On-off Control Design of a Class of Linear Systems via Lipschitz Switching Surfaces*  

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Abstract: In order to achieve more flexibility for state feedback control design, a new design method based on Lipschitz switching surfaces is proposed for a class of n-th order linear systems subject to on-off control input in this paper. The Lipschitz switching surface is piecewise smooth and developed by the combination of several available smooth subsurfaces due to the partitioned domains of interest. Due to the discontinuity of the on-off input, Filippov’s differential inclusion is adopted to describe the dynamics of trajectories of the closed-loop system. The globally asymptotic stability of the on-off control system with Lipschitz switching surfaces is illustrated by means of nonsmooth analysis and LaSalle’s invariant principle for nonsmooth systems.

1. INTRODUCTION

On-off control is frequently used due to its simplicity in the mechanism of actuator and high efficiency of energy consumption. In the fields of power electronics, spacecrafts, rockets and satellites, it involves great concerns on the subject. For systems actuated by this kind of on-off input, system design and analysis up to now are mostly performed by approximation methods which limit the attainable performance of the control system.

On-off control for a class of linear system with a linear part \( G \) and a nonlinear part \( \Phi \) is studied in the state feedback point of view(Akasaka and Liu [2007]). As shown in Fig.1, \( G \) and \( \Phi \) indicate the plant and the actuator respectively. They are connected to each other by the state feedback \( s(x) \). The actuator is modeled as an ideal discontinuous relay with deadzone. Then the global stability condition of the closed-loop system is derived, which is characterized by PDMI(partial differential matrix inequality) about the feedback law. The obtained feedback law is explicitly obtained by solving the PDMI analytically, which is constructed by a linear part and an explicit integral part including a nonlinear kernel. Based on the same approach, nonlinear state and output feedback control design problems are derived for linear systems with input saturation(Akasaka and Liu [2008]). It should be noted that the solution of PDMI is constructive, which means that the switching surface with on-off input for the closed-loop system is not unique. A drawback, however, is that the feedback law derived by this method is a smooth function, i.e., the switching surface for the on-off control system is a smooth surface.

The considered n-th order system \( G \) is given by

\[
G : \dot{x} = Ax + bu
\]

where control input \( u \) supplied by the discontinuous on-off actuator \( u = \Phi(\tau) \) is described by

\[
\Phi(\tau) = \begin{cases} 
\gamma, & \tau > \epsilon \\
0, & |\tau| < \epsilon \\
-\gamma, & \tau < -\epsilon 
\end{cases} \tag{2}
\]

Due to the on-off input, two assumptions are made for the purpose of globally asymptotic stability.

A.1 \((A, b)\) is controllable;

A.2 There exists \( P = P^T \geq 0, P \neq 0 \) such that

\[
A^T P + PA \leq 0 \tag{3}
\]

Close attention is being paid to nonsmooth control design methods (Cortés[2008], Clarke et al. [1998], Bernardo et al. [2007]). Sliding mode control with nonsmooth sliding switching surface is studied very recently. Wang and Lin [1999] designed prespecified trajectory, which is composed of several nonsocial segments in the phase plane and is regarded as the piecewise sliding surface. Based on the geometric properties of a general Lipschitz continuous surface, Zheng et al. [2009] developed a sliding mode controller for a class of nonlinear SISO systems, of which the switching surface is only linear Lipschitz. Yao et al. [2009] considered combined sliding surface for sliding mode
control design by proving that the available switching surface is not unique for the fixed reaching phase design, and proposed a new sliding surface, which is piecewise smooth.

