FREQUENCY DOMAIN DESIGN OF REDUCED ORDER $H_\infty$ FILTERS FOR DISCRETE TIME SYSTEMS

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Abstract

In this contribution the frequency domain design of reduced order $H_\infty$ filters of order $n-K$ is investigated for $n$th order discrete time systems with $m$ measurements of which $K$ are undisturbed. Starting from the known time domain results, the polynomial matrices parameterizing reduced order a priori and a posteriori $H_\infty$ filters are derived. A simple example demonstrates the proposed design procedure.

1 Introduction

The use of $H_\infty$ filters, which estimate some linear combination of the system states in the $H_\infty$ norm minimization sense, is appropriate when there is little knowledge of the statistics of the driving and of the measurement noise signals. Compared to minimum variance estimators (Kalman filters) they are less sensitive to uncertainty in the system parameters [10].

The $H_\infty$ filtering problem was first considered in [3] and in [9] using a frequency domain approach. A solution of the $H_\infty$ filtering problem in the framework of the Riccati equation approach is given in [14]. The corresponding theory has also been developed in the discrete time case (see e.g. [1], [13]).

This paper considers the frequency domain design of reduced order $H_\infty$ filters for discrete time systems, where $K$ of the $m$ measurements $y_M$ of the $n$th order plant are not affected by disturbances. The resulting filter is of order $n-K$, since it suffices to build an (n-$K$)th order observer to reconstruct the whole system state. The $H_\infty$ filter is characterized by polynomial matrices, parameterizing its discrete time transfer matrix.

The $H_\infty$ estimation problem can be solved under various patterns of information. In this contribution a priori and a posteriori $H_\infty$ filtering are considered. The a priori $H_\infty$ filter uses the measurements in a one step delay, whereas the a posteriori $H_\infty$ filter uses the current measurements in order to generate the desired estimate. As a consequence, the filter channels related to the disturbed measurements are strictly proper in the case of a priori estimates and proper in the case of a posteriori estimates. When using such a posteriori $H_\infty$ filters the $H_\infty$ norm bound may be lower than the one obtained by a priori $H_\infty$ filters.

After introducing the system descriptions in the time and in the frequency domain, the underlying $H_\infty$ estimation problems are formulated in Section 2. This section also contains the reduced order $H_\infty$ filter schemes in the time domain both for the a priori and the a posteriori estimation, forming the basis for the frequency domain solution derived in Section 3. A simple demonstrating example follows in Section 4.

2 Problem formulation and time domain results

Consider a time invariant, discrete time, linear system of order $n$ with $m_k$ unmeasurable outputs $y_k$, $m$ measurements $y_M$, and $q \geq m$ disturbances $w$ represented by

$$
x(k+1) = Ax(k) + Gw(k), \quad x(0) = 0
$$

$$
y_i(k) = C_i x(k)
$$

$$
y_M(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w(k)
$$

$$
y_M(k) = C_M x(k) + Dw(k)
$$

with $C_M$ having full row rank, $D, D_1^T > 0$, and $A$ invertible.

The output $y_M$ is subdivided such that $y_1$ contains the m-$K$ disturbed measurements and $y_2$ the $K$ perfect ones with $0 \leq K \leq m$. It is assumed that the pair $(C_M, A)$ is detectable.

In the sequel, the composite matrix $C$ will be used in different partitions

$$
C = [C_1^T \ C_2^T]^T = [C_1^T \ C_2^T \ C_2^T]^T = [C_1^T \ C_2^T]^T
$$

The frequency domain description of system (1) is

$$
y(z) = \begin{bmatrix} y(z) \\ y_M(z) \end{bmatrix} = \begin{bmatrix} C(zI-A)^{-1}G + [0] \\ D \end{bmatrix} w(z)
$$

and it is assumed that the strictly proper part of this transfer matrix is represented in a left coprime matrix fraction description

$$
C(zI-A)^{-1}G = \tilde{D}^{-1}(z)\tilde{N}(z)
$$
with $\bar{D}(z)$ row reduced.

