OBSEIVER BASED STABILIZATION
OF DISCRETE-TIME NONLINEAR SYSTEMS

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Keywords : Nonlinear systems, stabilization, observer-based control, système temps-discret, EKO.

Abstract
In this note we propose an observer based stabilization method for nonlinear discrete-time systems. The approach we use here is based on the stabilization method recently developed in [5] coupled with the EKO. From the Lyapunov approach, sufficient conditions for stability are deduced and expressed in terms of LMI that depend on arbitrary matrices fixed by the user. This has the advantage to enlarge the class of systems to be considered as it can be shown through numerical examples.

1. Introduction
Over the past four decades, stabilization of nonlinear dynamical systems has received a great attention in the literature as it can be shown through basic works in this field [3], [15] and [22]. Several design methodologies have been developed for local and global stabilization problems of continuous and discrete-time nonlinear systems, see for instance [1], [6], [18], [19], [21], [23] and the references inside.

When the control laws are designed, the state variables are assumed to be available. But in general, this is not true in practice and the current state must be estimated by another dynamical system, that is a state observer.

Thus, observer based stabilization of nonlinear systems has been studied in the past few years. The main contributions, however, concern continuous time systems ; this problem has been investigated by several authors, among them [2], [9], [12], [14] and [20].

For discrete-time nonlinear systems only few designing methods have been established [6], [8] and [17]. Relevant ones have been developed by Byrnes and Lin [7] and Lin [17]. In particular the work in [17], where a global stabilization is achieved via state and output feedback, the proposed technique is judicious but only systems with stable state unforced dynamics are considered, this may be seen as a conservative condition.

The aim of this work is to analyse behaviour of the state feedback stabilization method recently developed in [5] with the use of the EKO. Thanks to simple Lyapunov function, sufficient condition for stability are deduced and seem to work for a large clan of nonlinear systems even with unstable unforced dynamics. Two numerical examples are provided to show performances of the proposed method and easiness of the implementation.

2. Problem formulation
Consider a class of discrete-time multi-input multi-output (MIMO) nonlinear systems of the form
\[
\begin{align*}
    x_{k+1} &= A(x_k)x_k + g(x_k)u_k = f(x_k, u_k) \\
    y_k &= h(x_k)
\end{align*}
\]
where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^p \) and \( y_k \in \mathbb{R}^q \) denote the state, input and output vectors respectively. The matrix \( A(.) \), \( g(.) \) and the vector \( h(.) \) are continuously differentiable nonlinear maps.

Problem :
The problem is to find a dynamic compensator
\[
\begin{align*}
    \xi_{k+1} &= \eta(\xi_k, y_k) \\
    u_k &= \theta(\xi_k) \\
    x_{k+1} &= A(x_k)x_k + g(x_k)\theta(\xi_k)
\end{align*}
\]
so that the closed loop system (1)-(3) is asymptotically stable at the equilibrium \((x, x-\xi) = (0,0)\).

3. Main result
Consider the following EKO based stabilizer
\[
\begin{align*}
    \dot{\xi}_k &= \xi_{k+1} + K_{k+1}e_{k+1} \\
    u_k &= -L\xi_k
\end{align*}
\]
and the error vectors
\[
\begin{align*}
    \tilde{x}_{k+1} &= x_{k+1} - \xi_{k+1} \\
    \tilde{x}_{k+1} &= x_{k+1} - \xi_{k+1}
\end{align*}
\]
Where
\[ \xi_i = f_t(\xi_t) \]  
(7)
\[ f_{u_t}(x_t) = f(x_t, u_t) \]  
(8)
\[ L_t = \Omega_t + g_t^T P_t g_t \]  
(9)
\[ P_{t+1} = \tilde{A}_t(N_t P_t A_t + L_t \Omega_t L_t^T + Q) \]  
(10)
\[ K_{t+1} = \Sigma_{t+1} H_t R_t^{-1} (H_t \Sigma_{t+1} H_t^T R_t^{-1} + \Lambda_0) \]  
(11)
\[ \Sigma_{t+1} = F_{t+1} \Sigma_{t+1} F_{t+1}^T + \Delta \]  
(12)
\[ A_i = A(g(\xi_i), g_i(\xi_i)) \]  
(13)

With
\[ e_{x,t} = y_{x,t} - h(\xi_{x,t}) \]  
(14)
\[ F_x = f_u(x, \xi_i) = \frac{\partial f(x_t - L_t x_t)}{\partial x_t} \bigg|_{x_t} \]  
(15)
\[ H_x = H_x(\xi_{x,t+1}) = \frac{\partial h(x)}{\partial x} \bigg|_{x_t} \]  
(16)
\[ A_i = A(g_i(\xi_i), g_i(\xi_i)) \]  
(17)
\[ P_t = P_t(\xi_i) \]  
(18)

The main result of this paper is summarized in the following theorem.

