Flow Estimation of Boundary Layers Using Wall Shear Information

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Abstract:
This paper investigates the problem of obtaining a state-space model of the disturbance evolution that precedes turbulent flow and the associated increase in skin-friction drag on aircraft surfaces. This problem is highly challenging since the flow system is governed by nonlinear, partial differential-algebraic equations (the Navier-Stokes equations) for which there currently exists no efficient controller/estimator synthesis techniques. In this paper it is shown how a sequence of model approximations can be employed to yield a linear, low-order state-space model, to which the standard tools of control theory can be applied. One of the novelties of this paper is the application of a numerical routine that converts a system of differential-algebraic equations into one of ordinary differential equations. This enables straightforward satisfaction of boundary conditions whilst dispensing with the need for parallel flow approximations and velocity-vorticity transformations. The efficacy of the model is demonstrated by the synthesis of a Kalman filter that clearly reconstructs the characteristic features of the flow, using only wall velocity gradient (shear) measurements obtained from a high-fidelity nonlinear simulation.

Keywords: Turbulence, nonlinear equations, partial differential equations, descriptor systems, model approximation, boundary conditions, estimation, Kalman filter

1. INTRODUCTION

It has been recommended (Argüelles et al., 2001) that by the year 2020 all new aircraft should meet a 50% reduction in fuel consumption (per passenger kilometre), for obvious economic and environmental reasons. This target is unlikely to be met unless new technologies emerge, which are capable of controlling the surrounding airflow to reduce the associated drag forces exerted on aircraft surfaces (Gad-el-Hak, 2000).

In order to synthesise a controller or observer, a model of the system is required. In the present work the system is taken to be a boundary layer (White, 2003) evolving over a flat plate, as depicted in Figure 1. The term ‘boundary layer’ simply refers to the layer of fluid next to a bounding surface. Here, the bounding surface is a flat plate, which can be considered a simplified aircraft wing. At relatively low Mach numbers this type of flow is governed by the incompressible Navier-Stokes equations:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mu \Delta \mathbf{v} - \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p + \mathbf{f},$$  
$$0 = \nabla \cdot \mathbf{v},$$

where the velocity of the fluid is \( \mathbf{v} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3 \), \( p : \Omega \times \mathbb{R}_+ \to \mathbb{R} \) is the pressure, \( \mathbf{f} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3 \) is a vector of external forces, \( \mathbf{g} : \partial \Omega \times \mathbb{R}_+ \to \mathbb{R}^3 \) is a vector of boundary conditions, and \( \mathbf{v}_0 \in \mathbb{R}^3 \) is a vector of initial velocities. The density \( \rho \), \( \mu = \mu(\mathbf{v}) \) and \( \mathbf{f} \) are the gradient, Laplace and divergence operators, respectively. \( \Omega \subset \mathbb{R}^3 \) is a domain in three spatial dimensions with boundary \( \partial \Omega \), and \( \mathbf{v}_0 \) is a point within the domain. Throughout this paper sans serif fonts represent parameters used to describe the flow system, whilst serif fonts denote vectors and matrices.

The Navier-Stokes equations (1) are a coupled system of nonlinear, partial differential-algebraic equations, for which no efficient controller/estimator synthesis techniques currently exist. The literature on control system design for such systems is scant, and of those papers that exist, the majority consider the significantly less complicated case of channel flow, e.g. Hogberg et al. (2003); Baramov et al. (2004); McKernan et al. (2007), where the mean (time-averaged) flow is parallel to the walls and fully developed in the sense that it is invariant in the streamwise direction. A consequence of this fact is that it enables a straightforward analytic reformulation of (1) into an equivalent system expressed in terms of ‘divergence-free’ variables of wall-normal velocity and vorticity. These variables implicitly satisfy the incompressibility constraint (1b) thus al-

* This work was supported by the EPSRC.
Fig. 1. Sketch of the estimation problem. The observer constructs estimates \( \hat{x}(t) \) of the true velocity perturbation (shown in red) above the sensors, using only measurements \( y(t) \) of the streamwise and spanwise wall shears.

