Adaptive Forward Identification and Auto-Tuning for Motor Velocity Control

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Abstract: This paper presents a new on-line adaptive identification of physical model parameters by adjusting the parameters in the feedforward controller and feedback PI controller for a motor drive system. The proposed scheme is characterized by new set-up of a suitable reference model which generates signals needed to identify the physical parameters of a two-mass motor system, such as inertia moments of motor and load, shaft conductance, viscous friction terms proportional to motor velocity and squared velocity, and Coulomb friction. The adaptive algorithm not only can give the model parameters but also compensate for the unknown friction effects by adjusting the feedforward controller. Stability is also investigated via the ASPR property of the controlled system. The effectiveness of the proposed scheme is examined in simulation and experiment of speed control of a two-mass motor system.

1. Introduction

In motor velocity control, uncertain inertia moment and nonlinear frictions sometimes degrade its control performance [1]. Various robust control approaches have been exploited previously [2]-[4], however, fine tuning of the controller parameters is essentially needed in motor drive systems to attain high tracking performance and quick recovery response for stepwise torque disturbances [5]. The closed-loop identification using only input and output data (control torque input and motor velocity) can hardly attain the identifiability in practical situations. Instead of such a direct identification scheme, an indirect approach will be effective in the closed-loop identification directly using the input-output data of motor drive system, the proposed feedforward approach can have quick convergence and less effect of disturbances, since the adaptive algorithm uses only signals generated by the feedforward controller except the output tracking error. The convergence of the adaptation algorithm is also investigated via the almost strictly positive real (ASPR) property of the controlled system [6]. The effectiveness of the proposed approach is examined in numerical simulation and experiment for velocity control of two-mass induction motor with unknown inertia ratio.

2. Two-Degree-Freedom Velocity Control

Fig.1 illustrates a typical two degree-of-freedom motor velocity control system for a two-mass motor. The PI controller is employed to compensate for the closed-loop response to stepwise torque disturbances. If the behavior of the motor and load is described by an equivalent one-mass system with $1/(T_M + T_L)$ and no nonlinear frictions exist, the PI parameters $K_e$ and $T_i$ can be designed as follows: Since $K_e$ corresponds to the proportional gain of the PI controller, then the closed-loop dynamics consisting of the PI controller and the equivalent system can be described by a second-order linear transfer function with the denominator as

$$s^2 + \frac{T_i D_M + K_e T_i}{T_i (T_M + T_L)} s + \frac{K_e}{T_i (T_M + T_L)}$$

which is characterized by the natural frequency $\omega_n$ =
Parameters approximately 0 identified by using the input/output data the two-mass motor with nonlinear friction. The friction of two-mass motor. The block surrounded by the bro-
sator tuning in a feedforward manner in velocity control identification of physical model parameters and compen-
sation of motor and load respectively, $a$ and $c$ denote nonlinear friction constants. The units are normalized by

$$\sqrt{K_e/T_i(T_M + T_L)}$$ and the damping factor $\zeta = (K_e + D_M)/\sqrt{2\omega_M(T_M + T_L)}$. If the cutoff frequency $\omega_c$ of the loop transfer system is specified as $\sqrt{\gamma \cdot \omega_n}$ where $\phi$ is chosen as 2, for instance, the PI controller parameters can be given as $K_e = \omega_c T_M$ and $T_i = \phi/\omega_c$. Thus, the damping factor of the closed-loop system is given by $\zeta = \sqrt{\frac{\phi}{2\omega_c T_M}}$. Then it becomes approximately 0.7 if $D_M = 0$ and $\phi = 2$. However, if the motor parameters $T_M$ and $D_M$ are uncertain, the tuning of the PI controller parameters is needed. Moreover, if the motor is subjected to unknown nonlinear frictions, adaptive compensation is also needed to compensate for the degraded performance [1].