### 3.1 Theorem:

Assume that there exists an integer N such that:

H1) \[ \sqrt{\lambda_{1,2} \cdots \lambda_{N}} \| A_i A_{i-1} \cdots A_1 \| < 1. \]

H2) \[ P_t A_i \tilde{A}_t \left( P_t^+ \Omega_t^{-1} \tilde{g}_t^{-1} \tilde{A}_t + \Omega_t \right) A_i P_t^{+2} \leq \lambda_i (1 - \delta) L_t. \]

H3) \[ \text{rank} \left( \frac{\partial}{\partial x} \begin{bmatrix} h_1(x) \\ h_{x_1} \circ f_{-1}(x) \\ \vdots \\ h_{x_{N-1}} \circ f_{-2}(x) \cdots \circ f_{-1}(x) \end{bmatrix} \right) = n \]  
(19)

for all \( x_k \in K \) and N-tuple of controls \( (u_k, \cdots, u_{x_k}) \in U(K \times U) \) and U are two compact subsets of IR^2 and (IR^2)^n, respectively.

H4) \( F_t \) and \( H_t \) are uniformly bounded matrices and \( F_t \) exists.

H5) The instrumental matrices \( R_t, S_t \) are chosen so that \( [\alpha_{x,t} - 1] \leq \alpha_{x,t} = \sup \alpha_{x,t} = \frac{\sigma(R_{x,t})}{\sigma(H_{x,t} \Sigma_{x,t} H_{x,t}^T + R_{x,t})} \) for \( i = 1 \cdots p \)

\[ \left| \beta_i \right| \leq \beta_i = \sup \left| \beta_i \right| \]

\[ \leq \left( \frac{\sigma(R_{x,t})}{\sigma(H_{x,t} \Sigma_{x,t} H_{x,t}^T + R_{x,t})} \right)^{1/2} \]  
(20)

\[ \leq \left( \frac{\sigma(F_{x,t})}{\sigma(F_{x,t}) \sigma(F_{x,t})} \right)^{1/2} \]  
(21)

\( \bar{\sigma} \) and \( \sigma \) denote the maximum and minimum singular values, respectively.

Then, the EKO based stabilization method (4) renders the equilibrium \( (x_t, \tilde{x}_t) = (0,0) \) of the closed loop system (1)-(2)-(4.b) asymptotically stable.

The parameter \( \alpha \) and \( \beta \) will be detailed later. The arbitrary positive real parameters \( (\tilde{A}_t) \) and the positive definite matrices \( \Omega_k \) and \( Q_k \) are fixed by the user.

### 4. Convergence Analysis

In this section, the convergence analysis of the EKO based stabilization law (1)-(2)-(5) will be performed by the standard Lyapunov approach.

First, we define a candidate Lyapunov function

\[ V_{x,t} = V \left( x_{x,t}, \tilde{x}_{x,t}, \left( \begin{array}{c} P_t \\ 0 \\ \Sigma_t \end{array} \right) \right) \]

by substracting both sides of (4.a) from \( x_{x,t} \), we obtain

\[ \tilde{x}_{x,t} = \tilde{x}_{x,t} - K_{x,t} e_{x,t} \]  
(22)

Here, we introduce unknown diagonal matrices \( \beta_t = \text{diag}(\beta_{p+1}, \cdots, \beta_p) \) and \( \alpha_t = \text{diag}(\alpha_{x,t}, \cdots, \alpha_{x,t}) \), to model errors due to the first order linearization technique, so that we obtain the following exact equalities:

\[ \tilde{x}_{x,t} = \delta_t F_t \tilde{x}_t \]

\[ \alpha_t e_{x,t} = H_t \tilde{x}_{x,t} \]  
(23)

The importance of this choice is given in [4].

