Robust Output Stabilization of Time-Delay Nonlinear System*

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Abstract: This paper deals with the output stabilization of time-delay systems with sector-bounded nonlinearity. In this paper we will consider the problem of absolute stability for a class of time-delay systems which can be represented as a feedback connection of a linear dynamical system with unknown parameters and an uncertainty nonlinearity satisfying a sector constraint. For a class of output control algorithms a controller providing output asymptotic stability of equilibrium position is designed.

Keywords: Absolute stability, time-delay systems, stabilization methods, decision feedback, nonlinear control, output regulation, single-input/single-output systems, robust control, robustness.

1. INTRODUCTION

The absolute stability problem, formulated by Lurie and coworkers in the 40’s, has been a well-studied and fruitful area of research, as presented in (Lurie and Postnikov, 1944). Since works of Lurie (Lurie and Postnikov, 1944), interest of researches of control systems has been attracted by structures including linear block and nonlinear feedback static block. It is possible to allocate big enough number of works devoted to solution of problems of nonlinear systems stabilization for a case when output of the nonlinear block is given as control input of the linear block (see the review (Kokotovic and Arcak, 2001)). It is also possible to allocate a block of works (Byrnes and Isidori, 1991; Khalil and Esfandiari, 1993; Lin and Saberi, 1995) in which nonlinear part and static nonlinearity adjust with an input on control. However, approaches (Byrnes and Isidori, 1991; Khalil and Esfandiari, 1993; Lin and Saberi, 1995) are focused on stabilization of systems with nonlinearity, resulted to an input of system, and do not allow one to solve more general problems. Today problems of control of nonlinear systems in which nonlinearity is not coordinated with control are of interest. Among works devoted to these subjects one can allocate papers (Bobtsov, 2005a; Bobtsov and Nikolaev, 2005b; Krstic and Kokotovic, 1996; Praly, 2003; Qian and Lin, 2002) in which similar problem is considered. However, these results mentioned above are delay-independent.

During the last two decades, the problem of stability of linear time-delay systems has been the subject of considerable research efforts. Many significant results have been presented in the literature (see for example (Gu et al., 2003; Richard, 2003)). However little attention has been focused on nonlinear time delayed systems. The problem of stability for nonlinear systems with delay has been studied in (Bliman, 2001; Germani and Manes, 2001; Germani et al., 2003; Ge et al., 2003; Gouaisbaut et al., 2001; He and Wu, 2003; Hua et al., 2002; Ito et al., 2010; Marquez-Martinez and Moog, 2004; Maznec et al., 2008; Moog et al., 2000; Oguchi et al., 2002; Teel, 1998). In paper (Ge et al., 2003) adaptive neural control was presented for a class of strict-feedback nonlinear systems with unknown time delays. For a case of measurability of the state vector (Ge et al., 2003), using appropriate Lyapunov–Krasovskii functionals, the uncertainties of unknown time delays are compensated for such that iterative backstepping design can be carried out. In (Germani et al., 2003) the relationships between the internal state and input dynamics of a controlled nonlinear delay system are studied. An interesting result is that a suitable stability assumption on the internal state dynamics ensures that, when the output is asymptotically driven to zero, both the state and control variables asymptotically decay to zero. In (Oguchi et al., 2002) the input-output linearization problem for retarded non-linear systems is considered, which have time-delays in the state. By using an extension of the Lie derivative for functional differential equations, authors derive coordinates transformation and a static state feedback to obtain linear input-output behaviour for a class of retarded non-linear systems. The obtained coordinates transformation is allowed to contain not only the current value of the state.
variables but also the past values of ones. In (Hua et al., 2002), a robust control problem of a class of nonlinear time delay systems is considered under the case when nonlinear uncertainties are bounded by first-order function. Geometrical method is employed to investigate the control problem of time delay systems in (Germani and Manes, 2001; Marquez-Martinez and Moog, 2004; Moog et al., 2000). Corresponding state feedback controller and output feedback controller are designed, but the strict conditions should be imposed on the investigated systems.