3. Adaptive Identification of Physical Parameters

Fig.2 illustrates the proposed scheme for adaptive identification of physical model parameters and compensator tuning in a feedforward manner in velocity control of two-mass motor. The block surrounded by the broken line includes the PI controller with unity gain and the two-mass motor with nonlinear friction. The friction terms include viscous friction proportional to velocity and squared velocity, and the Coulomb friction as stated later. Conventionally these model parameters are identified by using the input/output data $\{u_p(t), \omega_M(t)\}$ of the controlled system. However, the convergence of the parameters is affected by disturbances, and adaptive algorithms for rejecting the influences tend to be complicated. In the proposed feedforward modeling approach, the adaptive parameters $K_d(t), K_u(t), K_w(t)$ and $K_x(t)$, and the PI gain $K_e(t)$ are adjusted by forcing the output error $e_y(t)$ to zero, where the output error is defined by

$$e_y(t) = y_m(t) - \omega_M(t)$$

(2)

where $\omega_M(t)$ is the motor velocity and $y_m(t)$ is the reference velocity. The proposed configuration is based on the simple adaptive control (SAC) scheme [1], however the ordinary SAC scheme has taken care of no converged values of the adjustable parameters. On the other hand, the purpose of this paper is to give a new algorithm for adaptively providing all the physical parameters in the two-mass system from the adjustable parameters in the feedforward paths, by sophisticatedly constructing the

3.1 Two-mass motor system

Fig.3 show the motor drive system consisting of the two-mass motor with nonlinear frictions and the PI controller with unity gain which is employed to compensate stepwise torque disturbances. The motor and load are connected by a flexible shaft, and $u_p(t)$ is input torque, $\tau_c(t)$ torsional torque, $d(t)$ stepwise disturbance torque, $\omega_M(t)$ motor velocity, $\omega_L(t)$ load velocity. $T_M[s]$ and $T_L[s]$ are time constants of motor and load respectively, and $T_C[s]$ is a shaft constant. The friction effects are also introduced such as viscous friction proportional to velocity and squared velocity and the Coulomb friction as illustrated in Fig.3, where $D_M$ and $D_L$ are motor damping constants of motor and load respectively, $a$ and $c$ denote nonlinear friction constants. The units are normalized by the maximum working velocity $\omega_{\text{max}}$. Hence $T_M[s]$ implies the time in which the motor velocity changes from zero to $\omega_{\text{max}}$ by the maximum working torque.

The purpose of the paper is to present an on-line identification algorithm for giving the above all physical model parameters $\{T_M, T_L, T_C, D_M, D_L, a, c\}$.

The state space representation of the above motor drive system is given by

$$\dot{x}_p(t) = A_p x_p(t) + B_p y_p(t) + A_{\gamma p} \gamma(\omega_M(t))$$

(3a)

$$y_p(t) = C_p x_p(t)$$

(3b)

where $x_p = [\omega_M(t), \tau_C(t), \omega_L(t), x_p(t)]^T$, and

$$A_p = \begin{bmatrix} -\frac{D_M}{T_M} & -\frac{1}{T_M} & 0 & \frac{1}{T_M} \\ -\frac{1}{T_C} & 0 & -\frac{T_D}{T_L} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{T_L} \end{bmatrix}, \quad B_p = \begin{bmatrix} \frac{1}{T_M} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_p = [1 \ 0 \ 0 \ 0], \quad A_{\gamma p} = \begin{bmatrix} -\frac{a}{T_M} & -\frac{c}{T_M} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where the stepwise disturbance term is omitted in the above expression.

3.2 Construction of Reference Model

The major contribution of the paper is to clarify the role and structure of the reference model in order to de-
termine the physical model parameters in the closed-loop control. As shown in Fig.4, the role of the reference model is to generate the important signals which correspond to the actual signals in the two-mass motor system, as well as give the ordinary desired reference response. \( u_m(t) \) is the reference input satisfying the PE property as mentioned before. The reference model consists of two roles: One is the reference input generation part for supplying the reference input \( u_M(t) \), and the other is the reference model with the input \( u_M(t) \) for adjusting the model parameters. The total desired reference input-output property from \( u_m(t) \) to \( y_m(t) \) is given by

\[
y_m(t) = \frac{1}{(Ts + 1)^2} u_m(t)
\]  

(4)

where \( T \) is a time constant of the total reference model.