Next, from (11) and (13), we have

\[ K_{x,t} = \Sigma_t H_t R_t^{-1} \]  
(24)

and

\[ \Sigma_t^{-1} = \Sigma_t^{-1} + H_t R_t^{-1} H_t \]  
(25)

Next, from (11) and (13), we have

\[ V_{x,t} = x_t^T P_{x,t} x_t + \tilde{x}_{x,t}^T \Sigma_{x,t}^{-1} \tilde{x}_{x,t} = \tilde{x}_{x,t}^T H_t^{-1} R_t^{-1} \tilde{x}_{x,t} + e_{x,t}^T R_t^{-1} H_t^{-1} \Sigma_{x,t}^{-1} H_t R_t^{-1} e_{x,t} \]

and (27) into (28)

\[ V_{x,t} = x_t^T P_{x,t} x_t + \tilde{x}_{x,t}^T \Sigma_{x,t}^{-1} \tilde{x}_{x,t} = \tilde{x}_{x,t}^T H_t^{-1} R_t^{-1} \tilde{x}_{x,t} + e_{x,t}^T R_t^{-1} H_t^{-1} \Sigma_{x,t}^{-1} H_t R_t^{-1} e_{x,t} \]

(27)

with

\[ V_{x,t} = x_t^T P_{x,t} x_t + \tilde{x}_{x,t}^T \Sigma_{x,t}^{-1} \tilde{x}_{x,t} \]  
(28)

(29)
From (24) and (25), (29) becomes

\[ V_{i+1} = x_i^T P_i x_i + V_{i+1} + e_i(A_i R_i^{-1} A_i - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i) e_{i+1} \]

On the other hand, \( V_{i+1} \) may be written as

\[ V_{i+1} = x_i^T F_i^T \beta_i (F_i \Sigma_i F_i^T + S_i)^{-1} \beta_i F_i x_i \]

(32)

A decreasing sequence \( \{ V_i \}_{i=0}^\infty \) means that there exists a positive scalar \( 0 < \delta < 1 \) such that

\[ V_i - V_{i+1} \leq -\delta V_i \]

or equivalently

\[ V_i - (1-\delta) V_i = x_i^T P_i x_i - (1-\delta) x_i^T P_i x_i + e_i(A_i R_i^{-1} A_i - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i) e_{i+1} \]

\[ + e_i(A_i R_i^{-1} A_i - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i) e_{i+1} \]

\[ = x_i^T P_i x_i - (1-\delta) x_i^T P_i x_i + e_i(A_i R_i^{-1} A_i - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i) e_{i+1} \]

\[ + \tilde{x}_i^T F_i^T \beta_i (F_i \Sigma_i F_i^T + S_i)^{-1} \beta_i F_i x_i - (1-\delta) \Sigma_i \tilde{x}_i \leq 0 \]

(34)

A sufficient condition to ensure (35) leads to the following nonlinear inequalities

\[ x_i^T P_i x_i - (1-\delta) x_i^T P_i x_i \leq 0 \]

(36)

\[ a_i R_i^{-1} A_i - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i \leq 0 \]

(37)

\[ F_i^T \beta_i (F_i \Sigma_i F_i^T + S_i)^{-1} \beta_i F_i - (1-\delta) \Sigma_i \leq 0 \]

(38)

It is easy to deduce, by matrix manipulations, that under the hypothesis H2, we ensure that :

\[ \lambda \Lambda^T \Lambda (P_i + g_i \Sigma_i^{-1} g_i^T)^{-1} \Lambda_i + Q_i \geq 0 \]

(39)

or

\[ \Lambda_i^T R_i^{-1} \Lambda_i - P_i \leq -\delta P_i \]

(40)

Which implies

\[ x_i^T \Lambda_i^T R_i^{-1} \Lambda_i - P_i \leq -\delta x_i^T P_i x_i \]

(41)

and, therefore, (36) is satisfied.

We notice that (37) may be written into an equivalent form. Indeed, by the use of (11), (26) and a simple factorisation technique, we obtain

\[ (\alpha_i - I_\rho) R_i^{-1} A_i - (\alpha_i - I_\rho) - R_i^{-1} H_i \Sigma_i H_i^T R_i \]

\[ \times (H_i \Sigma_i H_i^T + R_i) \leq 0 \]

(42)

\[ \Rightarrow (\alpha_i - I_\rho) R_i^{-1} (\alpha_i - I_\rho - R_i^{-1} H_i \Sigma_i H_i^T R_i) \leq 0 \]

\[ \times (H_i \Sigma_i H_i^T + R_i) \leq 0 \]

(43)

using the following identity in (43)

\[ I_\rho = (H_i \Sigma_i H_i^T + R_i) (H_i \Sigma_i H_i^T + R_i)^{-1} \]

(44)

Inequalities (37) and (38) become

\[ (\alpha_i - I_\rho) R_i^{-1} A_i - (\alpha_i - I_\rho) - \alpha_i R_i^{-1} - R_i^{-1} H_i \Sigma_i H_i^T R_i \]

\[ \leq 0 \]

(45)

and

\[ F_i^T \beta_i (F_i \Sigma_i F_i^T + S_i)^{-1} \beta_i F_i - (1-\delta) \Sigma_i \leq 0 \]

(46)

Using (20) and (21), we can deduce that \( \{ V_i \}_{i=0}^\infty \) is a decreasing sequence.