In this paper a robust version of the results on high-gain stabilization of nonlinear systems is extended to a class of time-delay nonlinear systems without matching conditions. The given work, developing approaches obtained in (Bliman, 2001; Germani and Manes, 2001; Germani et al., 2003; Ge et al., 2003; Gouaisbaut et al., 2001; He and Wu, 2003; Hua et al., 2002; Marquez-Martinez and Moog, 2004; Moog et al., 2000; Oguchi et al., 2002), offers new solution of stabilization problem of nonlinear system consisting of structures including a linear block and nonlinear feedback static block. The similar approach may be found in (Furtat and Tsykunov, 2010). Assuming, that parameters of the linear part and delay are unknown, the output is measured (but not its derivatives), and the characteristic of the nonlinear feedback block is unknown, a controller providing asymptotic stability of equilibrium position is designed. In this paper an interesting approach is offered, that does not use the procedure of linearization of nonlinear system, design of observer and iterative backstepping design.

3. CONTROL DESIGN

Let the system (1) be such that transfer function \( \frac{b(p)}{a(p)} \) has relative degree \( \rho = 1 \) and polynomial \( b(p) \) is Hurwitz. Choose control of the following form (Bobtsov and Nikolaev, 2005; Fradkov, 1974; Fradkov, 2003):

\[
u(t) = -\mu y(t) + v(t), \tag{3}\]

where \( v(t) \) is an additional input and parameter \( \mu > 0 \).

Lemma (Bobtsov, 2005; Bobtsov and Nikolaev, 2005; Fradkov, 1974; Fradkov, 2003; Kokotovic and Arcak, 2001).

There exists some positive number \( \mu_0 \), such that for any \( \mu \geq \mu_0 \) the system

\[
y(t) = \frac{b(p)}{a(p) + \mu b(p)} v(t) + \frac{c(p)}{a(p) + \mu b(p)} \phi(t) \text{ has the SPR (strictly positive real) transfer function}
\]

\[
H(p) = \frac{b(p)}{a(p) + \mu b(p)}.
\]

Let \( \rho > 1 \) and the derivatives of output \( y(t) \) be measured. Choose control in the following form

\[
u(t) = \alpha(p) \bar{u}(t), \tag{4}\]

where any polynomial \( \alpha(p) \) of \( \rho - 1 \) degree is Hurwitz and \( \bar{u}(t) \) is a new variable.

Then we can rewrite the model (1) in the following form:

\[
y(t) = \frac{b(p)\alpha(p)}{a(p)} \bar{u}(t) + \frac{c(p)}{a(p)} \phi(t), \tag{5}\]

where polynomial \( b(p)\alpha(p) \) is Hurwitz and the relative degree of the transfer function \( \frac{b(p)\alpha(p)}{a(p)} \) is \( \rho = 1 \).

Choose \( \bar{u}(t) \) according to the equation (3)

\[
\bar{u}(t) = -\mu y(t) + v(t). \tag{6}\]

Substituting (6) into the equation (5), we obtain closed-loop system

\[
y(t) = \frac{b(p)\alpha(p)}{a(p) + \mu b(p)\alpha(p)} v(t) + \frac{c(p)}{a(p) + \mu b(p)\alpha(p)} \phi(t). \tag{7}\]

Then by using lemma for some \( \mu > \mu_0 \) it is easy to see that the following transfer function is SPR

\[
W(p) = \frac{b(p)\alpha(p)}{a(p) + \mu b(p)\alpha(p)}. \tag{7}\]

However control of the form (4), (6) can not be applied because it is impossible to measure derivatives of function.
Choose the control
\[ u(t) = -\alpha(p)(\mu + \kappa)\dot{y}(t), \] (8)
where number \( \mu \) and polynomial \( \alpha(p) \) are such that the transfer function (7) is SPR, the positive parameter \( \kappa \) is used for compensation of the uncertainty \( \varphi(y(t-\tau)) \) (see proof of the theorem, inequality (24)) and the function \( \hat{y}(t) \) is the estimation of output \( y(t) \). The function \( \hat{y}(t) \) is calculated according to the following algorithm
\[
\begin{aligned}
\dot{\xi}_1(t) &= \sigma \xi_2(t), \\
\dot{\xi}_2(t) &= \sigma \xi_3(t), \\
\vdots \\
\dot{\xi}_{p-1}(t) &= \sigma(-k_1\xi_1(t) - \cdots - k_{p-1}\xi_{p-1}(t) + k_1y(t)), \\
\dot{y}(t) &= \xi_3(t),
\end{aligned}
\] (9)
where number \( \sigma > \mu + \kappa \) (see proof of the theorem, inequality (23)) and parameters \( k_i \) are calculated for the system (9) to be asymptotically stable.

