ADAPTIVE CHAOTIC SYNCHRONIZATION THROUGH DECENTRALIZED EXTENDED KALMAN-TYPE OBSERVERS

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Keywords: Decentralized adaptive observer, chaotic systems, synchronization, secure communications

Abstract

This paper deals with the problem of adaptive observer-based synchronization of chaotic dynamical systems. The proposed approach consists in estimating the states and the unknown parameters in a separate way to reduce computational requirements. A rigorous stability analysis is performed and a judicious parametrization of the algorithm is proposed to enlarge the basin of attraction with high tracking ability. The results are illustrated by numerical examples of adaptive synchronization of Chua’s system with application to the problem of encoded signal transmission in secure communications.

1 Introduction

Synchronization of dynamical chaotic systems has received growing attention during the last decade due to its potential applications to secure communications problems. From a control theory point of view, chaos synchronization can be considered as an observer design problem in the sense that the receiver system is an observer of the chaotic transmitter system [1, 2, 3]. Few results, however, have been established to deal with the problem of synchronizing chaotic systems when the designer of the receiver doesn’t know some of the parameters of the transmitter, this problem is referred to adaptive (non linear) observers [4, 5] or adaptive synchronization [6], the lack of knowledge of some parameters may corrupt the synchronization and even break it. On the other hand, adaptive synchronization is an essential topic in secure communications when parameter modulation technique is used for message transmission, this approach consists in encoding the information message into the chaotic transmitter and the receiver should be able to recover the message (see [7] and the references inside for an overview of decoding observer-based techniques).

In this contribution, we propose a useful and decentralized algorithm for the states and parameters estimation of a large class of chaotic transmitters in the deterministic context. We show that the proposed technique is equivalent to the global extended Kalman observer when the state vector is completed by the unknown parameters. Some connections between the basin of attraction of the observer and the design of weighting arbitrary matrices are established. Performance of the proposed approach will be shown trough numerical simulations on the classical Chua’s circuit with applications in secure communications.

2 The decentralized adaptive synchronization algorithm

Consider a chaotic discrete-time transmitter system written in the general form :

\[ x_{k+1} = f(x_k, y_{ik}, \ldots, y_{ik}, \theta) \]  
\[ y_k = C^T x_k \quad i = 1, \ldots, q \]

where \( x_k \in \mathbb{R}^n \) is the state vector of the transmitter, \( f \) is a differentiable nonlinear function, \( y_{ik} \) is the \( i \)th transmitted chaotic signal, \( C_i \) is the \( i \)th row of a known real matrices \( C \in \mathbb{R}^{q \times n} \) and \( \theta = \theta_k = \theta_{k+1} \in \mathbb{R}^p \) is an unknown constant parameter vector that possibly contains the encoding messages in case of secure communications.

To achieve adaptive chaotic synchronization, the objective is to design an observer-based receiver for transmitter system (1a)-(1b) that provides both estimates \( \hat{x}_k \) and \( \hat{\theta}_k \) of the unknown transmitter states and parameters respectively based on the transmission of one or more driving signals \( y_{ik}, i = 1, \ldots, q. \)

An intuitive approach consists in using nonlinear estimation techniques on the augmented system where the state vector has been extended with the unknown parameters. However, in addition to the large computational requirements, we notice that this approach is very sensitive to initial conditions with some divergence problems occur when the initialization is not sufficiently close to the actual values. In order to avoid the previous problems, we propose to estimate the state and the parameter vectors in a separate way. Therefore the dynamic of the receiver is :

\[ \hat{x}_{k+1} = l(\hat{x}_k, y_{1k}, \ldots, y_{qk}, \hat{\theta}_k) \]  
\[ \hat{\theta}_{k+1} = m(\hat{x}_k, y_{1k}, \ldots, y_{qk}, \hat{\theta}_k) \]

where we want to design the functions \( m(\cdot) \) and \( l(\cdot) \) in order to
ensure local asymptotic convergence:

\[
\lim_{k \to \infty} (x_k - \hat{x}_k) = 0 \quad (4)
\]
\[
\lim_{k \to \infty} (\theta - \hat{\theta}_k) = 0 \quad (5)
\]

for all \((x_0 - \hat{x}_0) \in B^x_0\) and \((\theta_0 - \hat{\theta}_0) \in B^\theta_0\) where \(B^x_0\) and \(B^\theta_0\) are large enough open subsets \((B^x = \{x : ||x|| \leq \epsilon^x, \epsilon^x \geq 0\})\).

