A DESIGN METHOD OF 2-DEGREE-OF-FREEDOM REPETITIVE CONTROL SYSTEMS

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Keywords: repetitive control, 2-degree-of-freedom control, feed-forward control, periodic signal

Abstract

In the present paper, we examine a design method of 2-degree-of-freedom repetitive control systems. The 2-degree-of-freedom control has good characteristics, such as the input-output characteristics for the reference input and feed-back control characteristics can be independently settled. 2-degree-of-freedom controllers consist of the feed-forward controller and the feed-back controller. Design methods of feed-back controller for the repetitive control systems have been considered sufficiently, but design methods of feed-forward controllers for repetitive control systems have not been considered yet. In order to design feed-forward controller for the repetitive control system, we must solve the interpolation problem. In addition, the interpolation problem for feed-forward controller for the repetitive control system has many interpolation points. No method has been proposed to solve the interpolation problem for feed-forward controller for the repetitive control system. The purpose of this paper is to propose a simple design method for feed-forward controllers and to present a design method of 2-degree-of-freedom repetitive control system.

1 Introduction

In this paper, we examine a design method of 2-degree-of-freedom repetitive control systems. The repetitive control system is a type of servo-mechanism for repetitive reference input. That is, the repetitive control system follows the periodic reference input without steady state error even if a periodic disturbance or uncertainty exists in the plant [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The repetitive control system was initially proposed for 'high accuracy control magnet power supply of proton synchrotron' [1]. Subsequently, several papers on the theory and application of repetitive control systems have been published [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Since a repetitive control system follows any periodic reference input without steady state error, the repetitive control system can be used for 'high accuracy control magnet power supply of proton synchrotron' [1].

Notations

\( R(s) \) the set of real rational function with \( s \).
\( RH_{\infty} \) the set of stable proper real rational functions.
\( H_{\infty} \) a set of stable causal functions.

The purpose of this paper is to give a design method for feed-forward controllers and to propose a design method of 2-degree-of-freedom repetitive control system. First, the structure of 2-degree-of-freedom repetitive control system is presented. Next, the problem considered in this paper is summarized. We describe the basic idea to solve the problem of design method of feed-forward repetitive controller, which is the fusion of the stable filtered inverse system [13, 14] and the idea of predictive control [15]. A design procedure of the feed-forward controllers for single-input/single-output systems is proposed. Next, we present a design method of 2-degree-of-freedom repetitive control systems. We expand this result and propose a design procedure of the feed-forward controllers for multiple-input/multiple-output systems. A numerical example is illustrated to show the effectiveness of the proposed method.
Without steady state error, then the controller described by

$$G(s)\hat{G}(s) = \hat{Q}(s).$$

(1)

Figure 1: 2-degree-of-freedom control system

is the feed-back controller, $\hat{Q}(s) \in R(s)$, $\hat{G}(s) \in R(s)$ and $F(s) \in R(s)$ are the feed-forward controllers and satisfying

$$G(s)\hat{G}(s) = \hat{Q}(s).$$

(1)

$G(s)$ is assumed to be controllable and observable. $y$ is the output, $u$ is the control input, $r$ is the periodic reference input with period $T$ satisfying

$$r(t + T) = r(t) \quad (\forall t \geq 0)$$

(2)

and has settled beforehand, $d$ is the periodic disturbance with period $T$.

The transfer function from $r$ to $y$ in Fig. 1 and that from $d$ to $y$ in Fig. 1 are given by

$$y = G(s)\hat{G}(s)F(s)r$$

$$= \hat{Q}(s)F(s)r$$

(3)

and

$$y = \frac{1}{1 + G(s)C(s)}d,$$

(4)

respectively. Therefore 2-degree-of-freedom control system in Fig. 1 can settle independently the input-output characteristics for the reference input and the feed-back control characteristics.