Given $m$ measurements $y_M$ find an $H_\infty$ filter for the system (1), (3) that generates an estimate $\hat{y}_s(k)$ for the unmeasurable $m$ linear combinations $y_2(k)$ of the state $x(k)$ in the $H_\infty$ norm minimization sense. With $l_2[0,\infty)$ denoting the set of real square summable functions on the interval $[0,\infty)$, define the (worst case) performance measure

$$M = \sup_{w \in l_2[0,\infty)} \| y_2 - \hat{y}_2 \|_2$$

(5)

where using the a priori estimate $\hat{y}_s(k)$, and in the case of an a posteriori estimate $\hat{y}_s^*(k)$ use

$$M^* = \sup_{w \in l_2[0,\infty)} \| y_2 - \hat{y}_2^* \|_2$$

(6)

Two (suboptimal) filtering problems are considered

1) A priori $H_\infty$ filtering problem. Given a $\gamma > 0$, find a stable filter if it exists such that $M < \gamma$.

2) A posteriori $H_\infty$ filtering problem. Given a $\gamma > 0$, find a stable filter if it exists such that $M^* < \gamma$.

The time domain solutions to these problems are presented in [8]. In the sequel, however, other solutions are used, as they are better suited for the frequency domain design of the $H_\infty$ filter.

Consider the system (1) with $\kappa$ perfect measurements $y_2$ and a reduced order filter of order $n-\kappa$ [2]

$$\tilde{\xi}(k+1) = T(A - F_1 C_1)\Theta\xi(k) + T[F_1 (A - F_1 C_1)\Psi_2 \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}]$$

(7)

giving the state estimate

$$\hat{x} = \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ \xi \end{bmatrix} = \begin{bmatrix} \Psi_2 \\ \Theta \end{bmatrix} \begin{bmatrix} y_2 \\ \xi \end{bmatrix}$$

(8)

Define the matrices

$$C = \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}; \quad R = \begin{bmatrix} -\gamma I_{m_1} \\ 0 \\ 0 \end{bmatrix}; \quad D = \begin{bmatrix} D_1 \end{bmatrix}$$

$$S_\infty = \begin{bmatrix} 0 \\ GD_1 \end{bmatrix}; \quad \tilde{R}_i = R + C_1 \tilde{P} C_1^T$$

(9)

(10)

$$\bar{P} = A \tilde{P} A^T + GG^T - L_1 \tilde{R}_1 L_1^T$$

(11)

$$X = C_2 \tilde{P} C_2^T$$

(12)

Then the optimal filter gain matrices result from

$$L_i = [1_{m_1} \ldots 1_{m_1}] = (A \tilde{P} C_1^T + S_\infty) \tilde{R}_i^{-1}$$

(13)

$$\Psi_2 = \tilde{P} C_1^T X^{-1}$$

(14)

and $\tilde{P} = \tilde{P}^T \geq 0$ is a stabilizing solution of the algebraic Riccati equation (ARE)

$$\tilde{P} = A \tilde{P} A^T + GG^T - \Psi_2 X \Psi_2^T - L_1 \tilde{R}_1 L_1^T$$

(15)

After solving $T \Psi_2 = 0$ with $T$ having full row rank the matrix $\Theta$ is obtained from

$$\begin{bmatrix} \Psi_2 \\ \Theta \end{bmatrix} = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1}$$

(16)

The a priori estimate $\hat{y}_s(k)$ is obtained from the above results with $F_1 = L_1$ as

$$\hat{y}_s(k) = C_2 \Psi_2 y_2(k) + C_2 \Theta \xi(k)$$

(17)

where

$$-\gamma^2 I + C_2 \tilde{P} C_2^T < 0$$

(18)

must hold.

The a posteriori estimate $\hat{y}_s^*(k)$ is obtained from the above results with $F_1 = A \lambda_1$ where

$$\hat{y}_s^*(k) = (A - L_2 C_1)^{-1} L_1$$

(19)

as

$$\hat{y}_s^*(k) = \hat{y}_s(k) + C_2 \lambda_1 (y_1(k) - C_1 \hat{x}(k))$$

(20)

(see (8)) where

$$-\gamma^2 I + C_2 [\lambda D_1 D_1^T \lambda^T + (1 - \lambda C_1) \bar{P} (1 - C_1^T \lambda^T)] C_2^T < 0$$

(21)

must hold.