It is obvious, that the control (8) – (10) is technically possible as contains known or measurable signals. Substituting (8) into equation (1), we obtain
\[ y(t) = \frac{b(p)}{a(p)}[-\alpha(p)(\mu + \kappa)\dot{y}(t)] + \frac{c(p)}{a(p)}\varphi(t), \]
\[ = \frac{b(p)}{a(p)}[-\alpha(p)(\mu + \kappa)y(t)] + \alpha(p)(\mu + \kappa)\varepsilon(t) + \frac{c(p)}{a(p)}\varphi(t), \] (11)
where the error \( \varepsilon(t) = y(t) - \dot{y}(t) \).

After simple transformations, for model (11) we have
\[ a(p)y(t) + \mu\alpha(p)b(p)y(t) = b(p)\alpha(p)(\mu + \kappa)\varepsilon(t) - ky(t) + c(p)\varphi(t) \]
and
\[ y(t) = \frac{b(p)\alpha(p)}{a(p) + \mu b(p)\alpha(p)}[-ky(t) + (\mu + \kappa)\varepsilon(t)] + \frac{c(p)}{a(p) + \mu b(p)\alpha(p)}\varphi(t), \] (12)
where transfer function \( W(p) = \frac{b(p)\alpha(p)}{a(p) + \mu b(p)\alpha(p)} \) is SPR (see equation (7)).

Let us present model (12) in the form
\[ \dot{x}(t) = Ax(t) + B(-ky(t) + (\mu + \kappa)\varepsilon(t)) + q\varphi(t), \] (13)
\[ y(t) = c^T x(t), \] (14)
where \( x(t) \in R^n \) is a state vector of system (13); \( A, B, q \) and \( c \) are appropriate matrices of transition from model (12) to model (13), (14).

Since transfer function \( W(p) \) is SPR then
\[ A^T P + PA = -R, \; PB = c, \] (15)
where \( R = R^T > 0 \) and parameters of matrix \( Q_i \) depend on \( \mu \) and do not depend on \( \kappa \).

Let us rewrite model (9), (10) in the form
\[ \dot{\xi}(t) = \sigma(\Gamma \xi(t) + dk_1y(t)), \]
\[ \dot{y}(t) = h^T \xi(t), \]
\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-k_1 & -k_2 & -k_3 & \cdots & -k_{p-1}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix},
\]
\[ d = 0 \text{ and } h^T = [1 \; 0 \; 0 \; \cdots \; 0]. \]
Consider vector
\[ \eta(t) = hy(t) - \dot{\xi}(t), \]
then by force of structure of vector \( h \) error \( \varepsilon(t) \) will become
\[ \varepsilon(t) = y(t) - \dot{y}(t) = h^T h y(t) - h^T \dot{\xi}(t) = h^T (hy(t) - \dot{\xi}(t)) = h^T \eta(t). \]

For derivative of \( \eta(t) \) we obtain
\[ \dot{\eta}(t) = h\dot{y}(t) - \sigma(\Gamma(\eta(t) - \eta(t))) + dk_1y(t)) = h\dot{y}(t) + \sigma(\Gamma \eta(t) - \sigma dk_1 + \Gamma h) y(t). \]
Since \( dk_1 = -\Gamma h \) (can be checked by substitution), then
\[ \dot{\eta}(t) = h\dot{y}(t) + \sigma(\Gamma \eta(t)), \; \varepsilon(t) = h^T \eta(t), \] (16)
where matrix \( \Gamma \) is Hurwitz by force of calculated parameters \( k_i \) of system (9) and
\[ \Gamma^T N + N \Gamma = -M, \] (17)
where \( N = N^T > 0, \; M = M^T > 0. \)

**Theorem.** Consider the nonlinear system (13), (14), (16). Let number \( \rho = n - m \geq 1 \) and unknown function \( \alpha(t) = \varphi(y(t - \tau)) \) be such that:
\[ |\varphi(y(t - \tau))| \leq C_0|y(t - \tau)| \text{ for all } y(t - \tau), \]
where \( \tau > 0 \) is the constant delay, \( y(\emptyset) = \varphi(\emptyset) \) for...
\[ \forall \theta \in [-\tau,0) \text{ and number } C_0 > 0 \text{ is unknown. There exist such parameters } \sigma_0 > 0 \text{ and } \kappa_0 > 0 \text{ that for all } \sigma \geq \sigma_0 \text{ and } \kappa \geq \kappa_0, \text{ the nonlinear system (13), (14), (16) is asymptotic stable.} \]