If the plant $G(s)$ has a periodic disturbance $d$ with period $T$ and uncertainty, and the output $y$ follows the periodic reference input $r$ with period $T$ and the frequency component

$$\omega_i = \frac{2\pi}{T}i \quad (i = 0, \cdots, N)$$

(5)

without steady state error, then the controller $C(s)$ must be described by

$$C(s) = C_r(s)\hat{C}(s)$$

(6)

[4], where $C_r(s)$ is an internal model for the periodic reference input $r$ with period $T$ written as

$$C_r(s) = \frac{1}{1 - q(s)e^{-sT}}$$

(7)

and $q(s)$ is strictly proper, asymptotically stable low pass filter satisfying

$$1 - q(j\omega_i) = 0 \quad (i = 0, \cdots, N).$$

(8)

For practically, in many cases, $q(s)$ is settled by

$$q(s) = \frac{1}{(1 + s\tau_q)^{\alpha_q}}$$

(9)

satisfying

$$1 - q(j\omega_i) \simeq 0 \quad (i = 0, \cdots, N),$$

(10)

where $\tau_q > 0$ is sufficiently small positive real number and $\alpha_q$ is an arbitrary positive integer [3, 4, 5, 6, 7, 8].

In addition, in order the output $y$ to follow the reference input $r$ with the frequency components in Equation (5), the feed-forward controller $\hat{G}(s)$, $\hat{Q}(s)$ and $F(s)$ must satisfy

$$1 - G(s)\hat{G}(s)F(s)\bigg|_{s = j\omega_i} = 1 - \hat{Q}(s)F(s)\bigg|_{s = j\omega_i} = 0 \quad (i = 0, \cdots, N).$$

(11)

In order the output $y$ to follow the reference input $r$ in Equation (2) without steady state error, it needs that control system in Fig. 1 is stable. According to [12], the necessary and sufficient stability condition of the control system in Fig. 1 is satisfying following expressions:

1. $C(s)$ stabilizes $G(s)$.
2. $\hat{G}(s)$ is stable.
3. $\hat{Q}(s)$ is stable.
4. $F(s)$ is stable.

In this paper, we examines a design method of feed-forward controller $\hat{G}(s)$, $\hat{Q}(s)$ and $F(s)$ for 2-degree-of-freedom control systems in Fig. 1 to make 2-degree-of-freedom repetitive control systems in Fig. 1 stable.

We adapt a design method of filtered inverse system in [14] for designing the feed-forward controller. Because the design method for feed-forward controllers in [14] has following advantages:

1. the output $y$ follows the step reference input $r$ without steady state error.
2. even if the plant is multiple-input/multiple-output, the transfer function from $r$ to $y$ is decoupled. This implies that the input-output characteristics is reduced to single-input/single-output control problem.

Therefore, we consider the design problem that the output $y$ follows the general reference input $r$ written in Equation (2) without steady state error maintaining the advantage of the method in [14]. In order to maintain the advantage of the method in [14], if $\hat{G}(s)$ in Fig. 1 is designed using the method in [14], a design problem of feed-forward controller for multiple-input/multiple-output is reduced to single-input/single-output control problem. $F(s)$ is designed so that
the output $y$ in Fig. 1 may follow the periodic reference input $r$ without steady state error. Note that if $\hat{r}$ in Fig. 1 is calculable even if $F(s)$ is not necessarily causal in Fig. 1, the control system in Fig. 1 can be built. That is, in the repetitive control system, the periodic reference input $r$ is given in advance as in Equation (2). A design method of $F(s)$ using the characteristics of the repetitive control system such that the reference input is also acquired in advance is proposed.

3 Basic idea of designing $F(s)$

In this section, we describe a basic idea to design method of feed-forward controller in Fig. 1.

For easy explanation, it is assumed that the reference input $r$ is written by

$$r = \frac{\bar{\omega}}{s^2 + \bar{\omega}^2},$$

where $\bar{\omega} > 0$.