The state space expression of the reference model with the input \( u_M(t) \) is described by

\[
\dot{x}_m(t) = A_m x_m(t) + B_m u_M(t) + A_{\gamma m} \gamma(y_m(t)) \quad (5a)
\]

\[
y_m(t) = C_m x_m(t) \quad (5b)
\]

where \( x_m = [x_{m1}, x_{m2}, x_{m3}, x_{m4}, x_{m5}, x_{m6}]^T \), and

\[
A_m = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{T_l} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{T_l} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B_m = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix}, \quad A_{\gamma m} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
C_m = [1, 0, 0, 0, 0, 0] \\
\gamma(y_m(t)) = \begin{bmatrix}
|y_m(t)| y_m(t) \\
tanh(\alpha y_m(t))
\end{bmatrix}
\]

From Fig.2, the control input is given as

\[
\xi(t) = K^T(t) r(t)
\]  

(6)

\[
r(t) = [e_y x_{m1} x_{m2} x_{m3} x_{m4} u_M \bar{u}_M x_{d1} x_{d2}]^T \\
K(t) = [K_c K_{x1} K_{x2} K_{x3} K_{x4} K_{uM} K_{uM} K_{d1} K_{d2}]^T
\]

3.3 Identification of Physical Model Parameters

The adjustable parameters in (6) are updated as follows:

\[
\dot{K}(t) = e_y \Gamma r(t), \quad \Gamma = \Gamma^T \geq 0
\]  

(7)

where \( e_y(t) = y_m(t) - \omega_M(t) \).

By employing the controlled system (3) and the reference model (5) in the proposed scheme given in Fig.4, we can give the following convergence property of the adjustable parameter vector.

Property 1: Let the control law be given by (6) and the parameter adaptation law be given by (7) in the controlled system in the proposed structure in Fig.2. If the control input can force the output error \( e_y(t) \) to zero, the adjusted parameters converge as

\[
\lim_{t \to \infty} K_{x1} = D_M + D_L \]

\[
\lim_{t \to \infty} K_{x2} = T_C + T_L - T_C D_L^2 \\
\lim_{t \to \infty} K_{x3} = -2T_C T_L D_L + D_L^3 T_C \\
\lim_{t \to \infty} K_{x4} = 0 \\
\lim_{t \to \infty} K_{uM} = -T_C T_L^2 - T_C D_L^3 + 3T_L T_C^2 D_L^2 \\
\lim_{t \to \infty} K_{d1} = 3T_C^2 T_L D_L - 4T_L C D_L^3 - T_C^3 D_L^2 \\
\lim_{t \to \infty} K_{d2} = a, \quad \lim_{t \to \infty} K_{d2} = c
\]  

(8)

See Appendix 1. The property can be proved by the aid of the command generator tracker (CGT) theory which plays an important role in the simple adaptive control [6]. See Appendix 1 as for its proof. By neglecting higher orders of smaller parameters, we can solve (8) with respect to the physical parameters to be obtained, as

\[
T_L \approx \frac{3K_{x3} K_{uM}}{2K_{uM}}, \quad T_M \approx K_{x2} - T_L, \quad T_c \approx -\frac{4K_{x3}^2}{9K_{x3}^2 K_{uM}} \\
D_L \approx \frac{3K_{x3}^2}{4K_{uM}}, \quad D_M \approx K_{x1} - D_L
\]  

(9)

Thus, the physical parameters can be calculated by (10) in an on-line way, and then the proportional gain \( K_c(t) \) of the PI controller is tuned as \( K_c(t) = \omega_C K_{x2}(t) = \omega_C(T_{M} + T_{L}) \).