Following the flow, after spatial discretisation, to be described by ordinary differential equations (ODEs), rather than differential-algebraic equations (DAEs). Hence, the flow can be modelled as a standard state-space system, rather than a descriptor state-space system, for which there exists less control theory. In contrast, the mean flow of a boundary layer is non-parallel, since it varies in the streamwise direction. In an effort to recast the system in terms of a divergence-free basis, a parallel flow assumption is commonly employed, e.g. Hoepffner and Brandt (2008). In the present work the need for this assumption is avoided by employing a numerical routine that yields a state-space model without the need for an analytical reformulation of the governing equations. To complete the state-space model, Direct Numerical Simulation (DNS) based measurements of the streamwise and spanwise wall shear (wall-normal velocity gradient) at three evenly spaced locations along the plate are included. Based on this model, a Kalman filter is synthesised that produces estimates of the in-plane streamwise velocity perturbations. The overall scheme is sketched in Figure 1.

This paper is organised as follows. Section 2 describes the boundary layer DNS database and the underlying model, whilst Section 3 discusses a linear approximation to the boundary layer equations. In Section 4 the linearised system is spatially discretised to yield a finite-dimensional descriptor state-space model and Section 5 describes a method for converting this into a standard state-space system, which, in Section 6 is augmented with a disturbance model and wall shear measurements. Based on the resulting model, a Kalman filter is synthesised and the velocity field estimates are presented in Section 7, with conclusions in Section 8.

2. DESCRIPTION OF THE DNS DATABASE

In the present investigation data is obtained from a boundary layer DNS (Zaki and Durbin, 2006), a high-fidelity nonlinear simulation of the flow. A snapshot of this data is shown in Figure 2. The domain extends \( 525\delta_0, 40\delta_0, 30\delta_0 \) in the streamwise \((x)\), wall-normal \((y)\) and spanwise \((z)\) directions, where \( \delta_0 \) is the boundary layer thickness at the upstream end of the domain (the boundary layer thickness is defined as the height at which the mean streamwise velocity is 99% of the free-stream velocity). The data were generated by spatially discretising (1) using a central, second order finite-volume method on a staggered grid of \( 1798 \times 194 \times 194 \) nodes in \( x, y \) and \( z \), and advancing the resulting system in time by using Adams-Bashforth, Crank-Nicolson and implicit Euler schemes for the convective, viscous and pressure terms, respectively (Rosenfeld et al., 1991). Sixty-one snapshots of the streamwise, wall-normal and spanwise velocity components were available at each grid point, separated by a sampling period of \( T_s = 2 \) (nondimensional time units). In total, the available fields span a time interval that a fluid particle would take to travel a total distance of \( 120\delta_0 \) at the free-stream velocity \( U_\infty \).

Figure 2 shows two planes parallel to the wall. The lower plane is inside the boundary layer and the second is in the free-stream. Shaded regions depict velocity perturbations in the streamwise direction. The flow is initially laminar and characterised by long streamwise perturbations, or ‘streaks’, that are initiated by free-stream disturbances traversing the boundary layer (Zaki and Saha, 2009). The streaks grow in magnitude before decaying into turbulent spots, followed by full turbulence further downstream. Laminar-to-turbulent transition is accompanied by a large increase in skin-friction drag.

3. TRANSIENT GROWTH AND LINEARISATION

The transition from laminar to turbulent flow can be explained by a ‘transient growth’ phenomenon (Butler and Farrell, 1992; Trefethen et al., 1993) whereby small, three-dimensional perturbations of the mean flow can briefly be amplified by several orders of magnitude, before decaying asymptotically. Transient growth is a linear mechanism, and can initiate so-called ‘bypass’ transition to turbulence. A transient growth model that has been shown to accurately predict the evolution of streaky boundary layer disturbances is given by the Linearised Boundary Region Equations (LBRE) (Leib et al., 1999):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -u \frac{\partial U}{\partial x} - U \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - v \frac{\partial u}{\partial z} - \frac{1}{R} \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
\frac{\partial v}{\partial t} &= -u \frac{\partial V}{\partial x} - U \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - v \frac{\partial v}{\partial z} - \frac{1}{R} \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
\frac{\partial w}{\partial t} &= -U \frac{\partial w}{\partial x} - v \frac{\partial w}{\partial y} - v \frac{\partial w}{\partial z} - \frac{1}{R} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right),
\end{align*}
\]

