Time Dependent Uncertainties and Optimal Control

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Abstract

Optimal control problems involve the difficult task of determining time-varying profiles through dynamic optimization. Such problems become even more complex in practical situations where handling time dependent uncertainties becomes an important issue. This work presents a new approach based on real option theory to the solution of optimal control problems under uncertainty. First, using the fundamentals provided by real option theory, time-dependent uncertainties in some model parameters are modeled as Ito Processes. Then, after applying the dynamic programming optimality conditions for stochastic problems, the resulting formulation is converted into a maximum principle formulation without the conventional two-point boundary value problem. A coupled maximum principle-nonlinear programming numerical optimization algorithm can later be used to solve this computationally intensive problem. In order to show the scope of this approach, the classical isoperimetric problem is solved and a chemical engineering application is described.

Keywords: Optimal Control, Real Option Theory, Time Dependent Uncertainties

1. Introduction: Solving Optimal Control Problems

Optimal control problems in engineering have received considerable attention in the literature. In general, solution to these problems involve finding the time-dependent profiles of the control variables so as to optimize a particular performance index. Systematic methods to solve these problems include calculus of variations, the maximum principle and the dynamic programming technique.

The method of dynamic programming is based on Bellman’s principle of optimality which states that the minimum value of the objective function depends on the initial state and the initial time. The application of the dynamic programming technique to a continuously operating system leads to a nonlinear partial differential equation, the Hamilton-Jacobi-Bellman (HJB) equation. On the other hand, the maximum principle technique transform the integral objective function into a Hamiltonian, which represents a nonlinear objective function for each time step that can be optimized using a (discretized) variable, for that step. The maximum principle also needs to include

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additional variables and their corresponding first order differential equations, referred to as adjoint variables and adjoint equations, respectively (Thompson and Sethi, 1994).

2. Stochastic Optimal Control Problems

A stochastic process evolves over time in an uncertain way. Stochastic processes do not have time derivatives in the conventional sense and, therefore, they cannot always be manipulated using the ordinary rules of calculus. As a result, the typical mathematical techniques used to solve optimal control problems cannot be directly applied for the stochastic case. To work with stochastic processes one must make use of Ito’s Lemma and the dynamic Programming formulation.

2.1 Time dependent uncertainties

As stated above, stochastic processes cannot be manipulated using the ordinary rules of calculus as needed to solve stochastic optimal control problems. Ito (Dixit and Pyndick, 1994) provided a way around this by defining a particular kind of uncertainty representation based on the Wiener process. An Ito process is a stochastic process on which the increment of a stochastic variable $x$, $dx$, is represented by the equation:

$$dz = a(x,t) \, dt + b(x,t) \, dz$$

(1)

where $dz$ is the increment of a Wiener process, and $a(x,t)$ and $b(x,t)$ are known functions. By definition, $E[(dz)]=0$ and $E[(dz)^2]=dt$, where $E$ is the expectation operator and $E[dz]$ is interpreted as the expected value of $dz$.

The simplest instance of Equation (1) is the Brownian motion with a drift given by:

$$dx = \alpha \, dt + \beta \, dz$$

(2)

where $\alpha$ is called the drift parameter and $\sigma$ is the variance parameter. Figure 1 shows three sample paths of Equation (2). Other examples of Ito processes are the Geometric Brownian motion with drift and the Mean reverting process. For details, please refer to Dixit and Pyndick (1994).

Figure 1 Brownian motion with a drift
An important aspect of the approach proposed in this paper is that, by using the concepts of real option theory (Amram and Kulatilaka, 1999), time-dependent uncertainties in some engineering model parameters are represented as Ito Processes. Examples of the application of this representation are the relative volatility parameter of batch distillation and the mortality rate of predator-prey fisher information models.