With the above motivations, in order to achieve more flexibility for state feedback control design, the main purpose of this paper is to present a control design method for an on-off control system via constructed combined Lipschitz continuous switching surfaces. Since there exist a set of smooth switching surfaces can be chosen as candidates for the closed-loop system, it is natural to construct a new switching surface by means of their combination. According to our design rules, the whole state space is divided into $r$ subspaces and we choose $r$ available smooth subsurfaces. Under some assumptions, a new Lipschitz continuous surface can be constructed by combining these subsurfaces. Since the system is actuated by on-off input, which is discontinuous, the dynamics of the closed-loop is expressed in the sense of Filippov’s differential inclusion [Filippov [1988]]. Furthermore, according to the differential equation with discontinuous right-hand side and constructed nonsmooth switching surface discussed in the paper, a mathematic framework based on nonsmooth analysis [Clarke et al. [1998], Clarke [1983]] and nonsmooth Lyapunov method [Shevitz and Paden [1994], Paden and Shevitz [1987]] is introduced to illustrate the globally asymptotic stability.

The rest of the paper is organized as follows. In Section 2, we give a brief review on some mathematical preliminaries used throughout the paper. The main results, including the methodology to construct the Lipschitz switching surfaces and the globally asymptotical stable condition, are proposed in Section 3. A numerical example in 3-dimensional case is illustrated to verify the design method in Section 4. Section 5 concludes the whole paper.

2. MATHEMATICAL PRELIMINARIES

In the sequel, some definitions and notations used throughout the paper are presented. We constrain our topic in $\mathbb{R}^n$. For a subset $\Omega \subset \mathbb{R}^n$, the interior and the boundary of $\Omega$ are denoted by $Int(\Omega)$ and $Bd(\Omega)$, respectively. The open ball of radius $\delta$ centered at $x$ is defined as $B_\delta(x) = \{y \in \mathbb{R}^n : ||y - x|| < \delta\}$, where $|| \cdot ||$ denotes the Euclidean norm. We denote by $f(\Omega, t)$ the set of value of the function $f(x, t)$ for $x \in \Omega$ and fixed $t$, that is $f(\Omega, t) = \{f(x, t) : x \in \Omega\}$. $A^{-1}$ denotes the generalized inverse of an $n \times n$ matrix $A$, i.e., $AA^{-1}A = A$. Ker$A$ denotes the null space of $A$.

2.1 Lipschitz Surfaces

Definition 1. Let $\Omega$ be an open and bounded subset of $\mathbb{R}^n$, $\Omega$ is said to have Lipschitz boundary, and is called a Lipschitz set, if, for every point $x \in Bd(\Omega)$, there corresponds a local coordinate system $Y$ in $B_\delta(x)$ and a map $\phi_Y : \mathbb{R}^{n-1} \to \mathbb{R}$ defined on $Y$ such that

1. $\phi_Y$ is Lipschitz continuous;
2. $B_\delta(x) \cap \Omega = B_\delta(x) \cap \{(y_n : y_n > \phi_Y(y'))\}$;
3. $B_\delta(x) \cap Bd(\Omega) = B_\delta(x) \cap \{(y_n : y_n = \phi_Y(y'))\}$.

where $y' = [y_1 y_2 \cdots y_{n-1}]^T$, $y = [y' y_n]^T$ is the coordinate of $Y$.

The boundary of $\Omega$, i.e., $S = Bd(\Omega)$, is defined to be a Lipschitz surface. In other words, the boundary of a Lipschitz domain can locally be expressed as the graph of Lipschitz continuous function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ after an appropriate rotation and translation. However, a Lipschitz continuous surface needs not to be smooth. Without loss of generality, the Lipschitz surface $S$ concerned in this paper is supposed to satisfy the following assumptions.

A.3 The coordinate system $Y$ is the original one, in other words, it is not necessary to find a new coordinate system in Definition 1;

A.4 $\phi_Y$ is not only defined on $B_\delta(x)$ but also on the whole state space;

A.5 The whole state space is divided by $S$ into three simply connected partitions, i.e., $S$ and two sides of $S$.

Thus a Lipschitz surface $S$ can be usually expressed as $S = \{x : s(x) = 0, x \in \mathbb{R}^n\}$ (4) where $s(x) : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. Two Lipschitz domains can be derived as follows

$S^+ = \{x : s(x) > 0, x \in \mathbb{R}^n\}$ (5)

$S^- = \{x : s(x) < 0, x \in \mathbb{R}^n\}$ (6)

2.2 Filippov Solution

Consider the system in form

$$\sum_{\Omega} : \dot{x} = f(x, t) \quad x(t_0) = x_0$$ (7)

where $x = [x_1 x_2 \ldots x_n]^T \in \mathbb{R}^n$ denotes the state variable of the system, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the vector field, real-valued measurable and essentially locally bounded.