\[
\begin{align*}
0 &= -u \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right), \\
0 &= -u \frac{\partial V}{\partial x} + v \frac{\partial V}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right), \\
0 &= -U \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right),
\end{align*}
\]

where \( R := U_\infty \delta_0 / \nu \) is the Reynolds number, \( \nu := \mu / \rho \) is the kinematic viscosity of the fluid, \( U \) and \( V \) are the average streamwise and wall-normal velocities, respectively, whilst \( u, v, w \) and \( p \) are the perturbation velocities (streamwise, wall-normal velocity, and pressure perturbations).
and spanwise) and pressure. For clarity, the spatial and temporal dependence of each of the variables is not shown here, but it should be noted that \(u, v, w\) and \(p\) are each real-valued functions of \(x, y, z\) and \(t\), whereas the mean-flow velocities are real valued functions of \(x\) and \(y\) only. Velocities and pressures have been made nondimensional by scaling with the inlet boundary layer thickness \(\delta_0\) and the free-stream velocity \(U_\infty\). The following boundary conditions are assumed (Andersson et al., 1999):

\[
\begin{align*}
    u(x, 0, z, t) &= 0, & v(x, 0, z, t) &= 0, & w(x, 0, z, t) &= 0, \\
    u(x, y_{\text{max}}, z, t) &= 0, & p(x, y_{\text{max}}, z, t) &= 0, & w(x, y_{\text{max}}, z, t) &= 0,
\end{align*}
\]

where \(y_{\text{max}} \to \infty\), although in practice this is set to a large but finite value. The initial condition is assumed to be zero:

\[
\begin{align*}
    u(x, y, z, 0), & \ \ \ \ v(x, y, z, 0), & \ \ \ \ w(x, y, z, 0), & \ \ \ \ p(x, y, z, 0) = 0.
\end{align*}
\]

The mean flow quantities in (2) are computed as follows:

\[
\begin{align*}
    U(x, y) &= F'(\eta), \\
    V(x, y) &= 0.5 \sqrt{[V/U_\infty]} (\eta F'(\eta) - F(\eta)), \\
    \partial U(x, y)/\partial x &= -\eta/(2\pi F'(\eta)), \\
    \partial V(x, y)/\partial x &= (1 - 4\pi^2 \eta^2 F''(\eta) + \eta^2 F'(\eta) - F(\eta)), \\
    \partial U(x, y)/\partial y &= \eta/(2\pi F'(\eta)) \\
    \partial V(x, y)/\partial y &= (1 - 4\pi^2 \eta^2 F''(\eta) + \eta^2 F'(\eta) - F(\eta)),
\end{align*}
\]

where \(\eta = y/(\sqrt{V/U_\infty})^{-1/2}\) and \(F(\eta)\) and its derivatives are the solutions of the Blasius Equation (Boyd, 1999). The streamwise region of validity for the linear model is deduced from the DNS data by studying the downstream evolution of the kinetic energy of the \(u\) perturbations.

4. SPATIAL DISCRETISATION

The set of equations (2) represents a system of linear, partial differential-algebraic equations. A finite dimensional approximation is obtained by discretising in the \(x\), \(y\) and \(z\) directions.

4.1 Spanwise discretisation

Referring to Figure 2, the streaks are periodic in the spanwise direction and so the flow variables are assumed periodic in \(z\), in which case the Fourier transform is employed as follows:

\[
u(x, y, z, t) \approx \Re \left[ \sum_{n_z=1}^{N_z} \hat{u}(x, y, t) e^{i n_z \beta z} \right],
\]

where \(i = \sqrt{-1}\), \(n_z\) is the harmonic number, \(\beta := 2\pi n_z/L_z\) is a wavenumber and \(L_z\) is the fundamental wavelength in the spanwise direction. \(N_z\) is finite and represents the truncation of the series. Similar expressions are obtained for the other perturbation variables and are substituted into (2). In the present work attention is restricted to \(\beta = 10\), corresponding to roughly 12 wavenumber streaks across the span (as can be inferred from Figure 2). Note that the Fourier expansion in (5) completely decouples the spanwise dynamics of the system (2), thus enabling the design of filters/controllers for individual models that each consist of a separate Fourier mode.