According to [14], $\hat{G}(s)$ in Fig. 1 is settled by

$$G(s)\hat{G}(s) = \hat{Q}(s) = Q(s)G_K(s),$$

where $G_K(s) \in RH_{\infty}$ is the inner function of $G(s)$ satisfying

$$G_K(-s)G_K(s) = 1$$

and

$$G_K(0) = 1.$$  \hspace{1cm} (15)

$Q(s)$ is

$$Q(s) = \frac{1}{(1 + s\tau)^\alpha}.$$  \hspace{1cm} (16)

and $\alpha$ is arbitrary positive integer which makes $\hat{G}(s)$ be proper and $\tau$ is sufficiently small positive real number.

Next we describe a design method of $F(s)$ to satisfy Equation (11). For easy explanation, $\tau$ is assumed to be settled to be sufficiently small positive real number and to satisfy

$$Q(j\bar{\omega}) = 1.$$  \hspace{1cm} (17)

From the assumption of Equation (17) and Equation (13), we have

$$1 - G(s)\hat{G}(s)F(s)|_{s=j\bar{\omega}} = 1 - Q(s)G_K(s)F(s)|_{s=j\bar{\omega}} = 1 - G_K(s)F(s)|_{s=j\bar{\omega}}.$$  \hspace{1cm} (18)

We define $F(s)$ by

$$F(s) = e^{sW},$$  \hspace{1cm} (19)

where $W \in R$ is settled by

$$W = -\frac{\bar{\omega}}{\omega} G_K(j\bar{\omega}).$$  \hspace{1cm} (20)

From the assumption of Equation (14) and simple manipulation, using $F(s)$ written by Equation (19), it is confirmed that

$$1 - G_K(s)F(s)|_{s=j\bar{\omega}} = 0.$$  \hspace{1cm} (21)

In this way, we can design $F(s)$ satisfying Equation (11).

Since $G_K(s)$ is an inner function satisfying Equation (14) and Equation (15), $\mathcal{L}G_K(j\bar{\omega}) < 0$ is confirmed. Therefore, since $W > 0$ in Equation (20), $F(s)$ in Equation (19) is not causal. However, from the assumption that $r$ is obtained beforehand, $\hat{r}$ can be calculated and the control system in Fig. 1 is evaluated.

$F(s)$ in Equation (19) works as the predictor, the proposed method is a method using predictor.

4 Design method for feed-forward controller

In this section, we expand the idea described in 3 and propose a design procedure of $F(s)$ such that the output $y$ follows the reference input $r$ in Equation (2) without steady state error.

In the same manner as the method in 3, $\hat{G}(s)$ in Fig. 1 is designed using the method in [14]. That is, $G(s)$ satisfies Equation (11).

From Equation (13), the problem of designing $F(s)$ satisfying Equation (11) is equivalent to hold

$$1 - Q(s)G_K(s)F(s)|_{s=j\bar{\omega}} = 0 \quad (i = 0, \cdots, N).$$  \hspace{1cm} (22)

The rest of designing problem is to find the design procedure of $F(s)$ satisfying Equation (22). Let $F(s)$

$$F(s) = \sum_{i=0}^{N} h_i(s)e^{sW_i},$$  \hspace{1cm} (23)

where $h_i > 0 \in R (i = 0, \cdots, N)$ and $W_i \in R (i = 0, \cdots, N)$. For simplicity, we rewrite Equation (23) to

$$F(s) = \sum_{i=0}^{N} h_i(s)e^{sW_i} = h_0(s)e^{sW_0} \left(1 + h_1(s)e^{sW_1} \left(1 + h_2(s)e^{sW_2} \left(1 + \cdots \left(1 + h_N(s)e^{sW_N}\right)\right)\right)\right),$$  \hspace{1cm} (24)

where

$$h_i(s) = \prod_{k=0}^{i} h_k(s) \quad (i = 0, \cdots, N)$$  \hspace{1cm} (25)

and

$$W_i = \sum_{k=0}^{i} W_k \quad (i = 0, \cdots, N).$$  \hspace{1cm} (26)

The design procedure for $F(s)$ satisfying Equation (22) is summarized following procedure.
1. Let \( i = 0 \).