Property 2: In the proposed control system in Fig.2, the output errors are bounded in a finite region specified by

\[
D(e_x(t)) = \begin{cases}
-\infty < e_{x1}(t) < \infty, \\
|e_{x2}(t)| + |e_{x3}(t)| + |e_{x4}(t)| < \beta
\end{cases}
\]

4. Discrete-Time Implementation

Fig.5 illustrates digital implementation of the continuous-time algorithm proposed in the paper. The
controlled system model discretized with a zero-order holder does not satisfy the ASPR property since the relative degree of the discrete-time model is one. Hence, a serial controller designed by $H_\infty$ design theory is introduced to compensate the relative degree one \cite{7}. In the simplified implementation, by replacing the designed transfer function from $\xi_k(k)$ to $u_k'(k)$ by a phase-lead compensator, we design the compensator as:

$$u_k'(k) = \frac{1 + (1 - z^{-1})}{\delta K_1(k) + 1} \xi_1(k)$$

where the frequency response of (10) is almost same as that of the compensator via $H_\infty$ design in the working frequency range.

We take a $\delta$-operation with lowpass filter to descretize the proposed continuous-time adaptive algorithm, where $\delta = (z - 1)/T$, $T$ is a sampling interval. To reduce noise effects, a lowpass filter $1/E(\delta)$ is adopted to filter the signals involving noise, as $e_{yf}(k) = (1/E(\delta))e_{yf}(k)$ and $r_f(k) = (1/E(\delta))r_f(k)$. Thus, the adaptive algorithm can be rewritten by using the filtered signals as

$$\hat{\xi}_1(k) = K^T(k)r_f(k)$$

$$\delta K_1(k) = e_{yf}(k)\gamma_2e_{yf}(k) - \sigma K_1(k)$$

$$\delta K_2(k) = e_{yf}(k)\gamma_2e_{yf}(k)$$

$$\delta K_3(k) = e_{yf}(k)\gamma_3e_{yf}(k) - \rho_3|e_{yf}(k)|K_3(k)$$

$$\delta K_4(k) = e_{yf}(k)\gamma_4e_{yf}(k) - \rho_4|e_{yf}(k)|K_4(k)$$

$$\delta K_5(k) = e_{yf}(k)\gamma_5e_{yf}(k)$$

$$\delta K_6(k) = e_{yf}(k)\gamma_6e_{yf}(k)$$

$$\delta K_7(k) = e_{yf}(k)\gamma_7e_{yf}(k)$$

$$\delta K_8(k) = e_{yf}(k)\gamma_8e_{yf}(k)$$

where $r_f(k) = [x_{yf}(k) x_{yf}(k) u_{MF}(k) u_{MF}(k) x_{yf}(k) x_{yf}(k)]^T$,

$$K(k) = [K_x(k) K_x^T(k) K_u(k) K_u(k)^T K_d(k)^T]^T, \quad x_{mf}(k) = [x_{mf1}(k) x_{mf2}(k) x_{mf3}(k) x_{mf4}(k) x_{mf5}(k) x_{mf6}(k) x_{mf7}(k) x_{mf8}(k)]^T, \quad x_{df}(k) = [x_{df1}(k) x_{df2}(k) x_{df3}(k) x_{df4}(k) x_{df5}(k) x_{df6}(k) x_{df7}(k) x_{df8}(k)]^T, \quad K_x(k) = [K_x1(k) K_x2(k) K_x3(k) K_x4(k)]^T, \quad K_d(k) = [K_d1(k) K_d2(k) K_d3(k)]^T.$$  

We also apply the $c_1$-adaptation law \cite{8} for updating the smaller parameters $K_2$, $K_3$ and $K_4$, $\sigma$ used in the $c_1$ modification \cite{9} is chosen as $0 < \sigma < 1$ and

$$K_x(k) = \omega_c K_2(k) \quad \text{if} \quad K_2(k) \leq \omega_c K_2(k)$$

Thus, $K_x(k)$ converges to $\omega_c K_2(k) = \omega_c(T_M + T_L)$ and then the PI controller is auto-tuned.