Definition 2. Consider the differential equation given in (7), whose right-hand side $f(x, t)$ may be discontinuous to $x$, a time dependant vector function $x(t)$ is called a solution in the sense of Filippov on $[t_0, t_1]$ if $x(t)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$

$$\dot{x} \in K[f](x, t)$$ (8)

where

$$K[f](x, t) = \bigcap_{\delta > 0} \bigcap_{\mu S = 0} \partial f(B_\delta(x) - S, t)$$ (9)

and $\bigcap_{\delta > 0} \bigcap_{\mu S = 0}$ denotes the interaction over all sets $S$ of Lebesgue measure zero, $\partial f$ denotes the closure convex hull, $K[f](x, t)$ denotes the Filippov set-valued map of $f(x, t)$.

2.3 Fundamentals of Nonsmooth Analysis

With the notations of Filippov solution [Filippov [1988]] and Clarke’s generalized gradient [Clarke [1983]], nonsmooth version of LaSalle’s theorem is given in the sequel.

Lemma 1 (Chain Rule). Let $x(t)$ be a Filippov solution of

$$\sum_{\Omega} : \dot{x}(t) \in V(x, t)$$ (10)

on an interval containing $t$ and $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz and in addition, regular function. Then $V(x, t)$ is absolutely continuous, $(d/dt)V(x, t)$ exists almost everywhere and

$$\frac{d}{dt}V(x, t) \in \dot{V}(x, t)$$ (10)
where
\[
\dot{\tilde{V}}(x, t) := \bigcap_{\xi \in \partial V(x,t)} \xi^T \left(K[f](x, t) 1 \right)
\]  
(11)
where \(\partial V(x, t)\) denotes the Clarke’s generalized gradient of \(V\) at \((x, t)\).

**Lemma 2**(LaSalle’s Invariant Principle). Let \(\Omega\) be a compact set such that every Filippov solution to the autonomous system \(\Sigma\) starting in \(\Omega\) is unique and remains in \(\Omega\) for all \(t \geq t_0\). Let \(V: \Omega \rightarrow \mathbb{R}\) be a time-independent regular function such that \(v \leq 0\) for all \(v \in \tilde{V}\) (if \(\tilde{V}\) is empty set then this is trivially satisfied). Define \(\mathcal{E} = \{x \in \Omega| 0 \in \tilde{V}\}\). Then every trajectory in \(\Omega\) converges to the largest invariant set, \(\mathcal{M}\), in the closure of \(\mathcal{E}\).

### 2.4 Analytical Solution of PDMI

**Lemma 3.** Consider a PDMI as follows
\[
\begin{align*}
A^TP + PA + Pb & + A^T \left(\frac{\partial s(x)}{\partial x}\right)A + b^T \left(\frac{\partial s(x)}{\partial x}\right)b \leq 0 \\
\end{align*}
\]  
(12)
where \(s(x)\) is a \(C^1\) function with \(s(0) = 0\), \(P = P^T \geq 0\) and \(Q\) is a semi-positive definite symmetric matrix satisfying the Lyapunov equation
\[
A^TP + PA = -Q
\]  
(13)
Let \(g(\cdot), h(\cdot) \in \mathbb{R}\) be piecewise \(C^1\) functions, and \(k, \tilde{k}, l, \tilde{l} \in \mathbb{R}^n\) be constant vectors. Then, \(s(x)\) given below satisfies the PDMI (12).
\[
s(x) = -[(b^TP + k^TQ)A^g + \tilde{l}^T(I - AA^g)]x - \int_{0}^{\alpha(x)} g(\upsilon)d\upsilon - \int_{0}^{\beta(x)} h(\omega)d\omega
\]  
(14)
where \(\alpha(x), \beta(x)\) are given by
\[
\alpha(x) = \tilde{k}^TQA^g x
\]  
(15)
and \(\beta(x) = \tilde{l}^T(I - AA^g)x\)
(16)
and the integral kernels \(g(\cdot), h(\cdot)\) as well as the constant vectors \(k, \tilde{k}, l, \tilde{l}\) satisfy the inequality below:
\[
\begin{align*}
(b^TP + k^TQ)A^g b & - \tilde{l}^T(I - AA^g)b \\
- \tilde{k}^TQA^gb \cdot g(\alpha(x)) - \tilde{l}^T(I - AA^g)b \cdot h(\beta(x)) & + \frac{1}{2}(k + \tilde{k}g(\alpha(x)))^T Q(k + \tilde{k}g(\alpha(x))) \leq 0
\end{align*}
\]  
(17)