4.2 Wall-normal discretisation

In the wall-normal direction a higher clustering of grid points is employed within the boundary layer compared to the free-stream. This ensures the boundary layer is adequately resolved by a system of modest state-dimension. This distribution of grid points is achieved as follows. Firstly, the perturbation variables are approximated on a grid of \(N_y\) Chebyshev collocation nodes:

\[
\gamma_{n_y} := \cos \left( \frac{n_y - 1}{N_y - 1} \pi \right), \ \ n_y = 1, \ldots, N_y.
\]

The wall-normal derivatives \(\partial / \partial y, \partial^2 / \partial y^2\) are approximated by Chebyshev differentiation matrices \(Y_{ch}, Y_{ch}^2\) (Weideman and Reddy, 2000). One could construct analogous finite differencing matrices on the same grid points, but spectral differentiation (e.g. Chebyshev) is known to be more accurate for fewer grid points (Trefethen, 2000), thus reducing the state-dimension of the model. To apply Chebyshev differentiation to (2), the interval \(0 \leq y \leq y_{\text{max}}\) is mapped to the Chebyshev domain \(-1 \leq y_{ch} \leq 1\) with the following mapping (Hanifi et al., 1996):

\[
y_{ch} := a \left( 1 + y_{\text{mid}} \right) / \left( b - y_{\text{mid}} \right),
\]

where \(a := y_{\text{mid}}y_{\text{max}}/(y_{\text{max}} - 2y_{\text{mid}})\) and \(b := 1 + 2a/y_{\text{max}}\). This mapping places half the nodes in the region \(0 \leq y \leq y_{\text{mid}}\). By setting \(y_{\text{mid}} = 2\) and \(y_{\text{max}} = 14\), a reasonable tradeoff is obtained between resolving the boundary layer whilst not wasting too many points in the free stream. Lastly, the chain rule and (6b) are used to obtain:

\[
\begin{align*}
    \frac{\partial u(x, y, t)}{\partial y} &\approx \gamma_{y_{ch}} (x, t), \\
    \frac{\partial^2 u(x, y, t)}{\partial y^2} &\approx \gamma_{y_{ch}^2} (x, t),
\end{align*}
\]

where \(\gamma_{y_{ch}} (x, t) := \hat{u}(x, y_{\text{max}}, t)\), \(Y_{ch} := (dy_{ch}/dy)^2 Y_{ch} + (dy_{ch}/dy) Y_{ch}^2\), with similar expressions for the other perturbation variables. The results presented in Section 7 used a model with \(N_y = 15\) wall-normal grid-points.

4.3 Streamwise discretisation

The present streamwise sensor locations are chosen purely on the basis that they are within the transient growth region and are spaced closely enough to resolve first-order velocity gradients in the streamwise direction. With this in mind, spanwise arrays of wall sensors are placed at three arbitrary streamwise locations within the transient growth region of the flow. A semi-staggered grid is used to evaluate the velocities at these streamwise locations, whilst pressures are resolved at intermediate spacings \(x_{12} = 47.5\) and \(x_{23} = 52.5\). Adopting the notation \(\tilde{u}_{x_1, y_{\text{ch}}, t} := \hat{u}(x_1, y_{\text{max}}, t)\) etc., the following three-point finite-differencing scheme is employed to approximate the streamwise derivative terms in (2a):

\[
\begin{align*}
    \frac{\partial \tilde{u}_{x_1, y_{\text{ch}}, t}}{\partial x} &\approx \frac{1}{\Delta x} \left( -3 \tilde{u}_{x_1, y_{\text{ch}}, t} + 4 \tilde{u}_{x_2, y_{\text{ch}}, t} - \tilde{u}_{x_3, y_{\text{ch}}, t} \right), \\
    \frac{\partial \tilde{u}_{x_2, y_{\text{ch}}, t}}{\partial x} &\approx \frac{1}{\Delta x} \left( -\tilde{u}_{x_1, y_{\text{ch}}, t} + \tilde{u}_{x_3, y_{\text{ch}}, t} \right), \\
    \frac{\partial \tilde{u}_{x_3, y_{\text{ch}}, t}}{\partial x} &\approx \frac{1}{\Delta x} \left( \tilde{u}_{x_1, y_{\text{ch}}, t} - 4 \tilde{u}_{x_2, y_{\text{ch}}, t} + 3 \tilde{u}_{x_3, y_{\text{ch}}, t} \right)
\end{align*}
\]