5. Simulation and Experimental Results

The effectiveness of the proposed scheme for adaptive identification and compensation for friction is investigated in experiments using an induction motor. The control purposes are the tracking ability to the reference velocity changes and the recovery performance to torque disturbances. We examined the effectiveness of the proposed algorithm in simulations and experiments of velocity control, where we used two induction motors connected by a flexible shaft, where one motor is a driving machine and the other is a load machine, and they constitute a two-mass system. By attaching a weight to driving motor or load motor, we can change an inertia ratio (load inertia/drive inertia) as 1/3, 1, 3, as shown in Fig.6. The motor velocity is measured by an incremental pulse encoder with resolution 1024[pulse/rev]. We validated the obtained identified physical parameters by comparison with the real parameter values which were obtained via off-line identification using the M-sequence method. The values are given as follows: $T_M = 0.23[s]$, $T_L = 1.29[s]$, $J_M = 1.255[\text{kgm}^2]$, $J_L = 0.25[\text{kgm}^2]$, $J_{\text{load}} = 0.5[\text{kgm}^2]$, and $\omega_c = 150[\text{rad/s}]$. We conclude that the proposed scheme is effective to compensate the velocity disturbances. The derived compensation algorithm can be carried out in a real-time manner. The desired acceleration can be achieved with the application of the algorithm in the current control system.
were obtained in off-line manners. Then, the experimental setup is given as follows: The sampling interval \( T_s = 0.001 \text{s} \), the cut-off frequency is \( \omega_c = 30 \text{rad/s} \), and then the integration constant is set at \( T_i = 2/30 \text{s} \). On the other hand, the time constant of reference model is chosen \( T = 0.08 \text{s} \). On the setup, we consider three cases with different inertia ratio \( R \): (1) \( R = 1 : (T_M = 0.23 \text{s}, T_L = 0.23 \text{s}) \), (2) \( R = 3 : (T_M = 0.23 \text{s}, T_L = 0.69 \text{s}) \), (3) \( R = 1/3 : T_M = 0.69 \text{s}, T_L = 0.23 \text{s} \).

Fig.7 shows simulation results of the adaptive identification and tuning algorithms. The reference output \( y_m(k) \) is changed in various speed modes consisting of acceleration, constant velocity, decleration and zero velocity. 100% stepwise torque disturbances are added at 395 [s] in constant velocity and at 896 [s] in zero velocity. All the adjustable parameters are shown in Figs.7(b)(c). From these estimated parameters in the case \( R = 1 \), we can calculate the physical parameters from (9), which are plotted in Fig.7(d). Other physical parameters can also be identified similarly as in Figs.7(e)(f). All of the parameters converge their true values depicted by dotted lines, even in the presence of disturbances.

Fig.8 shows the experimental results in velocity control of an induction motor. As shown in Fig.8(a), the reference velocity is changed in four modes consisting acceleration, positive constant velocity, decleration, negative constant velocity, in which the unity velocity is normalized by the working velocity. By adding the weight to the motor or load machine, we can change the inertia ratios as \( R = 1 \), \( R = 3 \) and \( R = 1/3 \). The inertia and damping constants of motor and load, and shaft constant can be calculated from the adjustable parameters as stated before. The dotted lines indicate the physical parameters identified in an off-line identification using the M-sequence. In the initial interval until 150 [s], only larger parameters are updated and then all the parameters are updated after 150 [s]. It is seen from Figs, all the physical parameters converge to the constants which were obtained in off-line manners.8(b)∼(d).

6. Conclusions

This paper has presented the new adaptive scheme for identifying the physical parameters of the motor drive system and compensating for friction effects in the feedforward manner. The proposed approach is characterized by the setup of the reference model which is derived from the physical structure of motor drive systems and generates the corresponding signals which are needed in the parameter adjustment. The convergence and stability have been investigated by assuming the ASPR property of the input-output relation of the controlled system. The serial compensation method has also been given to compensate for the non-ASPR property. Finally the experimental studies have validated the effectiveness of the proposed feedforward type of the adaptive auto-tuning algorithm in velocity control of two-mass induction motor.