**Remark 1.** It is worth noting that the structure of the explicit formula of \(s(x)\) varies with the property of the system (1). If \(A\) is nonsingular, \(I - A^g A = I - AA^g = 0\). Thus, the third term of (14) is vanished and we lose a degree of freedom, i.e., the design parameter \(h(\cdot)\). On the other hand, if \(A\) is singular, i.e., \(A\) has at least an eigenvalue at the origin, \(h(\cdot)\) increases the freedom of the design.

### 3. MAIN RESULTS

Due to the problem formulated in Section 1 with on-off control input and state feedback, how to design a switching condition is by no means a main problem. In this paper, we consider the switching condition to be two switching surfaces. The main contributions in this section are two folds: the methodology to construct Lipschitz switching surfaces is derived; and the globally asymptotic stability is illustrated for the system with on-off control input.

Consider the \(n\)-th order system (1) with the controllable canonical form, i.e.,
\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
-a_1 & -a_2 & \cdots & -a_n
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
\vdots \\
\vdots \\
1
\end{bmatrix}
\]  
(18)
where \(K[\Phi](s(x))\) denotes the Filippov set-valued map of \(\Phi(s(x))\)
\[
\begin{align*}
\gamma_{sgn}(s(x)), & \quad \text{if } |s(x)| > \epsilon \\
0, & \quad \text{if } |s(x)| < \epsilon \\
\begin{cases}
\mathbb{C}[0, \gamma], & \quad \text{if } s(x) = \epsilon, \epsilon \neq 0 \\
\mathbb{C}[-\gamma, 0], & \quad \text{if } s(x) = -\epsilon, \epsilon \neq 0 \\
\mathbb{C}[-\gamma, \gamma], & \quad \text{if } s(x) = 0, \epsilon = 0
\end{cases}
\end{align*}
\]  
(19)

### 3.1 The Construction of Lipschitz Switching Surfaces

By Akasaka and Liu [2007], the switching surface of the on-off control system is a smooth surface, and it is derived by solving the PDMI (12). Since the analytical solution of PDMI is not unique, thus the available candidate switching surface is not unique, it is natural to construct a new switching surface by means of their combination. Without loss of generality, the whole state space is divided into \(r\) domains, i.e., \(D_i \subseteq \mathbb{R}^n, i = 1, \ldots, r\), such that
\[
A.6 \cup_{i=1}^r D_i = \mathbb{R}^n;
\]

**A.7** They have pairwise disjoint interior, i.e., \(\text{Int}\{D_i\} \cap \text{Int}\{D_j\} = \emptyset\) for any \(i \neq j, i, j = 1, \ldots, r\);

**A.8** \(S = \{x : s(x) = 0, x \in \mathbb{R}^n\} = \cup_{i=1}^r S_i\) defines a Lipschitz surface in the state space, where \(S_i = \{x : s_i(x) = 0, x \in D_i, i = 1, \ldots, r\}\) denotes the subsurface of \(S\) and \(s_i(x)\) satisfies Lemma 3.