Indeed, under (20) and (21) and using the fact that \( (\alpha_i - I_\rho) \) and \( \beta_i \) are diagonal matrices, we have

\[ \frac{\bar{\sigma}(\alpha_i - I_\rho)^2 \Sigma_i}{\sigma(H_i \Sigma_i H_i^T + R_i)} \leq \frac{1}{\delta} \frac{\sigma(F_i \Sigma_i F_i^T + S_i)}{\sigma(F_i \Sigma_i F_i^T)} \]

(47)

\[ \Rightarrow \frac{\bar{\sigma}(\alpha_i - I_\rho)^2 \Sigma_i}{\sigma(H_i \Sigma_i H_i^T + R_i)} \leq \frac{1}{\delta} \frac{\sigma(F_i \Sigma_i F_i^T + S_i)}{\sigma(F_i \Sigma_i F_i^T)} \]

(48)

we have then

\[ \bar{\sigma}(\alpha_i - I_\rho)^2 \Sigma_i \leq \sigma((H_i \Sigma_i H_i^T + R_i) \beta_i F_i \Sigma_i F_i^T + S_i) \beta_i F_i \]

(53)

\[ \leq \sigma((H_i \Sigma_i H_i^T + R_i) \beta_i F_i \Sigma_i F_i^T + S_i) \beta_i F_i \]

(49)

\[ \bar{\sigma}(\alpha_i - I_\rho)^2 \Sigma_i \leq \sigma((H_i \Sigma_i H_i^T + R_i) \beta_i F_i \Sigma_i F_i^T + S_i) \beta_i F_i \]

(50)

which induce that (45) and (46) are satisfied, and consequently \( V_i \) is a strictly decreasing sequence.

Now, we will prove that the matrices \( P_i \) and \( \Sigma_i \) are bounded from above and below for all \( k \), i.e. there exists \( \gamma, \nu, \eta \) and \( \bar{\eta} \) such that:

\[ 0 < \nu I \leq P_i \leq \gamma \]

(55)

and

\[ 0 < \eta I \leq \Sigma_i \leq \bar{\eta} I \]

(56)

We can verify easily, from (10), that since \( \lambda \) is positive definite, we have \( 0 < \nu I \leq P \). The second inequality \( P_i \leq \gamma I \) may be deduced from the sufficient hypothesis H1.

Indeed, if we consider the following auxiliary Riccati equation

\[ \tilde{P}_{i+1} = \lambda_i (\Lambda_i^T \tilde{P} \Lambda_i + Q_i) \]

(57)
we notice that, under hypothesis H1, and for a large parameter \( \gamma \), we have
\[
\bar{P}_k \leq \gamma I_k
\]
for all \( k \)  \hspace{1cm} (58)

Thus, when we choose the initial condition as follow
\[
P_0 \leq \bar{P}_0 \ (< \bar{P}_k)
\]
\hspace{1cm} (59)

we obtain, by the use of (10) and (17):
\[
P_{k+1} = \lambda_i \left( P_k - P_k \bar{g}_k (g_k \bar{g}_k + R_k)^T g_k^T P_k \right) + \bar{Q}_k
\]
\[
\leq \lambda_i \left( P_0 A_k + Q_k \right) \leq \bar{P}_0 \equiv \lambda_i \left( P_0 A_k + Q_k \right) \leq \bar{P}_k \hspace{1cm} (60)
\]

So the boundness of \( P_k \) is proved.

The proof of (56) is obtained from the local observability hypothesis H3, which ensure the boundness of \( \hat{S} \) (see [10] and [24]).

Since \( V_k \) is a strictly decreasing sequence and the couple \( (P_k, \hat{S}_k) \) is bounded, it follows that
\[
0 \leq \mu \left( \frac{x_k}{x_k} \right)^T \left( \frac{x_k}{x_k} \right) \leq V_k \leq (1 - \delta)^k V_0
\]
\hspace{1cm} (61)

\[
\Rightarrow 0 \leq \mu \lim_{K \to \infty} \left( \frac{x_k}{x_k} \right)^T \left( \frac{x_k}{x_k} \right) \leq \lim_{K \to \infty} V_k \leq \lim_{K \to \infty} (1 - \delta)^k = 0
\]
\hspace{1cm} (62)

with
\[
0 < \mu I_k \leq \left( P_k 0 \right) \left( P_k 0 \right)^{-1}
\]

Therefore the convergence of both the state space and the error dynamics to zero is ensured.