Using standard Kalman theory, the decoupled states and parameters estimation algorithm that we propose is of the following form:

**Stage 1: states estimation**

\[
\begin{align*}
\hat{x}_{k+1} &= \hat{x}_{k+1/k} + K_{k+1}^x e_{k+1} \\
K_{k+1}^x &= \Phi_{k+1} \Delta_{k+1} \\
P^x_{k+1/k} &= (F_k^x F_k^\theta P_{k+1/k}^\theta P_{k+1/k}^\theta F_k^x + Q_k^x) \\
P^x_{k+1} &= P^x_{k+1/k} - K_{k+1}^x \Phi_{k+1}^T 
\end{align*} \quad (6-9)
\]

**Stage 2: parameters estimation**

\[
\begin{align*}
\hat{\theta}_{k+1} &= \hat{\theta}_k + K_{k+1}^\theta e_{k+1} \\
K_{k+1}^\theta &= \Psi_{k+1} \Delta_{k+1} \\
P^\theta_{k+1/k} &= P^\theta_{k+1/k} + Q_k^\theta \\
P^\theta_{k+1} &= P^\theta_{k+1/k} - K_{k+1}^\theta \Psi_{k+1}^T \quad (10-13)
\end{align*}
\]

where

\[
\begin{align*}
\hat{x}_{k+1/k} &= f(\hat{x}_k, \hat{\theta}_k, y_{k}, \cdots, y_{k}, u_k) \\
\Delta_{k+1} &= (C^x P^x_{k+1/k} C^{xT} + R_{k+1})^{-1} \\
\Phi_{k+1} &= P^x_{k+1/k} C^{xT} \\
\Psi_{k+1} &= P^\theta_{k+1/k} C^{xT} \\
P^\theta_{k+1/k} &= F_k^\theta P_k^\theta F_k^\theta + Q_k^\theta \\
P^\theta_{k+1} &= P^\theta_{k+1/k} - K_{k+1}^\theta \Psi_{k+1}^T 
\end{align*} \quad (14-19)
\]

with

\[
F_k^x = \left. \frac{\partial f(x_k, y_k, \cdots, y_k, u_k)}{\partial x_k} \right|_{\hat{x}_k, \hat{\theta}_k} \\
F_k^\theta = \left. \frac{\partial f(x_k, y_k, \cdots, y_k, u_k)}{\partial \theta} \right|_{\hat{x}_k, \hat{\theta}_k} \\
e_{k+1} = y_{k+1} - C^x \hat{x}_{k+1/k} = C^x (x_{k+1} - \hat{x}_{k+1/k}) 
\]

**3 Convergence analysis**

Before we give the main result in this section, we show first that the proposed separate-bias observer is equivalent to the global state and bias estimation algorithm obtained from the Extended Kalman Observer (EKO) where the states are augmented by the bias vector. Indeed, (7), (11), (15), (16) and (17) may be written as:

\[
\begin{align*}
\begin{bmatrix} K_{k+1}^x \\ K_{k+1}^\theta \end{bmatrix} &= \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} \begin{bmatrix} C^x \\ 0 \end{bmatrix} \times \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} \begin{bmatrix} C^{xT} \\ 0 \end{bmatrix} + R_{k+1}^{-1} \\
\begin{bmatrix} F_k^x \\ F_k^\theta \end{bmatrix} &= \begin{bmatrix} F_k^x \\ F_k^\theta \end{bmatrix} \begin{bmatrix} Q_k^x \\ Q_k^\theta \end{bmatrix} \\
C &= (C^x) \\
Q_k &= \begin{bmatrix} Q_k^x \\ Q_k^\theta \end{bmatrix}\end{align*} \quad (23-24)
\]