The above presented optimal solution differs from the one presented in [8]. The matrix $\tilde{P}$ used here is related to the matrix $\bar{P}$ used in [8] by

$$\tilde{P} = (I - \Psi_2 C_2) \bar{P} (I - C_2^T \Psi_2)$$

(22)
which has as a consequence
\[ C_2 \tilde{P} = 0 \] (23)
For more details see [6].

3 Frequency domain design of discrete time H_\infty filters

3.1 The polynomial matrix equation for the fictitious H_\infty filter

The frequency domain design of the reduced order H_\infty filter is based on the left coprime factorization (4) of the system (1). As an intermediate result, the “fictitious” filter with

\[ L_z = \begin{bmatrix} L_z & L_1 \end{bmatrix} \]
\[ \xi_1(k+1) = T(A - L_z C_x) \Theta \xi_1(k) + \]
\[ + T \tilde{L}_m (A - L_z C_x) \Psi_2 \begin{bmatrix} y_z(k) \\ y_2(k) \end{bmatrix} \] (24)

(with y_z as an input) is considered and the “realizable” filter (7) (case F = L_z) results for L_z = 0.

Assumption 1. The factorization (4) is such that the highest row degree coefficient matrix \( \Pi_z \tilde{D}^*(z) \) has full rank, with

\[ \tilde{D}^*(z) = \Pi_z \begin{bmatrix} D(z) & 0 \\ 0 & 0 \end{bmatrix} \]
and \( \Pi_z \) denoting taking the polynomial part. This can always be assured by appropriate left unimodular operations [12].

Assumption 2. The matrices \( \tilde{R}_1 \) and X (see (10) and (12)) have full rank. They can be computed from the frequency domain results (see below).

In [5] the relations between the time and the frequency domain parameterizations of reduced order observers have been presented. With \( \tilde{D}_1(z) \) parameterizing the fictitious filter (24) in the frequency domain, one has

\[ \tilde{D}_1(z) \tilde{D}_1(z^-1) = C(zI - A)^{-1} \begin{bmatrix} L_z & \Psi_2 \end{bmatrix} + \begin{bmatrix} I_{m_z + \kappa + 6} & 0 \\ 0 & 0 \end{bmatrix} \] (26)

Theorem 1. With the system representation (4) such that Assumptions 1 and 2 hold, the polynomial matrix \( \tilde{D}_1(z) \) parameterizing the fictitious filter in the frequency domain solves the polynomial matrix equation

\[ \tilde{D}_1(z) \begin{bmatrix} \tilde{R}_1 & 0 \\ 0 & X \end{bmatrix} \tilde{D}_1(z^-1) = D(z) \begin{bmatrix} R_0 & 0 \\ 0 & 0 \end{bmatrix} D^T(z^-1) + \]
\[ + \tilde{N}(z) \tilde{N}^T(z^-1) + \tilde{N}(z) \begin{bmatrix} 0 & D_1^T \end{bmatrix} \tilde{D}^T(z^-1) + \]
\[ + D(z) \begin{bmatrix} 0 & D_1^T \end{bmatrix} \tilde{N}^T(z^-1) \] (27)

Proof: Observing that

\[ \tilde{P} - A \tilde{P} A^T = (zI - A) \tilde{P} (z^-1 I - A^T) + (zI - A) \tilde{P} A^T + \]
\[ A \tilde{P} (z^-1 I - A^T) \]

the ARE (15) can be written as

\[ (zI - A) \tilde{P} (z^-1 I - A^T) + (zI - A) \tilde{P} A^T + A \tilde{P} (z^-1 I - A^T) + \]
\[ + \Psi_1 X \Psi_2^T + L_1 \tilde{R}_1 L_1^T = G G^T \]