where \(\Delta x = 5\) is the separation between the streamwise locations. Similar expressions are obtained for the other perturbation velocities. The streamwise derivative term in (2b) is approximated at the pressure nodes as follows:

\[
\begin{align*}
    \frac{\partial \tilde{u}_{x_1, y_{\text{ch}}, t}}{\partial x} &\approx \frac{1}{\Delta x} \left( -\tilde{u}_{x_1, y_{\text{ch}}, t} + \tilde{u}_{x_2, y_{\text{ch}}, t} \right), \\
    \frac{\partial \tilde{u}_{x_2, y_{\text{ch}}, t}}{\partial x} &\approx \frac{1}{\Delta x} \left( -\tilde{u}_{x_2, y_{\text{ch}}, t} + \tilde{u}_{x_3, y_{\text{ch}}, t} \right)
\end{align*}
\]

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The resulting finite-dimensional system of ordinary differential and algebraic equations is expressed in (8), where \( x_D \in \mathbb{C}^m \) is the state vector, \( m = 11N_g, A_e \in \mathbb{C}^{N \times N} \) are the submatrices of \( AD_{DBC} \) (defined in Appendix A), \( E_{\cdot \cdot} := I_N \) are the submatrices of \( ED_{DBC} \) (where \( I \) is the identity matrix), 0 is a matrix of zeros and all other unmarked entries are zeros. The subscript ‘\( D \)’ denotes vectors and matrices associated with a descriptor state-space system, while the subscript ‘noBCs’ indicates that boundary conditions (3) have yet to be satisfied.

Enforcing these boundary conditions is straightforward and amounts to modifying the relevant rows of (8). For example, to enforce the condition \( \tilde{u}(x_1, y_{\max}, t) = 0 \), the top rows of \( E_{1,1}, A_{1,1}, A_{1,2}, A_{1,3} \) and \( A_{1,9} \) are set to zero, except for the (1,1) element of \( A_{1,1} \) (corresponding to \( \tilde{u}(x_1, t) \)), which is set equal to 1. This ease of enforcing boundary conditions is one of the main benefits of the descriptor system approach to modelling. By comparison, traditional velocity-vorticity methods require considerable care be taken in constructing wall-normal derivative operators of up to fourth order. Unless the basis functions of these operators each satisfies the boundary conditions, the discretised system will be contaminated by so-called ‘spurious eigenvalues’ that typically reside in the complex right-half-plane (Bewley and Liu, 1998a).

The next section describes a method for converting the autonomous descriptor state-space system:

\[
E_D \dot{x}_D(t) = A_D x_D(t),
\]

where \( E_D \) and \( A_D \) are the matrices in (8) after the inclusion of boundary conditions, into a standard state-space system of the form \( \dot{x}(t) = Ax(t) \).

5. DEALING WITH DESCRIPTOR SYSTEMS

The divergence constraint (1b) and imposition of boundary conditions (3) causes \( E_D \) to be rank deficient. Therefore, it is not possible to obtain a standard state-space system by simply premultiplying both sides of (9) by \( E_D^{-1} \). The system (9) is an example of a descriptor state-space system (also known as a singular, implicit or generalised state-space system), the control and estimation of which are still an open research field. In this section a convenient numerical method (Schön et al., 2003; Gerdin, 2006; Shahzad et al., 2011) is summarised for converting (9) into a standard state-space system.