also from (9), (13) and (19) we deduce that:

\[
\begin{align*}
\begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} &= \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} \begin{bmatrix} C^x \\ 0 \end{bmatrix} \times \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix} + \begin{bmatrix} F_k^x \\ F_k^\theta \end{bmatrix} \begin{bmatrix} Q_k^x \\ Q_k^\theta \end{bmatrix} \\
F_k &= \begin{bmatrix} F_k^x \\ F_k^\theta \end{bmatrix}, C = (C^x) \quad \text{and} \quad Q_k = \begin{bmatrix} Q_k^x \\ Q_k^\theta \end{bmatrix} \end{align*} \quad (25)
\]

By using the notations \(K_{k+1} = \begin{bmatrix} K_{k+1}^x \\ K_{k+1}^\theta \end{bmatrix}\), \(P_{k+1} = \begin{bmatrix} P_{k+1/k}^x \\ P_{k+1/k}^\theta \end{bmatrix}\), \(F_k = \begin{bmatrix} F_k^x \\ F_k^\theta \end{bmatrix}\), \(C = (C^x)\) and \(Q_k = \begin{bmatrix} Q_k^x \\ Q_k^\theta \end{bmatrix}\), it’s easy to verify that (23), (24) and (25) are the propagation equations of the augmented EKO.

In the following we show, by the use of the exact linearization technique presented in [8], that the design of the instrumental matrices, in particular \(Q_k^x\) and \(Q_k^\theta\), plays a central role to improve tracking and rate of convergence of the proposed separate-bias observer. In the rest of the paper and without loss of generality, we set \(Q_k^\theta = 0\).

Let us introduce the error vectors \(\tilde{x}_{k+1/k}, \tilde{x}_{k+1/k} / \tilde{\theta}_{k+1/k} :\)

\[
\begin{align*}
\tilde{x}_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\
\tilde{x}_{k+1/k} &= x_{k+1} - \hat{x}_{k+1/k} \\
\tilde{\theta}_{k+1} &= \theta_{k+1} - \hat{\theta}_{k+1} \\
\tilde{\theta}_{k+1/k} &= \theta_{k+1} - \hat{\theta}_{k+1/k} \end{align*} \quad (26-28)
\]

and consider the following candidate Lyapunov function \(V_{k+1} :\)

\[
V_{k+1} = \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix}^T P_{k+1} \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} \quad (29)
\]

The goal is to point out conditions so that \(V_{k+1}\) is a decreasing sequence.
First of all, we introduce an unknown diagonal matrix \( \beta_k \) to parametrize all the errors due to the first order linearization technique of the nonlinear function \( f \) \[8\]. We have then the following exact relation:

\[
\hat{x}_{k+1/k} = \beta_k (F_k^T \hat{x}_k + F_k^T \hat{\theta}_k) \tag{30}
\]

instead of the following approximation usually used in the literature:

\[
\hat{x}_{k+1/k} \approx F_k^T \hat{x}_k + F_k^T \hat{\theta}_k \tag{31}
\]

which are correct only at the neighborhood of the actual trajectories.

The unknown diagonal matrix \( \beta_k \) depends on how far \( \hat{x}_k \) and \( \hat{\theta}_k \) are from zeros. In the following, (30) will be used in the Lyapunov function in order to evaluate the propagation errors and to point out connections between convergence of the proposed observer and the linearisation errors. From (23) to (25), we have:

\[
\begin{align*}
K_{k+1} &= P_{k+1}C^TR_{k+1}^{-1}P_{k+1}C^T + R_{k+1}^{-1} \\
\bar{P}_{k+1}^{-1} &= P_{k+1}C^T \left( CP_{k+1}C^T + R_{k+1}^{-1} \right) \\
P_{k+1}^{-1} &= P_{k+1}C^TR_{k+1}^{-1}C
\end{align*} \tag{32} \tag{33}
\]