In the sequel, an examples in 3-dimensional case is presented to illustrate how to construct a combined Lipschitz surface by the method developed in this paper. Fig.2(a) illustrates an example to partition the 3-dimensional space into three separated domains with three half planes as their boundaries. Fig.2(b) shows the projection of the boundaries and partitioned domains on \(x_1-x_2\) plane. As a result, which is shown in Fig.2(c), a 3-dimensional space can be simply divided into two separated domains with the corresponding Lipschitz surface \(S\), combined by three half planes \(S_1, S_2\) and \(S_3\), which are laying in the three separated domains respectively.

Based on the combined Lipschitz surface, two Lipschitz switching surfaces for on-off control design are developed as follows, of which the threshold value \(\pm \epsilon\) are considered.
The whole state space is divided by $\hat{S}$ and $\tilde{S}$ into five separated simple connected domains, i.e., $\hat{S}^+, \hat{S}^-, \hat{S}^- \cap \hat{S}^+$, $\tilde{S}$ and $\hat{S}$. According to the system formulated in (1), the actuator is implemented by

$$u = \begin{cases} 
\gamma, & x \in \hat{S}^+ \\
\gamma, & x \in \hat{S}^+ \cap \tilde{S}^+ \\
-\gamma, & x \in \hat{S}^- \\
0, & x \in \hat{S}^- \cap \tilde{S}^+ \\
-\gamma, & x \in \tilde{S} 
\end{cases}$$

Remark 2. In 3-dimensional case, the number of partitioned domains $r$ is not limited to three, e.g., $r = 2$ is allowed. In fact, the number of the separated domains can be freely chosen if all of the assumptions 3–8 are satisfied.

3.2 Stability Analysis

A. Equilibria Set

The equilibria set of (18) consists of all points at which $\dot{x} \equiv 0$ and can be indicated by

$$\mathcal{N} = \{ x \in \mathbb{R}^n : 0 \in Ax + bK[\Phi](s(x)) \}$$

Due to the closed-loop dynamics of the system, any equilibrium point must have the form $x = x_1, 0, \cdots, 0, 0$ if $x_1$ is the only $x_1$ needs to be specified.

In the sense of Filippov, the equilibria set of system (18) is calculated as follows

\begin{itemize}
  \item [i)] $a_1 \neq 0$, i.e., Ker $A = \{ 0 \}$
  \item [ii)] $a_1 \neq 0$, i.e., Ker $A = \{ 0 \}$
\end{itemize}

$$\mathcal{N} = \{ x \in \mathbb{R}^n : x_2 = \ldots = x_n = 0, |s(x)| \leq \epsilon \}$$

Note that the state vectors satisfying $|s(x)| = \epsilon$ are also considered as equilibria in Filippov framework.

\[ 10896 \]
\[ \dot{V}(x) = (2P_i x + 2\Phi(s_i)\frac{\partial s_i}{\partial x}^T)(Ax + bK[\Phi]) \]
\[ = x^T (A^T P_i + P_i A)x + b^T P_i x bK[\Phi] \]
\[ + 2\frac{\partial s_i}{\partial x} (Ax + bK[\Phi]) \]
\[ = z^T \begin{bmatrix}
A^T P_i + P_i A & P_i b + A^T \left( \frac{\partial s_i}{\partial x} \right)^T \\
b^T P_i + \frac{\partial s_i}{\partial x} A & \frac{\partial s_i}{\partial x} b + b^T \left( \frac{\partial s_i}{\partial x} \right)^T
\end{bmatrix} z \]
\[ (31) \]

where \( z = [x^T K[\Phi]]^T \). According to Lemma 3, \( s_i(x) \) is an analytical solution to the PDMI (12). Thus, \( \dot{V}(x) \leq 0 \).

