delta \| u \|_{2,T} + \| v \|_{2,T} \\
& \leq (1 + \delta) \| v \|_{2,T} + \delta kL_m Q_m \tilde{y} \|_{2,T} \\
& = (1 + \delta) \| v \|_{2,T} + \delta k\lambda_N \| \tilde{y} \|_{2,T} \quad (10)
\end{align*}
\]
In the last inequality, we have used the facts that \( \| Q_m \| = 1 \) and \( \lambda_N = \| L_m \| \).

By combining (9) and (10), we conclude the following theorem as the main result of this paper.

**Theorem 1.** Under Assumptions 1 and 2, assume that
\[ k\lambda_2 + \varepsilon > k\lambda_N \delta. \quad (11) \]
Then, the relative output feedback law (6) achieves the robust synchronization. That is, the closed-loop map from \( v \) to \( \tilde{y} \) satisfies (8) with
\[ \alpha = \frac{1 + \delta}{\varepsilon + k\lambda_2 - k\lambda_N \delta}. \quad (12) \]

**Remark 1.**
(i) We have derived a robust synchronization condition for undirected communication graphs. The result can be easily generalized to the case of directed graphs by slightly modifying the definition of the graph Laplacian.

(ii) A more general control strategy than (5) is
\[ u_i = -\sum_{(i,j) \in E} k_{ij} (y_i - y_j), \quad i \in \mathcal{V}. \quad (13) \]

We have taken the control law (5), namely \( k_{ij} = k \forall (i, j) \in E \), for ease of analysis. If the control law of (13) is applied, the synchronization condition will involve the eigenvalues of the weighted graph Laplacian instead of \( k\lambda_2 \) and \( k\lambda_N \).

**4. DISCUSSIONS**

**4.1 Stability Margin**

Define
\[ \delta = \frac{k\lambda_2 + \varepsilon}{k\lambda_N}. \quad (14) \]
We refer to \( \delta \) as the estimated stability margin, since it gives a lower bound of the stability margin in the sense that, for a fixed \( k \), the closed-loop map from \( v \) to \( \tilde{y} \) is robustly finite gain \( L_2 \) stable for any \( \delta < \delta_0 \).

Fig. 2 illustrates the relation between the estimated stability margin \( \delta \) and the feedback gain \( k \).

If \( \varepsilon < 0 \), \( \lambda_2 > 0 \) implies that \( G \) is a connected graph. Then, \( \delta \) is monotone increasing with respect to \( k > -\varepsilon/\lambda_2 \), and is bounded above by \( \lambda_2/\lambda_N \).

If \( \varepsilon \) is equal to zero, i.e. \( P \) is incrementally passive, then we have \( \delta = \lambda_2/\lambda_N \), which is independent of the feedback gain \( k \).

Moreover, in the case where \( \varepsilon \) is positive, we see that, for any \( \delta > 0 \), there always exists a feedback gain \( k > 0 \) which achieves the robust synchronization against \( (\Delta_1, \ldots, \Delta_N) \in B^N \mathcal{V} \). Furthermore, the robust synchronization is achieved even if we take \( k = 0 \), since the input-output map of each agent from \( v_i \) to \( y_i \) is already finite gain \( L_2 \) stable in this case. Additionally, since \( k = 0 \) implies
Fig. 2. Feedback gain vs estimated stability margin

\[ u = 0 \text{ and } w = \Delta v, \] it is shown by similar calculations to the previous section that, for any \((\Delta_1, \ldots, \Delta_N) \in \mathbb{R}_N^{N}\),

\[ \| \tilde{y} \|_{2,T} \leq \varepsilon^{-1}(1 + \delta)\|v\|_{2,T}, \quad \forall v \in \mathcal{L}_{2N}^{\mathbb{R}_T}, \quad \forall T \geq 0. \]

To study the relation between \( \delta \) and the network topology of \( \mathcal{G} \), we consider the 4-agent system with incrementally passive agents, i.e. \( N = 4 \) and \( \varepsilon = 0 \). We compute \( \lambda_2, \lambda_3, \) and \( \delta \) for six network topologies (A)–(F) in Fig. 3. The result is shown in Table 1.

It is seen from the table that the closed-loop system is more robust, i.e. \( \delta \) is large, for a denser communication graph: the linear graph (A) is the weakest, and the complete graph (F) is the most robust. Moreover, comparison of the results for (C) and (D) suggests that, when \( |E| \) is fixed, the robustness can be improved by forming a loop which goes through all nodes.

Fig. 3. Communication graphs for 4-agent system

Next, we examine the relation between the size of the communication graph and the stability margin, i.e. between \( N \) and \( \delta \). As shown in Table 2, analytic formulae for the eigenvalues of the Laplacian \( L \) are known for some typical network topologies in the literature (e.g. Chung (1997); de Abreu (2007)).

It is seen by substituting these formulae into (14) that, as \( N \) goes to infinity, \( \delta \) for complete graphs converges to 1, while \( \delta \) for the other topologies decays to zero or a negative value if \( \varepsilon \leq 0 \). Though \( \delta \) is a lower estimate of the stability margin, this observation suggests that, for a larger number of agents, the closed-loop multi-agent system may become less robust against \( \Delta_i \)'s unless the communication graph is complete.

4.2 Worst-Case \( \mathcal{L}_2 \) Performance

As a measure of the performance of robust synchronization, we adopt and examine the value of \( \alpha \) in (12), which is an upper bound on the worst-case \( \mathcal{L}_2 \)-gain under the condition of (11).

Since

\[ \frac{\partial \alpha}{\partial \delta} = \frac{\varepsilon + k\lambda_2 + k\lambda_N}{(\varepsilon + k\lambda_2 - k\lambda_N)\delta^2}, \]

\( \alpha \) is monotone increasing with respect to \( \delta \) satisfying (11), for fixed \( \varepsilon \) and \( k \). In particular, \( \alpha \) is unbounded around \( \delta = \delta_0 \). Moreover, as \( \delta \to 0 \), \( \alpha \) converges to 1/(\( \varepsilon + k\lambda_2 \)), which is the nominal \( \mathcal{L}_2 \)-gain bound obtained by Scardovi and Leonard (2009).

It is seen from (12) that, if \( \delta < \lambda_2/\lambda_N \) (This is the case when \( \varepsilon > 0 \)), then the feedback gain \( k \) cannot be arbitrarily large, and must satisfy

\[ k < \frac{\varepsilon}{\lambda_N \delta - \lambda_2}. \]

In this case, \( \alpha \) is monotone increasing with respect to \( k \), and is unbounded around \( k = \varepsilon/(\lambda_N \delta - \lambda_2) \).

Finally, to study the relation between the worst-case \( \mathcal{L}_2 \)-gain performance and the network topology of \( \mathcal{G} \), Fig. 4 illustrates the \( \delta-\alpha \) plots for the communication graphs in Fig. 3 for the 4-agent system with \( \varepsilon = 0 \) and \( k = 1 \). It is seen from the figure that the worst-case \( \mathcal{L}_2 \) performance is improved as the graph \( \mathcal{G} \) becomes more dense.
5. CONCLUSIONS

In this paper, we have derived a sufficient condition for achieving the robust synchronization of networked multiple agents with uncertain dynamics. The obtained condition is characterized in terms of the largest and second smallest eigenvalues of the Laplacian of the communication graph as well as the feedback gain and uncertainty bound. By this condition, we have also clarified the trade-off between the worst-case $L_2$-gain and the uncertainty bound, and the effect of the network topology on the stability margin and the worst-case $L_2$ performance. This trade-off provides an insight for the design of the communication graph.

REFERENCES