Using (32), (6) and (10), the Lyapunov function \( V_{k+1} \) (29) becomes:

\[
\begin{align*}
V_{k+1} &= \left( \hat{x}_{k+1/k} \right)^T P_{k+1}^{-1} \hat{x}_{k+1/k} \\
&= \left( \hat{x}_{k+1/k} \right)^T C^TR_{k+1}^{-1} e_{k+1} - e_{k+1}^T R_{k+1}^{-1} C \left( \hat{x}_{k+1/k} \right) \\
&\quad + e_{k+1}^T R_{k+1}^{-1} C P_{k+1} C^T R_{k+1}^{-1} e_{k+1}
\end{align*} \tag{34}
\]

Using (33), (30), (22) and the special structure of \( C = (C^T 0) \) we obtain:

\[
\begin{align*}
V_{k+1} &= \left( \hat{x}_k \right)^T \hat{\beta}_k (F_k P_k F_k^T + Q_k)^{-1} \hat{\beta}_k F_k \left( \hat{x}_k \right) \\
&\quad + e_{k+1}^T (-R_{k+1}^{-1} + R_{k+1}^{-1} C S P_{k+1} C^T R_{k+1}^{-1}) e_{k+1}
\end{align*} \tag{35}
\]

with the extended matrix \( \hat{\beta}_k = \left( \beta_k 0 \right) \).

The Lyapunov sequence \( \{V_k\}_{k=1}^\infty \) is a decreasing one if there exists a positive scalar \( \zeta \in [0, 1] \) so that:

\[
V_{k+1} - V_k \leq -\zeta V_k, \tag{36}
\]

or equivalently:

\[
\begin{align*}
V_{k+1} - (1 - \zeta) V_k &= \left( \hat{x}_k \right)^T \left( F_k^T \hat{\beta}_k \times \right. \\
&\quad \left. (F_k P_k F_k^T + Q_k)^{-1} \hat{\beta}_k F_k - (1 - \zeta) V_k \right) \left( \hat{x}_k \right) \\
&\quad + e_{k+1}^T (\cdots) e_{k+1}
\end{align*} \tag{37}
\]

We notice that a sufficient condition to ensure (37) consists in verifying the following couple of inequalities:

\[
\begin{align*}
F_k^T \hat{\beta}_k (F_k P_k F_k^T + Q_k)^{-1} &\leq (1 - \zeta) P_k^{-1} \tag{38} \\
R_{k+1}^{-1} C S P_{k+1} C^T &\leq I_q \tag{39}
\end{align*}
\]

In the following theorem we give sufficient conditions to ensure asymptotic convergence of the proposed adaptive observer.

**Theorem 1** If we assume that:

\( i. \) The augmented – when the states are completed by the parameters vector – linearized system along the estimator’s trajectory is \( N \)-locally observable, i.e. there exist a finite integer \( N \geq 1 \) and positive real numbers \( \gamma_1 \) and \( \gamma_2 \) such that, for all \( k \geq N - 1 \), we have:

\[
\gamma_1 I_{n+q} \leq O_{k\rightarrow N+1,k}^T \cdot \left( \begin{array}{c}
\gamma_2 I_{n+q} \\
\gamma_2 I_{n+q}
\end{array} \right) \tag{40}
\]

where \( 0 < \gamma_1, \gamma_2 < \infty \) and,

\[
O_{k\rightarrow N+1,k} = \left( \begin{array}{c}
C \\
CF_k \gamma_1 \\
\vdots \\
CF_k \gamma_1 - F_k - 2 \ldots F_k - N + 1
\end{array} \right) \tag{41}
\]

for all \( (\hat{x}_k, \hat{\theta}_k) \in N(x_{x,0}) \) which denotes a neighborhood of \( (x_k, \theta_k) \).

\( ii. \) \( F_k, F_k^\theta \) are uniformly bounded matrices and \( F_k^{e^{-1}} \) exists.