If \( x \) is located at the common boundary of two domains, \( V(x) \) is nonsmooth, the generalized gradient is not equal to the classical gradient. Without loss of generality, we take account of the boundary of \( D_i \) and \( D_j \), where \( i \neq j, i, j = 1, \ldots, r \), and \( x \in Bd\{D_i\} \cap Bd\{D_j\} \), the generalized gradient can be represented as

\[ \partial V(x) = \lambda(2P_i x + 2\Phi(s_i(x))\left( \frac{\partial s_i(x)}{\partial x} \right)^T) \]
\[ + (1 - \lambda)(2P_j x + 2\Phi(s_j(x))\left( \frac{\partial s_j(x)}{\partial x} \right)^T) \]
\[ (32) \]

where \( \lambda \in [0, 1] \) such that

\[ \dot{V}(x) = \lambda^2(2P_i x + 2\Phi(s_i(x))\left( \frac{\partial s_i(x)}{\partial x} \right)^T) \]
\[ + (1 - \lambda)^2(2P_j x + 2\Phi(s_j(x))\left( \frac{\partial s_j(x)}{\partial x} \right)^T)(Ax + bK[\Phi]) \]
\[ = \lambda^2 z^T \begin{bmatrix}
A^T P_i + P_i A & P_i b + A^T \left( \frac{\partial s_i}{\partial x} \right)^T \\
b^T P_i + \frac{\partial s_i}{\partial x} A & \frac{\partial s_i}{\partial x} b + b^T \left( \frac{\partial s_i}{\partial x} \right)^T
\end{bmatrix} z \]
\[ + (1 - \lambda)^2 z^T \begin{bmatrix}
A^T P_j + P_j A & P_j b + A^T \left( \frac{\partial s_j}{\partial x} \right)^T\\
b^T P_j + \frac{\partial s_j}{\partial x} A & \frac{\partial s_j}{\partial x} b + b^T \left( \frac{\partial s_j}{\partial x} \right)^T
\end{bmatrix} z \]
\[ (33) \]

According to Lemma 3 and \( \lambda \in [0, 1] \), \( s_i(x) \) and \( s_j(x) \) are two analytical solutions to the PDMI (12). Thus, \( \dot{V}(x) \leq 0 \).

By the above argument, we can conclude that \( \dot{V}(x) \leq 0 \) for all \( x \in \mathbb{R}^n \). Obviously, the set of \( x \) satisfying \( \dot{V}(x) \equiv 0 \) is given by

\[ \mathcal{E} = \{ x \in \mathbb{R}^n : Qx = 0 \} \]
\[ (34) \]

According to Lemma 2, the trajectory of the closed-loop system \( x(t) \) converges globally to the largest invariant set \( M \subset \mathcal{E} \cup \mathcal{N} \).

Remark 3. The threshold values \( \epsilon \) and \( -\epsilon \) are not used in the proof of the theorem 1, thus they have no influence on the result. In fact, there exist some on-off control systems in which the forward threshold is different from the backward one.

4. A NUMERICAL EXAMPLE

In order to illustrate the design method, a numerical example is presented in this section. In the simulation, parameters are set as \( \gamma = 1 \) and \( \epsilon = 10 \).

\[ 10897 \]
A1 and A2 are two arbitrary points in the state space, which are laying in $\hat{S}^-$ and $\hat{S}^+$. B1 and B2 are two arbitrary points located on the switching surfaces $\hat{S}$ and $\check{S}$, respectively. C1 and C2 are two arbitrary points in the state space, which are laying in $\hat{S}^+ \cap \check{S}^-$. It is clear that the system is globally asymptotically stable.

5. CONCLUSION

In this paper, a new control design method with Lipschitz switching surfaces for a class of $n$-th on-off control system has been studied. The switching surface is piecewise smooth and constructed by combination of several available smooth subsurfaces due to the partitioned domains of interest. The dynamics of the closed-loop system is described in the sense of Filippov’s differential inclusion. Nonsmooth analysis and LaSalle’s invariant principle for nonsmooth system are presented to illustrate the globally asymptotic stability. A numerical example is provided to show the effectiveness of the proposed approach. The flexibility of the control design for linear systems subject to on-off control input is enhanced by the method illustrated in this paper, further, the achievable performance with Lipschitz switching surfaces should be analyzed in future.

REFERENCES