4.1 Remarks :

1. we introduce the weighting factor \( \lambda_i \) to control boundedness of \( P_k \) and, by the way, to relax the Lyapunov stability condition of the unforced dynamic system without preliminary coordinate transformations of the initial system. A simple method to design \( \lambda_i \) consists to set \( \lambda_i = 1 \) as long as \( P_k \leq \bar{P}_k \) and \( \lambda_i < 1 \) otherwise.

2. the two inequalities (20) and (21), cannot be always checked, since the parameter \( \alpha_k \) and \( \beta_k \) are unknown ; however, they give us the domains to which \( \alpha_k \), \( \beta_k \) should belong such that \( V_k \) is a decreasing sequence.

5. Simulation results

In order to show the high performances of the theory developed so far, we consider two numerical examples chosen from the literature.

The EKO control law given in (4), makes the origin asymptotically stable as we can see in the different figures.

5.1 Example 1 :

The following nonlinear discrete-time system has been considered in [7]
\[
x_{k+1} = A x_k + g(x_k) u_k
\]
\hspace{1cm} (63)

\[
y_k = C x_k
\]
\hspace{1cm} (64)

With
\[
A = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g(x_k) = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

and
\[
C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

where \( A \) is unstable. Fig. 1 and Fig. 2 show the convergence behaviour of \( x \) and \( x - \xi \) to the equilibrium \( (0,0) \), where the initialization vectors are \( x_0 = [-10 -7.5 10] \) and \( \xi_0 = [4 3 4] \).

![Fig. 1. The state \( x_k \) with respect to sampling time \( k \).](image1)

![Fig. 2 : State estimation error \( (x_k - \xi_k) \) with respect to sampling time \( k \).](image2)
5.2 Example 2:

Here, we study the example of the planar vertical take-off and Landing (PVTOL) aircraft. This example is treated in several papers in the literature [11] and [13] and [16]. The equations of motion are given by [11]:

\[
\begin{align*}
\ddot{\theta} &= u_2, \\
\ddot{y} &= \cos(\theta)u_1 + \varepsilon\sin(\theta)u_2 - 1 \\
\ddot{x} &= -\sin(\theta)u_1 + \varepsilon\cos(\theta)u_2
\end{align*}
\]

(65)

Where \(x, y\) denote the horizontal and the vertical position of the aircraft center of mass and \(\theta\) is the roll angle that the aircraft makes with the horizon. The control inputs \(u_1 \) and \(u_2\) are the thrust (direct out of the bottom of the aircraft) and the angular acceleration (rolling moment). The parameter \(\varepsilon\) is a small coupling coefficient between the rolling moment and the lateral acceleration of the aircraft. The coefficient “-1” is the normalized gravitational acceleration. Let us use (65) with \(u_2=1+v_1\), after Euler discretization of step \(T\), we obtain the next state space model

\[
x_{k+1} = A(x_k)x_k + g(x_k)u_k
\]

(66)

with

\[
A(x_k) = \begin{bmatrix}
1 & T & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -T\sin(\theta_1)/\theta_2 & 0 \\
0 & 0 & 1 & T & 0 & 0 \\
0 & 0 & 0 & 1 & T & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
g(x_k) = \begin{bmatrix}
0 & 0 \\
-\sin(\theta_1) & \varepsilon\cos(\theta_1) \\
0 & 0 \\
\cos(\theta_1) & \varepsilon\sin(\theta_1) \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
x_k = \begin{bmatrix} x_k \\ x_2 \\ y_k \\ \theta_k \\ \theta_2 \end{bmatrix}
\]

and

\[
u_k = \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}
\]

We assume that only, the vertical and horizontal positions and the roll angle are available

\[
y_k = C x_k \quad \text{with} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

(67)

We consider the discrete-time model, with \(T=0.007\), \(\varepsilon =0.02\) and the initializations \(x_0= [200 \ 0 \ 10 \ 0 \ 0.5 \ 0]\), \(\bar{x}_0=[75 \ 30 \ 15 \ 3 \ 30 \ 0\]}. The simulations confirm the high quality of our approach.
6. Conclusion

In this paper, we have presented an observer based control law which asymptotically stabilize a class of discrete-time affine nonlinear systems. A simple and useful approach, using the EKO and a modified Riccati equation is given. We establish a separation principle, and the EKO based stabilization method we use gives a good results, even when the free dynamics are not Lyapunov stable. Finally, the proposed control law was successfully applied to a large numerical examples treated in the literature , and two of them are detailed in this paper.

References