Let \( E_D, A_D \in \mathbb{C}^{m \times m} \). The pair \( (E_D, A_D) \) is defined as regular if \( l = m \) and there exists an \( s \in \mathbb{C} \) such that \( \det(sE_D - A_D) \neq 0 \) (Dai, 1989). Regularity of a matrix pair ensures the transfer function of a system is well-defined. Next, a result is employed that reveals how the slow and fast subsystems of (9), containing the finite and infinite generalised eigenvalues, respectively, can be decoupled to yield the so-called standard form. According to Gerdin (2006, Lem. 2.3), if the pair \( (E_D, A_D) \) in (9) is regular, there exist nonsingular matrices \( T, S \in \mathbb{C}^{m \times m} \) such that the transformation

\[
T E_D S S^{-1} x_D(t) = T A_D S S^{-1} x_D(t)
\]

(10a)

gives the system in standard form:

\[
\begin{bmatrix} I & 0 \\ N & I \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},
\]

(10b)

where \( N \in \mathbb{C}^{(m-n) \times (m-n)} \) is nilpotent, \( A \in \mathbb{C}^{m \times m} \) is an identity matrix of compatible dimensions and \( x(t) = S^{-1} x_D(t) \).

The matrices in (10) are computed as follows (Gerdin, 2006; Schön et al., 2003):

(i) Compute the generalised Schur form of the matrix pencil \( \lambda E_D - A_D \) so that:

\[
T_1 (\lambda E_D - A_D) S_1 = \lambda \left[ \begin{array}{c} E_1 \\ E_2 \end{array} \right] + \left[ \begin{array}{c} A_1 \ A_2 \\ A_3 \ A_4 \end{array} \right],
\]

(11)

where \( T_1 \) and \( S_1 \) are unitary matrices i.e. \( T_1^* T_1 = T_1 T_1^* = I \), and are not to be confused with \( T \) and \( S \) in (10a). The generalised eigenvalues should be sorted so that the diagonal elements of \( E_1 \) contain only non-zero elements. Computation of the generalised Schur form and the subsequent reordering can be accomplished using MATLAB’s \texttt{dlyap} routine or more efficiently by the algorithm described in Shahzad et al. (2011).

(ii) Solve the following coupled Sylvester equation to obtain the matrices \( L \) and \( R \):

\[
\begin{align*}
E_1 R + L E_3 &= -E_2, \\
A_1 R + L A_3 &= -A_2.
\end{align*}
\]

(12a)

(12b)

The solution to (12) can be obtained by solving for \( L \) in:

\[
A_1 E_1^{-1} L E_3 A_3^{-1} - L - (A_2 - A_1 E_1^{-1} E_2) A_3^{-1} = 0,
\]

(13a)

and substituting to obtain \( R \):

\[
R = -E_1^{-1} E_2 - E_1^{-1} L E_3.
\]

(13b)

(13a) is a type of Sylvester equation and can be computed in MATLAB using the \texttt{dlyap} routine, or more efficiently by the algorithm described in Shahzad et al. (2011).

(iii) Form the matrices in (10) as follows:

\[
T = \begin{bmatrix} E_1^{-1} \\ A_1^{-1} \end{bmatrix} \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} T_1, \quad S = S_1 \begin{bmatrix} R \\ 0 \end{bmatrix},
\]

(14a)

\[
A = E_1^{-1} A_1, \quad N = A_3^{-1} E_3.
\]

(14b)
Thus, the autonomous state-space system $\dot{x}(t) = Ax(t)$ is obtained from the top row of (10b). Temporal discretisation of the resulting system yields the following discrete-time system:

$$x_{k+1} = \tilde{A}x_k,$$

where $x_k$ is the state of the system at time $t_k$ and $\tilde{A} := e^{AT}$, where $T_k = 2$ is the sample period. The next section augments this system with a disturbance model and measurements of the velocity gradients at the wall, to produce a system of the form:

$$x_{k+1} = \tilde{A}x_k + \tilde{B}w_k,$$

$$y_k = \tilde{C}x_k + \tilde{D}w_k + v_k.

Again, the approach will be to model in terms of the states of the descriptor system, before transforming to those of (15).