Multiplying this with \( C(zI - A)^{-1} \) from the left and with \( (z^-1 I - A^T)^{-1} C^T \) from the right (see (2)), then substituting (see (10) and (13))

\[ \tilde{C} \tilde{P} C^T = \begin{bmatrix} \tilde{R}_1 - R_0 & 0 \\ 0 & 0 \end{bmatrix} \] and \( A \tilde{P} C^T = [L_1 \tilde{R}_1 - S_1 \tilde{S}_1 \tilde{S}_1 \tilde{S}_1 \tilde{R}_1 - S_1 \tilde{C}] \]

where we have used \( C_2 \Psi_2 = 1 \) following from (8) and (23), and reordering the result gives

\[ \begin{bmatrix} C(zI - A)^{-1} [L_1 \Psi_2] + \begin{bmatrix} 0 & 0 \end{bmatrix} \tilde{R}_1 \end{bmatrix} \tilde{R}_1 \]
\[ \begin{bmatrix} L_2 & \Psi_2 \end{bmatrix} (z^{-1} I - A^T)^{-1} C^T + \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} R_0 & 0 \\ 0 & 0 \end{bmatrix} \] (28)
\[ + C(zI - A)^{-1} G G^T (z^{-1} I - A^T)^{-1} C^T + \]
\[ + C(zI - A)^{-1} G [0 D_1^T 0] + C(zI - A)^{-1} G [0 D_1^T 0] \]

which in view of (4) and (26) can be written as

\[ \tilde{D}_1(z) \begin{bmatrix} \tilde{R}_1 & 0 \\ 0 & X \end{bmatrix} \tilde{D}_1(z^-1) = D(z) \begin{bmatrix} R_0 & 0 \\ 0 & 0 \end{bmatrix} D^T(z^-1) + \]
\[ + \tilde{N}(z) \tilde{N}^T(z^-1) + \tilde{N}(z) \begin{bmatrix} 0 & D_1^T \end{bmatrix} \tilde{D}^T(z^-1) + \]
\[ + D(z) \begin{bmatrix} 0 & D_1^T \end{bmatrix} \tilde{N}^T(z^-1) \] (29)

after the result has been multiplied by \( \tilde{D}(z) \) from the left and by \( D^T(z^-1) \) from the right.

Since the matrix \( \begin{bmatrix} \tilde{R}_1 & 0 \\ 0 & X \end{bmatrix} \) in (29) is indefinite, one must use J-spectral factorization [11] to obtain the (stable) spectral factor \( \tilde{D}(z) \) of the known right hand side of (29), i.e.
\[ \tilde{D}(z) J \tilde{D}^\top(z^{-1}) = \tilde{D}(z) \left[ \begin{array}{cc} R_0 & 0 \\ 0 & 0 \end{array} \right] \tilde{D}^\top(z^{-1}) + \\
+ \tilde{N}(z) \tilde{N}^\top(z^{-1}) + \tilde{N}(z) \left[ \begin{array}{cc} D_1^\top & 0 \end{array} \right] \tilde{D}^\top(z^{-1}) + \\
+ \tilde{D}(z) \left[ \begin{array}{cc} 0 & D_1^\top \end{array} \right] \tilde{N}^\top(z^{-1}) \]  

(30)

where \( J \) is a diagonal matrix with entries 1 and –1 on the main diagonal and \( \det \tilde{D}(z) \) has its roots inside the unit circle. The factorization result for \( \tilde{D}(z) \) may be such that \( \det \tilde{D}(z) \) contains \( \mu \neq 0 \) superfluous roots at \( z = 0 \) which is often the case for discrete time systems [7]. These must be extracted by right operations

\[ \tilde{D}_{\text{red}}(z) = \tilde{D}(z) V_{\text{ext}}(z) \]  

(31)

det \( V_{\text{red}}(z) = z^{-\kappa} \) such that \( V_{\text{ext}}(z) V_{\text{ext}}^\top(z^{-1}) = J \) (for an algorithm see [4]) to obtain a polynomial matrix \( \tilde{D}_{\text{red}}(z) \) with \( \text{deg}[\det \tilde{D}_{\text{red}}(z)] = n - \kappa \).