(a) Let \( h_i(s) \) is designed so that 
\[
|Q(j\omega_i)G_K(j\omega_i)h_i(j\omega_i)| = 1 \quad \text{are satisfied.}
\]
\( h_i(s) \) is established by
\[
h_i(s) = 1.
\] (27)

(b) A design of \( W_i \)
\( W_i \) is established to hold
\[
\{ \mathcal{L}(Q(s)G_K(s)) + \mathcal{L}e^{sW_i} \} |_{s=j\omega_i} = 0.
\] (28)

That is, \( W_i \) is established by
\[
W_i = 0.
\] (29)

(c) Add 1 to \( i \) and go to next step.

2. From Equation (24), \( h_i(s) \) is settled as
\[
h_i(s) = \prod_{k=0}^{i} \hat{h}_k(s),
\] (30)

where
\[
\hat{h}_i(s) = l_i \hat{h}_i(s),
\] (31)

\( l_i \in R, \hat{h}_i(s) \in RH_{\infty} \). When \( i = 1 \), \( \hat{h}_i(s) \) is settled by
\[
\hat{h}_i(s) = \frac{s}{b_i(s)}
\] (32)
to hold \( \hat{h}_i(\omega_0) = \hat{h}_i(0) = 0 \). When \( i > 1 \), \( \hat{h}_i(s) \) is settled by
\[
\hat{h}_i(s) = \frac{s^2 + \omega_i^2 - 1}{b_i(s)}
\] (33)
to hold \( \hat{h}_i(j\omega_{i-1}) = 0 \). Here, \( b_i(s) \) is any stable polynomial expression that makes \( \hat{h}_i(s) \) in Equation (32) or Equation (33) proper. Let \( l_i \)
\[
l_i = \frac{1 - Q(j\omega_i)G_K(j\omega_i)\sum_{k=0}^{i-1} h_k(j\omega_i)e^{j\omega_iW_k}}{Q(j\omega_i)G_K(j\omega_i)\hat{h}_i(j\omega_i)\prod_{k=0}^{i-1} \hat{h}_k(j\omega_i)}.
\] (34)

Next, we present a design method for \( W_i \).
\( W_i \) is established so that the phase angle of 
\( Q(j\omega_i)G_K(j\omega_i)h_i(j\omega_i)e^{j\omega_iW_i} \) is equal to that of 
\( 1 - Q(j\omega_i)G_K(j\omega_i)\sum_{k=0}^{i-1} h_k(j\omega_i)e^{j\omega_iW_k} \). That is, \( W_i \) is designed to hold
\[
\mathcal{L}(Q(j\omega_i)G_K(j\omega_i)h_i(j\omega_i)e^{j\omega_iW_i})
= \mathcal{L}
\left( 1 - Q(j\omega_i)G_K(j\omega_i)\sum_{k=0}^{i-1} h_k(j\omega_i)e^{j\omega_iW_k} \right).
\] (35)

\( W_i \) to hold Equation (35) is given by
\[
W_i = \frac{1}{\omega_i} \left\{ -\mathcal{L}(Q(j\omega_i)G_K(j\omega_i)h_i(j\omega_i)) 
+ \mathcal{L}
\left( 1 - Q(j\omega_i)G_K(j\omega_i)\sum_{k=0}^{i-1} h_k(j\omega_i)e^{j\omega_iW_k} \right) \right\}.
\] (36)

3. If \( i + 1 > N \), end up. In other cases, add 1 to \( i \) and return preceding step.

5 A design procedure of 2-degree-of-freedom repetitive control system

In this section we describe a design procedure of 2-degree-of-freedom repetitive control system.

A design procedure of 2-degree-of-freedom repetitive control system using design method of feed-forward controller in 4. is as follows:

1. A frequency component Equation (5) of the periodic reference input \( r \) with period \( T \) which the output \( y \) should follow is decided. That is, \( N \) in Equation (5) is settled.