2.2 Ito’s Lemma

Ito’s Lemma is easier to understand as a Taylor series expansion. Suppose that x follows Equation (1), and consider a function that is at least twice differentiable in x and once in t. We would like to find the total differential of this function, dF. The usual rules of calculus defines this differential in terms of first-order changes in x and t and the higher-order terms all vanish in the limit. For an Ito process following Equation (1), it can be shown that the differential dF is given in terms of first-order changes in t and x and second-order changes in x. Hence, Ito’s Lemma gives the differential dF as:

\[
dF = \left[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{1}{2} \left( \frac{\partial^2 F}{\partial x^2} \right) \right] dt + b(x,t) \frac{\partial F}{\partial x} dz
\]

2.2 Stochastic dynamic programming

The principle of optimality in dynamic programming states that, for the optimal control problem in the deterministic case:

Maximize _θ_ \( L = \int_0^T k(x, \theta) dt \) subject to \( \frac{dx}{dt} = f(x, \theta) \)

the optimality conditions are given by the HJB equation:

\[
0 = \text{Maximize}_\theta \left[ k(x, \theta) + \sum_i \frac{\partial L}{\partial x_i} f_i + \sum_i \frac{\sigma_i^2}{2} \frac{\partial^2 L}{\partial (\partial x_i)^2} \right]
\]

where \( x \) is the vector of state variables and \( \theta \) the vector of control variables. As an extension of this approach to stochastic optimal control problems, Merton and Samuelson (1990) found that, if the state variables are represented as Ito processes:

\[
dx = f dt + \sigma dz
\]

then the optimality conditions for dynamic programming are given by:

\[
0 = \text{Maximize}_\theta \left[ k(x, \theta) + \sum_i \frac{\partial L}{\partial x_i} f_i + \sum_i \frac{\sigma_i^2}{2} \frac{\partial^2 L}{\partial (\partial x_i)^2} + \sum_i \sum_j \sigma_i \sigma_j \frac{\partial^2 L}{\partial x_i \partial x_j} \right]
\]

In Equation (8), \( \sigma_i \) is the variance parameter of the state variable \( x_i \).
3. The Stochastic Maximum Principle

Although the mathematics of dynamic programming look different from the maximum principle formulation, in most cases they lead to the same results as reported by Diwekar (1992) for the deterministic case. However, Notice that, as explained above, the literature reports an extension of the HJB equation for the stochastic case. Hence, the mathematical equivalence between dynamic programming and the maximum principle can be made extensive to the stochastic case, and that is one of the main issues of our approach. Here we describe how the stochastic dynamic programming formulation is converted into a stochastic maximum principle formulation.

3.1 The main result of the reformulation

Seeking simplicity in the representation, on this section we assume that only one state variable and one control variable exist in the formulation; the generalization to a multivariable problem could be easily achieved. Also, it will be assumed that the optimal control problem has been rewritten in Mayer linear form \( (k=0) \), as required by the maximum principle formulation. Consider the optimality conditions of dynamic programming for stochastic optimal control problems, Equation (8):

\[
0 = \max_\theta \left[ L_t + L_x f + \frac{\sigma^2}{2} L_{xx} \right] = \max_\theta \left[ L_t + H \right]
\]

where \( H \) is the Hamiltonian function for stochastic optimal control problems. The main aspect of the derivations consists on obtaining the expressions for the adjoint equations. The adjoint equations provides the dynamics of the adjoint variables in the maximum principle. Since it can be shown that the adjoint variables of the maximum principle are equivalent to the derivatives of the objective function with respect to the state variables of the dynamic programming approach, let us name \( \mu = L_x \) and \( \omega = L_{xx} \) as the adjoint variables of our problem. Then we can apply a procedure similar to the one used for obtaining the adjoint equations of the maximum principle formulation in the deterministic case (Thompson and Sethi, 1994). The main difference is now that, since the state variable, \( x \), is a stochastic variable behaving as an Ito process, the second order contributions of the objective function with respect to \( x \) have to be included in the analysis. The resulting adjoint equations are:

\[
H = \mu \sigma^2 + \frac{\sigma^2}{2} \omega
\]

\[
\frac{d\mu}{dt} = -f_x \mu - \frac{1}{2} \sigma^2 \omega \quad \mu(T) = c
\]

\[
\frac{d\omega}{dt} = -2 \omega f_x - \mu f_{xx} - \frac{1}{2} \sigma^2 \omega \omega(T) = 0
\]

where \( c \) is the coefficient of the state variable in the objective function (Mayer form).