\( iii. \) The instrumental matrix \( \hat{\beta}_k \in \mathbb{R}^{n \times 1} \) satisfies:

\[
\hat{\beta}_k \leq \left( \frac{1 - \zeta}{\sigma_{\min}(P_k)\sigma_{\max}(P_k)} \sigma_{\min}(F_k^T) \right)^{1/2} \Pi_k^c \tag{42}
\]

with \( 0 < \hat{\beta}_k = \max_{i=1,\ldots,n} (|\hat{\beta}_k|, 1) < \infty \),

\( iv. \) The weighting matrix \( R_{k+1} \) is chosen as:

\[
R_{k+1} = \lambda C S P_{k+1} C^T, \quad \lambda \geq 1 \tag{43}
\]

then (6)-(22) is a decentralized adaptive observer for (1a)-(1b) and:

\[
\lim_{k\to\infty} (x_k - \hat{x}_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} (\theta - \hat{\theta}_k) = 0
\]

\( \sigma_{\max} \) and \( \sigma_{\min} \) denote the maximum and minimum singular values respectively.

**Proof:** First of all, hypothesis \( ii. \) and the local observability assumption (40) are introduced to assure that the matrix \( P_k \) is bounded from above and below, for more details see \[9\], while
hypothesis iii. and iv. lead to obtain a decreasing Lyapunov function \( \{ V_k \}_{k=1}^{\infty} \). Indeed, using the fact that \( \beta_k \) is diagonal matrices and from (42), we obtain:

\[
(\sigma_{\text{max}}(\beta_k))^2 \leq \frac{(1 - \zeta)\sigma_{\text{min}}(P_k^{-1})}{\sigma_{\text{max}}(F_k)\sigma_{\text{max}}((F_k P_k F_k^T + Q_k)^{-1})\sigma_{\text{max}}(F_k)} \quad (44)
\]

As,

\[
\sigma_{\text{max}}(F_k^T \beta_k (F_k P_k F_k^T + Q_k)^{-1} \beta_k F_k) \leq (\sigma_{\text{max}}(\beta_k))^2 \sigma_{\text{max}}(F_k^T)\sigma_{\text{max}}((F_k P_k F_k^T + Q_k)^{-1}) \times \sigma_{\text{max}}(F_k)
\]

We have then:

\[
\sigma_{\text{max}}(F_k^T \beta_k (F_k P_k F_k^T + Q_k)^{-1} \beta_k F_k) \leq (\sigma_{\text{max}}(\beta_k))^2 \sigma_{\text{max}}(F_k^T)\sigma_{\text{max}}((F_k P_k F_k^T + Q_k)^{-1}) \times \sigma_{\text{max}}(F_k) \leq (1 - \zeta)\sigma_{\text{min}}(P_k^{-1})
\]

which induce that (38) is satisfied.

On the other hand, substituting (43) into (39), we have:

\[
\lambda^{-1}(C_k P_k x_k^T C_k^T)^{-1} (C_k P_{k+1} x_k^T C_k^T) \leq I_q
\]

which is satisfied for all \( \lambda \geq 1 \).

Therefore, (45) and (47) induce that \( V_k \) is a decreasing Lyapunov function.

Finally, as \( P_k \) is a bounded matrix, it follows from (36) that:

\[
0 \leq \mu \begin{pmatrix} \tilde{x}_k^T \\ \tilde{\theta}_k^T \end{pmatrix} \begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix} \leq V_k \leq (1 - \zeta)^k V_0
\]

which implies:

\[
0 \leq \mu \lim_{k \to \infty} \begin{pmatrix} \tilde{x}_k^T \\ \tilde{\theta}_k^T \end{pmatrix} \begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix} \leq \lim_{k \to \infty} (V_k) \leq V_0 \lim_{k \to \infty} (1 - \zeta)^k = 0
\]

with \( \mu I_{n+p} \leq P_k^{-1} \).