References

(c) Inertia ration: \( R = 3 \)  
(d) Inertia ration: \( R = 1/3 \)

**Figure 8:** Experimental results


**Appendix: Proof of Properties 1 and 2**

**Proof of Property 1:**

From the CTG theory, the ideal control input \( \xi^*(t) \) can be given for the controlled system (3) and the reference model (5), as

\[
x_p^*(t) = S_{11}x_m(t) + S_{12}u_M(t) + S_{13}\dot{u}_M(t)
\]

\[
\dot{\xi}^*(t) = S_{21}x_m(t) + S_{22}u_M(t) + S_{23}\dot{u}_M(t) + S_r\gamma(y_m(t))
\]

From (3), (5) and the above expression, it follows that \( \dot{x}_p^*(t) \) satisfies

\[
\dot{x}_p^*(t) = S_{11}\dot{x}_m + S_{12}\dot{u}_M(t)
\]

\[
= S_{11}A_{m}x_m(t) + S_{12}B_{m}u_M(t)
\]

\[
+ S_{11}A_{\gamma}y_m(t) + S_{12}\dot{u}_M(t)
\]

\[
= A_p x_p^*(t) + B_p \xi^*(t) + A_{\gamma}\gamma(y_m(t))
\]

\[
= [A_p S_{11} + B_p S_{21}]x_m(t) + [A_p S_{12} + B_p S_{22}]u_M(t)
\]

\[
+ [A_p S_{13} + B_p S_{23}]\dot{u}_M(t) + [B_p S_{12} + A_{\gamma}]\gamma(y_m(t))
\]

(13)

Furthermore, it leads from \( y_m(t) \equiv \omega_M(t) \) that

\[
y_m(t) = C_p x_p^*(t)
\]

\[
= C_p S_{11}x_m(t) + C_p S_{12}u_M(t) + C_p S_{13}\dot{u}_M(t)
\]

\[
= C_m x_m(t)
\]

(14)

Then we have from (13) and (14) that

\[
S_{11}A_{m} = A_p S_{11} + B_p S_{12}, \quad S_{12}B_{m} = A_p S_{12} + B_p S_{22}
\]

\[
S_{12} = A_p S_{13} + B_p S_{23}, \quad S_{11}A_{\gamma} = B_p S_{12} + A_{\gamma p}
\]

\[
C_p S_{11} = C_m, \quad C_p S_{12} = 0, \quad C_p S_{13} = 0
\]

Then it follows that \( S_{21}, S_{22}, S_{23}, S_r \) are given by

\[
S_{21} = [D_M + D_L T_M + T_L - D_{21}^L T_C - D_{11}^L T_C + 2T_L] \sigma a c
\]

\[
S_{22} = -T_C (D_L^2 + 3D_L^2 T_C) - 3D_L^2 T_C T_L
\]

\[
S_{23} = 3T_C^2 D_L^2, \quad S_r = [\sigma a c]
\]

then the ideal control input \( \xi^*(t) \) is

\[
\dot{\xi}^*(t) = S_{21}x_m(t) + S_{22}u_M(t) + S_{23}\dot{u}_M(t) + S_r\gamma(y_m(t))
\]

\[
= (D_M + D_L) x_m(t) + (T_M + T_L - D_{21}^L T_C) x_m(t)
\]

\[
+ (-D_L T_C - D_{11}^L T_C + 2T_L) x_m(t) + 0 \cdot x_m(t)
\]

\[
+ a \cdot x_m(t) + c \cdot x_n(t)
\]

\[
+ (-T_C D_L^2 T_C - 3D_L^2 T_C T_L) u_M(t)
\]

\[
+ (3T_C^2 D_L^2 - 4T_C T_L D_L - 4D_L^2 T_L D_L) \dot{u}_M(t)
\]

\[
+ a[y_m(t)] y_m(t) + c \cdot \gamma(y_m(t))
\]

\[
\cong (D_M + D_L) x_m(t) + (T_M + T_L) x_m(t)
\]

\[
+ (-2T_C T_L D_L) x_m(t) + (T_C T_L^2) u_M(t)
\]

\[
+ (3T_C^2 D_L^2) \dot{u}_M(t) + a \cdot x_d(t) + c \cdot x_d(t)
\]