6. DISTURBANCE MODEL AND WALL-SHEAR MEASUREMENTS

It is assumed, for convenience, that the states $x_{D_k}$ and the measurements $y_k$ of the system are perturbed by zero-mean, Gaussian, uncorrelated white noise sequences, $w_k$ and $v_k$, with covariances $Q_k$ and $R_k$, respectively. In theory, an estimate of $Q_k$ can be obtained from measurement data (Rajamani and Rawlings, 2009). However, obtaining good estimates via this approach would have required significantly more time samples than were available. Therefore, a process noise model $\tilde{B}_D \in \mathbb{C}^{m \times g}$, $w_k \in \mathbb{C}^g$, was directly obtained from the DNS data. The forcing term $\tilde{B}$ in (16a) is obtained from the following:

$$\begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} = T \tilde{B}_D,$$

where $\tilde{G} \in \mathbb{C}^{(n-m) \times g}$ and $T$ is the matrix in (10a). With respect to (16b), spanwise Fourier transformed DNS data (at streamwise locations $x_1$, $x_2$ and $x_3$) provides 'measurements', taken at the plate surface, of the wall-normal gradients of the streamwise and spanwise velocities:

$$\begin{bmatrix} \frac{\partial u_k}{\partial y} \\ \frac{\partial v_k}{\partial y} \end{bmatrix} = \begin{bmatrix} Y_{1_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{1_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_{D_k},$$

$$y_k = \tilde{C}_D \begin{bmatrix} \frac{\partial u_k}{\partial y} \\ \frac{\partial v_k}{\partial y} \end{bmatrix} + v_k = \begin{bmatrix} \tilde{H} \\ \tilde{1} \end{bmatrix} x_k + v_k = \tilde{C}_D x_k + \tilde{D}w_k + v_k,

where $\tilde{H} \in \mathbb{C}^{(p-n) \times (m-n)}$ and $D := -\tilde{H}G$ (Schön et al., 2003). With all terms in (16) defined, a steady-state Kalman filter was synthesised for the system. This filter produces estimates $\hat{x}_k$, but it is straightforward to interpret these states in terms of the velocities and pressures in $\hat{x}_{D_k}$ via the transformation:

$$\hat{x}_{D_k} = S \begin{bmatrix} \hat{x}_k \\ z_k \end{bmatrix}.

7. RESULTS AND DISCUSSION

The streamwise velocity perturbation fields above each of the sensing locations are shown in Figure 3 for the initial and final times of the available data. Clearly, the Kalman filter, using only wall shear information and a low-order, linear model is reconstructing the characteristic streaky disturbances within the transient growth region of the boundary layer. It should be noted that the estimated streaks are of approximately the correct shape, location and magnitude, despite uncertainty in the initial conditions. Quantitatively, the estimates differ slightly from the DNS data, but it is worth emphasising that the latter (obtained from a nonlinear model) took approximately one week of wall-clock time on 96 processors of the HLRB-IISupercomputer to compute. In contrast, and owing to the relative simplicity of the estimation model, the velocity estimates (right-hand plots) could easily be generated in real time by a microcontroller connected to wall-shear sensors. The quality of the estimates is dependent on factors such as sample rate, level of spatial discretisation and noise model. Assessing how good the model is clearly depends on one’s objectives. With respect to closed-loop control, given that feedback reduces the effects of uncertainties such as plant/model mismatch, then depending on the particular closed-loop performance specifications (Jones and Kerrigan, 2010), a model such as the one employed to produce the estimates in Figure 3 could well prove satisfactory.

8. CONCLUSIONS

Motivated by the problem of aircraft drag control, this paper began with Navier-Stokes Equations and employed a series of modelling approximations to yield a linear, low-order state-space model describing disturbance evolution within the transient growth region of a boundary layer. Based on this model and DNS-based wall shear measurements, a steady-state Kalman filter was synthesised that reproduced the characteristic streaky disturbances that are known precursors of turbulence, and hence increased drag. Such a model could easily be used to...
for closed-loop controller synthesis. Furthermore, it was argued that the numerical method described in this paper, for converting a system of DAEs into an equivalent system of ODEs, significantly reduced the modelling burden by allowing straightforward satisfaction of boundary conditions whilst dispensing with the need for commonly employed parallel-flow assumptions and velocity-vorticity transformations. Finally, it should be noted that control and estimation of fluid flows poses challenging research questions, many of which have not been tackled in this paper. The optimal design and location of sensors and actuators are but two obvious examples.

REFERENCES


Rajamani, M.R. and Rawlings, J.B. (2009). Estimation of the disturbance structure from data using semidefinite program-