Therefore:

\[
\lim_{k \to \infty} \begin{pmatrix} x_k - \tilde{x}_k \\ \theta - \tilde{\theta}_k \end{pmatrix} = 0
\]

**Remark 1**: The sufficient conditions (42) represents, in a way, the attraction region in the state and parameters spaces. Indeed, in the body of the paper we introduced a linearisation technique in the form of (30) to quantify the distance between the actual and estimated trajectories and in the same time we deduce a sufficient condition (42) on the norm of \( \beta_k \) so that we obtain a decreasing Lyapunov function, i.e. asymptotic convergence of the observer. Contrary to the standard EKO where \( \beta_k \) is assumed to be very close to the identity matrix (that is very small initialization errors), the proposed approach ensures asymptotic convergence for all matrix \( \beta_k \) (which represent the linearisation errors) bounded by \( \Pi_k^\beta \). On the other hand, as long as \( (\tilde{x}_k \theta_k) \) goes to \( (x_k \theta_k) \), the matrix \( \beta_k \) converges to the identity and the approximation (31) usually used in the literature becomes valid. The stability analysis is then similar to the linear case.

Design of the arbitrary matrices \( Q_k^x \) and \( Q_k^\theta \). In order to satisfy the condition (42), depends on the evaluation of the upper bound \( \beta_k \) on the norm of \( \beta_k \) in particular at the initialization of the algorithm. In practice, from our knowledge of the system to synchronize like Lorenz system, Rossler system or Chua circuit, most often we have an idea on the bounds of the state and parameter vectors to be estimated and by the use of the non-linearities \( f(.) \) we may evaluate realistic values of \( \beta_k \). In addition, we should recall that the proposed approach in only local but improve the convergence significantly.

**Remark 2**: Hereafter, by a quantitative analysis, the aim is to design the weighting matrices \( Q_k^x \) and \( Q_k^\theta \) so that the bound \( \Pi_k^\beta \) is as large as possible in order to satisfy condition (42).

Before we introduce a simple design of \( Q_k^x \) and \( Q_k^\theta \), we notice that the upper bound \( \Pi_k^\beta \) is large when \( Q_k \) (consequently \( Q_k^x \) and \( Q_k^\theta \)) is large. Indeed, since the matrices \( P_k \) and \( F_k \) depend on \( Q_k \) computed at time instant \( k-1 \), \( Q_k \) is a free weighting matrix to enlarge the above upper bound in the sense that we have:

\[
\frac{(1 - \zeta)\sigma_{\text{min}}(F_k P_k F_k^T + Q_k^\beta)}{(\sigma_{\text{max}}(F_k^T)\sigma_{\text{max}}(P_k)\sigma_{\text{max}}(F_k))^{1/2}} \geq \frac{(1 - \zeta)\sigma_{\text{min}}(F_k P_k F_k^T + Q_k^\beta)}{(\sigma_{\text{max}}(F_k^T)\sigma_{\text{max}}(P_k)\sigma_{\text{max}}(F_k))^{1/2}} = \frac{1}{Q_k^\beta} 
\]

for \( Q_k^\beta \geq Q_k^x > 0 \). However, we should have in mind that matrices \( P_k \) and \( P_{k+1} \) have to remain bounded and, in order to avoid numerical instabilities, \( Q_k \) can not be chosen extremely large.

**Remark 3**: In the following we propose a simple and efficient design, which is not unique, for \( Q_k^x \) and \( Q_k^\theta \):

\[
Q_k^x = \zeta^e e^e e_k I_n + e^e e_k I_n
\]

\[
Q_k^\theta = \zeta^e e^e e_k I_p + e^e I_p
\]

where \( e_k = y_k - C^e \tilde{x}_k \) and the positive scalars \( \zeta^e \) and \( \zeta^\theta \) have to be chosen sufficiently large especially in case of bad initial conditions and high non-linearities of \( f(.) \), \( e^e \) and \( e^\theta \) are small enough scalars introduced to avoid numerical instabilities.