2. design of feed-back controller

(a) Low pass filter \( q(s) \) is given by Equation (9). \( \tau_q > 0, \alpha_q \) is settled satisfying \( 1 - q(j\omega_i) \approx 0 \) \( (i = 0, \cdots, N) \). \( C_r(s) \) is give by Equation (7).

(b) \( \hat{C}(s) \) in Equation (6) is settled satisfying
\[
\left\| \frac{q(s)}{1 + G(s)\hat{C}(s)} \right\|_{\infty} < 1.
\] (37)

According to [10], if \( \hat{C}(s) \) in Equation (6) is settled satisfying Equation (37), the feed-back controller in Equation (6) stabilizes the plant \( G(s) \).

3. design of feed-forward controller

(a) The feed-forward controller \( \hat{G}(s) \) is \( RH_{\infty} \), \( \hat{Q}(s) \) is \( RH_{\infty} \) are designed satisfying Equation (13).

(b) \( F(s) \) is designed by the design procedure in 4.

4. The 2-degree-of-freedom repetitive control system in Fig. 1 is designed by using \( C(s), \hat{G}(s), \hat{Q}(s) \) and \( F(s) \).

6 Feed-forward controller for multiple-input/multiple-output systems

In this section, we expand the result in 4 and propose a design method of feed-forward controller for multiple-input/multiple-output systems.
Let us consider the 2-degree-of-freedom repetitive control systems in Fig. 1. Here $G(s) \in \mathbb{R}^{p \times p}(s)$ is the plant. $G(s)$ is assumed to be controllable and observable, and has no pole on the imaginary axis, and satisfies
\[ \text{rank } G(s) = p. \]  
(38)

$C(s) \in \mathbb{R}^{p \times p}(s)$ is the feed-back controller, $\hat{Q}(s) \in \mathbb{R}^{p \times p}(s)$, $\hat{G}(s) \in \mathbb{R}^{p \times p}(s)$ and $F(s) \in \mathbb{R}^{p \times p}(s)$ are the feed-forward controllers and satisfying Equation (1). $y \in \mathbb{R}^p$ is the output, $u \in \mathbb{R}^p$ is the control input, $r \in \mathbb{R}p$ is the periodic reference input with period $T$ written as Equation (2).

In the same manner as 4, the problem considered in this section is to find the design procedure of $F(s)$ satisfying
\[ I - G(s)\hat{G}(s)F(s) \bigg|_{s=j\omega_i} = 0 \quad (i = 0, \cdots, N). \]  
(39)

$\hat{G}(s)$ in Fig. 1 is settled by
\[ G(s)\hat{G}(s) = \hat{Q}(s) = Q(s)G_K(s), \]  
(40)

where $G_K(s) \in \mathbb{R}^{\alpha \times \alpha}$ is the diagonal inner function of $G(s)$ satisfying
\[ G_K(s) = \text{diag} \left\{ G_{K1}(s), \cdots, G_{Kp}(s) \right\} \]  
(41)

\[ G_{Ki}(-s)G_{Ki}(s) = 1 \quad (i = 1, \cdots, p), \]  
(42)

and
\[ G_K(0) = I, \]  
(43)

$Q(s)$ is
\[ Q(s) = \text{diag} \left\{ \frac{1}{(1+s\tau_1)^{q_1}}, \cdots, \frac{1}{(1+s\tau_p)^{q_p}} \right\} \]  
(44)

$\alpha_i (i = 1, \cdots, p)$ is arbitrary positive integer which makes $\hat{G}(s)$ be proper and $\tau_i (i = 1, \cdots, p)$ is sufficiently small positive real number.

From Equation (40), the problem of designing $F(s)$ satisfying Equation (39) is equivalent to hold
\[ I - Q(s)G_K(s)F(s) \bigg|_{s=j\omega_i} = 0 \quad (i = 0, \cdots, N). \]  
(45)

We settle $F(s)$ by
\[ F(s) = \text{diag} \left\{ F_1(s), \cdots, F_p(s) \right\}. \]  
(46)

From Equation (41), Equation (44) and Equation (46), Equation (45) is equivalent to
\[ 1 - q_m(s)G_{Km}(s)F_m(s) \bigg|_{s=j\omega_i} = 0 \quad (i = 0, \cdots, N : m = 1, \cdots, p). \]  
(47)

Equation (47) is a designing problem of feed-forward controller for single-input/single-output systems. Therefore, for multiple-input/multiple-output systems, we can design feed-forward controller $F(s)$ using the same procedure in 4.