**Proof.** Choose a Lyapunov-Krasovskii functional candidate as

\[ V(t) = x^T(t)Px(t) + \eta^T(t)N\eta(t) + \kappa \int_{t-\tau}^{t} y^2(\theta)d\theta. \]  \hfill (18)

Differentiating (18) along the trajectories of (16), we obtain

\[ \dot{V}(t) = x^T(t)(A^T + PA)x(t) + 2(\mu + \kappa)x^T(t)PBh^T\eta(t) + \kappa \eta^2(t) - \kappa \eta^2(t - \tau) + 2x^T(t)P\omega(t) - 2x^T(t)PB\omega(t) + \delta x^T(t)P\omega(t) + \delta x^T(t)PB\omega(t) \]

Substituting (21) into the inequality (20), we obtain

\[ \dot{V}(t) \leq -x^T(t)(Qx(t) - \sigma\eta^2(t)M\eta(t)) + \delta x^T(t)P\omega(t) + \delta x^T(t)PB\omega(t) \]

Let number \( \sigma \) be such that the following ratio is executed

\[ \eta \leq \sigma \]

Substituting (23) into the inequality (22), we obtain

\[ \dot{V}(t) \leq -x^T(t)(Qx(t) - \sigma\eta^2(t)M\eta(t)) + \delta x^T(t)P\omega(t) + \delta x^T(t)PB\omega(t) \]

where the number \( \delta > 0 \). Let the number \( 0 < \delta < 0.5 \) be such that

\[ -R + \delta I + (\delta \mu + 2\delta \kappa - \kappa)PBB^TP + \delta Pq\eta^2(t)P \leq -Q_\eta \eta. \]  \hfill (21)

We note that inequality (23) is executed for all \( \sigma \geq \sigma_0 \) where \( \sigma_0 \) corresponds to equality in (23). Substituting (23) into the inequality (22), we obtain

\[ \dot{V}(t) \leq -x^T(t)(Qx(t) - \sigma\eta^2(t)M\eta(t)) + \delta x^T(t)P\omega(t) + \delta x^T(t)PB\omega(t) \]

Let number \( \kappa \) be such that

\[ \kappa \geq C_0^2(\kappa^{-1} + \delta^{-1}) \].  \hfill (24)

It is straightforward to show that there exists such a positive number \( \kappa_0 \) and inequality (24) is executed for all \( \kappa \geq \kappa_0 \). Then we have

\[ \dot{V}(t) \leq -x^T(t)(Qx(t) - \sigma\eta^2(t)M\eta(t)) + \delta x^T(t)P\omega(t) + \delta x^T(t)PB\omega(t) \]

From the expression (25) follows asymptotic stability of system (13), (14), (16).

**Remark.** In some cases, the problem of choosing the parameters \( \kappa, \mu \), and \( \sigma \) of regulator (8), (9) which satisfy theorem can arise (see (21), (23) and (24)). This choice presents no appreciable difficulties for known polynomials \( a(p), b(p) \) and \( c(p) \) of the plant (1) and also for a definite number \( C_0 \). However, if the parameters of the plant (1) are unknown, the problem of calculating \( \kappa, \mu \), and \( \sigma \) may
prove to be problematic. As was demonstrated by theorem, if condition (23) is met, then there exists a number \( \sigma > \mu + \kappa \) such that \( \lim_{t \to \infty} |y(t)| = 0 \). A possible variant of adjustment of \( \kappa, \mu \) and \( \sigma \) lies in increasing them until the following goal condition is met

\[
|y(t)| < \delta_0 \quad \text{for} \quad t \geq t_1,
\]

where the number \( \delta_0 \) is set by the designer of the control system. This concept may be realized by using the following adjustment algorithm

\[
\tilde{k}(t) = \int_{0}^{t} \lambda(\theta)d\theta,
\]

where \( \tilde{k} = k + \mu \) and the function \( \lambda(t) \) is as follows

\[
\lambda(t) = \begin{cases} 
\lambda_0 & \text{for} \quad |y(t)| > \delta_0 \\
0 & \text{for} \quad |y(t)| \leq \delta_0,
\end{cases}
\]

where the number \( \lambda_0 > 0 \).