The original idea of designing \( Q_k^x \) and \( Q_k^\theta \) is, in fact, the presence of the output error \( e_k \) which controls automatically width of the upper bound on \( \beta_k \). Indeed, in case of high non-linearities and/or arbitrary large initialization errors, the terms \( \zeta^e e^e e_k \) and \( \zeta^\theta e^\theta e_k \) (and consequently the bound) become large.

**Remark 4**: Considering the weighting matrix (43), we remark that setting high values of \( \lambda \) leads to a very slow convergence rate or even a divergence of the algorithm (the gains \( K_{k+1}^x \) and \( R_{k+1}^\theta \) quickly go to zero) so much so that the choice of \( \lambda \) must
be a judicious compromise between stability and rate of convergence of the proposed observer.

Since the proposed approach is only local but may be applied to a very large class of non-linear systems, it is very difficult to give an idea, a priori, on the values of the parameters \( \zeta \), \( \zeta^0 \) and \( \lambda \) which depend on many factors such as initializations or degree of non-linearity of the system. In practice, suitable values of these parameters are determined, by an iterative approach. From our experience (we have tried this approach on several chaotic systems and on a large variety of physical processes under severe conditions) \( \zeta^0 \), \( \zeta^0 \) and \( \lambda \) belong in general to the intervals \([10^0, 10^{20}]\), \([10^0, 10^{20}]\) and \([1, 20]\) respectively.

Remark 5 : Using the special structures of \( F_k = \begin{pmatrix} F^x_k & F^y_k \\ 0 & I_q \end{pmatrix} \) and \( C = \begin{pmatrix} C_x & 0 \end{pmatrix} \), the observability matrix \( Q_{[k-N+1,k]} \) may be partitioned as follows:

\[
Q_{[k-N+1,k]} = [Q_{[k-N+1,k]}^y Q_{[k-N+1,k]}^x]
\]

with

\[
Q_{[k-N+1,k]}^y = \begin{pmatrix} C^x F^y_{k-N+1} \\ C^x F^y_{k-1} \ldots F^y_{k-N+1} \\ \vdots \\
C^x(F^y_{k-1} \ldots F^y_{k-N+1} + F^y_{k-2} \ldots F^y_{k-N+2} + \ldots + F^y_{k-1}) \end{pmatrix}
\]

As we can expect, if the original system (1a)-(1b) isn’t in some sense “observable” or “persistently excited” that is if \( \text{rang}(Q_{[k-N+1,k]}^y) = \cdot \) or \( \text{rang}(Q_{[k-N+1,k]}^x) = \cdot \) then the observability condition (40) is not verified. In particular, when a state has null initial condition and is not reachable, then all parameters multiplying only this state variable couldn’t be identified because they are not persistently excited i.e. the corresponding columns in \( F^y_k \) are zero and then we have rank \( \text{rang}(Q_{[k-N+1,k]}^y) < p \).

Remark 6 : In order to reduce the computational requirements and then the numerical instabilities, in particular, for large scale systems the proposed two stage algorithm may be implemented, in real time applications, on a separate processor system.

4 Numerical examples

Example 1 : Synchronization of Chua’s circuit

Consider the example of chaotic synchronization where both transmitter and receiver systems are implemented as Chua circuits with unknown constant parameters [10]. The transmitter system, so-called double scroll chaotic attractor (see Figure 1), is described by the following set of nonlinear differential equations:

\[
\dot{x}_1(t) = \alpha(-x_1(t) + x_2(t) - g(x_1(t))) \\
\dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t) \\
\dot{x}_3(t) = -\gamma x_2(t)
\]

where \( g(x_1) = m_1 x_1 + \frac{m_0 - m_1}{2}(|x_1 + 1| - |x_1 - 1|) \) with \( m_0 = -8/7 \) and \( m_1 = -5/7 \). The transmitted driven signal is \( y(t) = x_1(t) \). We suppose that the parameters \( \alpha \) and \( \lambda \) are unknown but constants (i.e. \( \theta = (\alpha \gamma) ^T, \theta = 0 \)).