7 Numerical example

In this section, a numerical example is illustrated to demonstrate the effectiveness of the proposed method.

Let the unstable plant $G(s)$ be
\[ G(s) = \frac{-s + 250}{s^2 + 17s - 200}. \]  
(48)

Let us consider to design repetitive control system for the plant $G(s)$ in Equation (48). The periodic reference $r$ with period $T = 1[\text{sec}]$ is written by
\[ r = \sin(2\pi t) + \sin(4\pi t). \]  
(49)

The feed-back controller $\hat{C}(s)$ is given by Equation (6). Where $C_r(s)$ and $q(s)$ are settled by Equation (7) and by
\[ q(s) = \frac{1}{1 + 0.01s}, \]  
(50)

respectively.

If $\hat{C}(s)$ in Equation (6) selected by
\[ \hat{C}(s) = \frac{100s^4 + 7000s^3 + 170000s^2 + 1570700s + 2779000}{s^4 + 500s^3 + 24800s^2 + 99341s}, \]  
(51)

satisfies stability condition in Equation (37). It is confirmed that feed-back controller $C(s)$ in Equation (6) stabilizes the plant $G(s)$.

Next we design feed-forward controller. Since the reference input $r$ is given by Equation (49), $N = 2$. $G_K(s)$ satisfying Equation (41) and Equation (42) is given by
\[ G_K(s) = \frac{-s + 250}{s + 250}. \]  
(52)

The low pass filter $Q(s)$ is settled by
\[ Q(s) = \frac{1}{1 + 0.01s}. \]  
(53)

From Equation (13), the feed-forward controller $\hat{G}(s)$ and $\hat{Q}(s)$ are written by
\[ \hat{G}(s) = \frac{100s^2 + 1700s - 20000}{s^2 + 350s + 25000}, \]  
(54)

and
\[ \hat{Q}(s) = \frac{-100s + 25000}{s^2 + 350s + 25000}, \]  
(55)

respectively.

When we design feed-forward controller $F(s)$ method in 4, $h_i(s)(i = 0, 1, 2)$, and $W_i(s)(i = 0, 1, 2)$ in Equation (23) are calculated by
\[ h_0(s) = 1, \]  
(56)
\[ h_1(s) = \prod_{k=0}^{1} \bar{h}_k(s), \quad (57) \]
\[ h_2(s) = \prod_{k=0}^{2} \bar{h}_k(s), \quad (58) \]
\[ \bar{h}_1(s) = \frac{0.018s}{0.01s + 1}, \quad (59) \]
\[ \bar{h}_2(s) = \frac{10^{-3} \left(0.045s^2 + 1.80\right)}{(0.01s + 1)^2}, \quad (60) \]
\[ W_0 = 0, \quad (61) \]
\[ W_1 = 0.0162, \quad (62) \]
\[ W_2 = 0.2781. \quad (63) \]

Next we show the difference of the response for the reference input between \( F(s) = 1 \) and the proposed method (\( F(s) \) is settled by Equation (23)). The error \( e = r - y \) for the reference input \( r \) is shown in Fig. 2. Here, the solid line shows time response in the case that \( F(s) \) is settled by Equation (23). The dotted line show time response in the case of \( F(s) = 1 \). From Fig. 2, we find that the proposed feed-forward controller \( F(s) \) in Equation (23) has better characteristics than \( F(s) = 1 \) such as the output follows without steady state error and the convergence speed is high.

**8 Conclusion**

In this paper, we gave a design method for feed-forward controllers using the stable filtered inverse systems in [13, 14] and the predictive control, and proposed a design method of 2-degree-of-freedom repetitive control system.

**References**