We take \( \sigma \) as follows

\[
\sigma = \sigma_0 \tilde{k}^2,
\]

where \( \sigma_0 > 0 \).

Obviously, for this calculation of \( \sigma \) there will be a time instant \( t_1 > t_0 \) such that the following conditions of theorem are satisfied

\[
\begin{align*}
\sigma M + \delta^{-1}(\mu + \kappa)hh^T + (\mu + \kappa)Nh \varepsilon^T BB^T ch^T N \\
+ (\mu + \kappa)hh^T + \delta^{-1}Nh \varepsilon^T AA^T ch^T N \\
+ \delta^{-1}Nh \varepsilon^T qq^T ch^T N + \delta^{-1}Nh \varepsilon^T BB^T ch^T N \leq -Q_2,
\end{align*}
\]

where the number \( 0 < \delta < 0.5 \) is chosen so that

\[
\begin{align*}
-R + \delta I + (\delta \mu + 2\delta \kappa - \kappa)PBB^T P \\
+ \delta PPP^T \leq -Q_1 < 0,
\end{align*}
\]

and the parameters \( \kappa \geq C_0^{-2}(\kappa^{-1} + \delta^{-1}) \).

4. SIMULATION RESULTS

Consider the following nonlinear system

\[
y(t) = \frac{b(p)}{a(p)}u(t) + \frac{c(p)}{a(p)}\theta(t),
\]

where polynomials \( b(p) = b_1 \), \( a(p) = p(p-a_1)(p-a_2) \), \( c(p) = c_0p - c_1 \) with unknown parameters \( b_0, a_1, a_2, c_0, c_1 \), delay \( h = 3 \) and nonlinear function \( \theta(t) = \varphi(y(t-\tau)) = y(t-\tau) \sin y(t-\tau) \).

Choose the control according to equation (8)

\[
u(t) = -\alpha(p)(\mu + \kappa)\dot{y}(t),
\]

where for relative degree \( \rho = 3 \) the polynomial \( \alpha(p) \) we choose as

\[
\alpha(p) = (p+1)^3.
\]

Then we can rewrite the control in the following form

\[
u(t) = -(p+1)^3(\mu + \kappa)\dot{y}(t) = \mu(\mu + \kappa)\dot{y}(t) = -2\mu(\mu + \kappa)\dot{y}(t) + \mu(\mu + \kappa)\dot{y}(t),
\]

where the positive numbers \( \mu > 0 \), \( \kappa > 0 \) and the function \( \dot{y}(t) \) is calculated by the following algorithm

\[
\begin{align*}
\dot{\xi}_1(t) &= \sigma \dot{\xi}_2(t), \\
\dot{\xi}_2(t) &= \sigma(-k_1\xi_1(t) - k_2\dot{\xi}_2(t) + k_1y(t)),
\end{align*}
\]

\[
\dot{y}(t) = \xi_1(t),
\]

where the number \( \sigma > \mu + \kappa \) and parameters \( k_i \) are \( k_1 = 1, k_2 = 2 \). Choose the parameters \( \tilde{k} = \mu + \kappa = 5 \) and \( \sigma = 12 \).

The results of a computer simulation for variable \( y(t) \) for the case \( b_0 = 1 \), \( a_1 = 1, a_2 = 0.5, c_0 = 1, c_1 = 0.3, \tau = 3 \), \( y(0) = 2 \) and \( \dot{y}(0) = 0 \) are presented in Fig. 1. The results of a computer simulation for variable \( y(t) \) for the case \( b_0 = 1 \), \( a_1 = -1, a_2 = -0.5, c_0 = 1, c_1 = -0.3, \tau = 3 \), \( y(0) = 5 \) and \( \dot{y}(0) = 0 \) are presented in Fig. 2. We can see that for two different groups of parameters presented control provides asymptotic stability of equilibrium \( y = 0 \).

5. CONCLUSION

In the paper the problem of synthesis of control for time-delay nonlinear system (1) was considered. This paper has extended the theory of output feedback control of time-delay nonlinear systems. The new control law (8) – (10), providing output asymptotic stability of system (1) was designed. Only measurements of the output, but not its derivatives were used. Dimension of the robust controller (8) – (10) is \( \rho - 1 \) and dimension of the adaptive controller (8) – (10), (27), (28) is \( \rho \).

REFERENCES


Fig. 1. Transients in control system for variable $y(t)$.