Using Euler discretization method, the discrete-time model of Chua’s circuit is:

\[
x_{1k+1} = (1 - \alpha T)x_{1k} + \alpha T(x_{2k} - g(x_{1k})) \\
x_{2k+1} = T x_{1k} + (1 - T)x_{2k} + T x_{3k} \\
x_{3k+1} = x_{3k} - \gamma T x_{2k}
\]

where \( T \) is the sampling period and the driven signal is \( y_k = x_{1k} \).

The initial conditions for the receiver were taken as \( x_0 = (0.1 \ 0.1 \ 0.1)^T \). The true values of the supposed unknown parameters are: \( \alpha = 15.6 \) and \( \lambda = 27 \). The sampling period is \( T = 0.01 \) second.

For the receiver, the initial conditions are very bad for the states as well as for the unknown parameters: \( \hat{x}_0 = (1 \ 1 \ 1)^T \) and \( \hat{\theta}_0 = (10 \ 20)^T \) with \( F_0^x = 10^{10} I_3 \) and \( F_0^y = 10^{10} I_2 \). In order to ensure convergence of the algorithm, the weighting matrices \( Q_k^x, Q_k^y \) and \( R_k+1 \) are chosen as in (51), (52) and (43) with \( \zeta^x = \zeta^y = 10^6 \), \( \lambda = 1 \) and \( \epsilon^x = \epsilon^y = 10^{-3} \).

The results of the simulations are reported in Figure 2 and 3. A comparison has been realized with the classical extended Kalman filter when the weighting matrices \( Q_k^x, Q_k^y \) and \( R_k+1 \) are chosen as \( Q_k^x = Q_k^y = 10^{-3} I_3 \), \( Q_k^y = 10^{-3} I_2 \) and \( R_k+1 = R = 10^{-3} \). The plots clearly show high performances of the proposed adaptive observer and the important role of the instrumental matrices \( Q_k^x, Q_k^y \) and \( R_k+1 \) to ensure strong convergence even with very bad initial conditions.

Example 2 : Secure communications using Chua’s circuit

In the current example, we are going to present the potential applications of synchronized Chua’s systems in secure transmissions. In this case, the driven signal \( y_k \) contains a binary
Then we use the proposed decentralized algorithm to estimate the modulated parameters in order to recover the information signal \( s_k \). Here we suppose that the message modulate the parameters \( \alpha \) of the transmitter with the following modulation rule:

\[
\alpha_k = \alpha_0 + \alpha_1 s_k
\]

where \( s_k = 0.5(1 - \text{sgn} (\sin (\frac{2\pi k T}{T_0}))) \) with \( \alpha_0 = 15.6, \alpha_1 = 0.5 \) and the period of \( s_k \) is \( T_0 = 20 \text{ s} \).

The initial conditions for the transmitter are the same as in the first example, we suppose that only the parameter \( \alpha \) is unknown.

The initial conditions of the receiver are: \( \hat{x}_0 = (1 \ 1 \ 1)^T \) and \( \hat{\alpha}_0 = 16.5 \) with \( P_{\hat{x}}^0 = 10^{20} I_3 \) and \( P_{\hat{\alpha}}^0 = 10^{20} \). The weighting matrices \( Q^0_k, Q^0 \) and \( R_{k+1} \) are chosen as in (51), (52) and (43) with \( \zeta^x = 1, \zeta^\theta = 10^{15}, \lambda = 15 \) and \( \varepsilon^x = \varepsilon^\theta = 10^{-3} \).

Figure 4 shows the secure communication via binary parameter modulation. We easily verify that the proposed algorithm recovers the information message \( s_k \) at the end of the communication link without any information on the dynamics of \( s_k \).

5 Conclusion

In this contribution a simple decoupled adaptive observer for chaos synchronization has been derived. Sufficient conditions for local asymptotic stability were established. Convergence and strong tracking ability were illustrated by chaotic synchronization using the classical Chua’s circuit.

References