The optimal control profile will result from extremizing the Hamiltonian with respect to the control variable; also a two-point boundary value problem has to be solved.

3.2 Coupled maximum principle and non linear programming approach

Sometimes, the solution to the two-point boundary value problem can be avoided. This can be done mainly when there exist algebraic constraints in the optimization problem.
One way to approach a constrained optimal control problem is to use the lagrangian form of the objective. However, if we do not include such constraint, the end boundary conditions for the adjoint equations would no longer be valid and would no longer have to be satisfied. Hence, what we actually would need to solve is an initial value problem, using the satisfaction of the constraints as an external criteria for convergence in an iterative procedure. Also notice that we can discretize the resulting problem and using a nonlinear optimization technique to obtain the corresponding solution. This coupled approach has been successfully used in several applications has proved to significantly reduce the computational effort of the solution method.

4. Illustrative Examples

In this section, the classical isoperimetric problem is solved and the maximum distillate problem, as a chemical engineering example of application, is also briefly described.

4.1 The isoperimetric problem

The isoperimetric problem consists on finding the maximum area that can be covered by a rope whose length (perimeter) is fixed ($\lambda$). In Mayer linear form, the deterministic version of the problem is formulated as:

\[
\begin{align*}
\text{Maximize} & \quad x_3(T) \\
\text{subject to:} & \quad \frac{dx_1}{dt} = u \quad x_1(0) = 0 \quad x_1(T) = 0 \\
& \quad \frac{dx_2}{dt} = \sqrt{1 + u^2} \quad x_2(0) = 0 \quad x_2(T) = \lambda \\
& \quad \frac{dx_3}{dt} = x_3 \quad x_3(0) = 0
\end{align*}
\]

For the stochastic version of the problem, we assume that $x_1$ is normally distributed with a variance parameter $\sigma=0.5$ and follows brownian motion (Equation (7)). The perimeter is given, $\lambda=16$. Figure 2 shows the deterministic and stochastic paths (one trajectory) numerically solved by using the proposed approach.

![Figure 2 The solution to the isoperimetric problem](image-url)
It can be seen that, although the stochastic solution follows a circular path, the expected area obtained is less than the area for the deterministic case for the same perimeter.

### 4.2 The maximum distillate problem

In the stochastic maximum distillate problem, the source of uncertainty can be incorporated, among other factors, through the behavior of the relative volatility parameter. In our approach to that problem, the definition of the optimal control problem does not explicitly consider the purity constraint of the product. If one wants to include such a constraint, then the objective function would have to be reformulated in lagrangian form. Hence, the adjoint equations are obtained by maximizing a Hamiltonian that does not incorporate the purity constraint on the distillate product. Therefore, the use of the final boundary conditions provides the limiting solution resulting in all the reboiler charge instantaneously going to the distillate pot with the lowest overall purity; hence, the final boundary conditions are no longer valid. Instead, the actual final boundary condition will be automatically imposed when the purity constraint is satisfied. This is an example of the procedure explained in section 4.2.

### 5. Conclusions

Solution to stochastic optimal control problems involves the difficult task of determining time-varying profiles through dynamic optimization under uncertainty. This work proposes a new and efficient approach (based on real option theory) to stochastic optimal control problems. First, time dependent parameters are assumed to behave as Ito processes. Then, the stochastic dynamic programming formulation is converted into a stochastic maximum principle formulation. Therefore, the solution to the partial differential equations involved in the dynamic programming are no longer necessary. In order to show the scope and efficiency of this approach, the classical isoperimetric problem has been solved; finally, a previously analyzed problem in chemical engineering is briefly described to illustrate an application in which the solution to the involved two-boundary value problem is also avoided and, therefore, the computational effort of the solution method can be further reduced.

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### References