Inspection of (26) shows, that (see [5])

\[ \Gamma_1 \left[ \begin{array}{c} \tilde{D}_r(z) \\ \tilde{D}_l(z) \end{array} \right] = \Gamma_1 \left[ \begin{array}{c} \tilde{D}_l(z)^\top(z^{-1}) \end{array} \right] \]  

(32)

The time domain quantity \( C_2 L_r \) can be obtained from the frequency domain parameters. Using (29), (30) and (31) gives

\[ \tilde{D}_{\text{red}}(z) J \tilde{D}_{\text{red}}^\top(z^{-1}) = \tilde{D}_r(z) \left[ \begin{array}{cc} R_0 & 0 \\ 0 & X \end{array} \right] \tilde{D}_r^\top(z^{-1}) \]  

(33)

If one knew the highest row degree coefficient matrix \( \Gamma_r \left[ \begin{array}{c} \tilde{D}_r(z) \end{array} \right] \) one could compute \( \tilde{D}_r(z) \) as

\[ \tilde{D}_r(z) = \tilde{D}_{\text{red}}(z) \Gamma^{-1}_r \left[ \begin{array}{c} \tilde{D}_{\text{red}}(z) \end{array} \right] \Gamma_1 \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \]  

Substituting this and (32) in (33) one obtains

\[ \tilde{D}_{\text{red}}(z) \tilde{D}_{\text{red}}^\top(z^{-1}) = \tilde{D}_{\text{red}}(z) \Gamma^{-1}_r \left[ \begin{array}{c} \tilde{D}_{\text{red}}(z) \end{array} \right] \Gamma_r \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \]  

which after appropriate rearranging gives

\[ \Gamma_r \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \Gamma_1 \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] J \Gamma_r^{-1} \left[ \begin{array}{c} \tilde{D}_{\text{red}}(z) \end{array} \right] \Gamma_r \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \]  

(34)

At the left hand side of (34) are known frequency domain quantities, and from the right hand side the unknown quantity \( C_2 L_r \), follows as

\[ \left[ \begin{array}{c} 1 \\ C_2 L_r \end{array} \right] = \left[ \begin{array}{c} \tilde{R}_r \\ C_2 L_r \tilde{R}_r \end{array} \right] \tilde{D}_{\text{red}}(z) \Gamma_r^{-1} \left[ \begin{array}{c} \tilde{D}_{\text{red}}(z) \end{array} \right] \Gamma_r \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \]  

(35)

Consequently, Assumption 2 can be verified using frequency domain results. With \( C_2 L_r \) from (35) the polynomial matrix \( \tilde{D}_r(z) \) can be computed from \( \tilde{D}_{\text{red}}(z) \) via

\[ \tilde{D}_r(z) = \tilde{D}_{\text{red}}(z) \Gamma^{-1}_1 \left[ \begin{array}{c} \tilde{D}_{\text{red}}(z) \end{array} \right] \Gamma_r \left[ \begin{array}{c} \tilde{D}_l(z) \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \]  

(36)

3.2 The a priori filtering case

Assumption 3. With \( \tilde{R}_r \) resulting from (34) \( [I_{m_0}, 0] \tilde{R}_r \) \( [I_{m_0}, 0]^\top < 0 \) holds (this is the condition (18), assuming that the a priori filtering problem is solvable for the chosen \( \gamma \)).

In the realizable \( H_{\gamma} \) filter (7), only the measurable outputs \( y_M \) are used. The polynomial matrix \( \tilde{D}(z) \) parameterizing the realizable reduced order \( H_{\gamma} \) filter is related to the time domain parameters (case \( F_1 = L_1 \)) by

\[ \tilde{D}^{-1}(z) \tilde{D}(z) = C(zI - A)^{-1}[0 \ F_1 \ \Psi_r] + \left[ \begin{array}{c} m_0 - \kappa \end{array} \right] \]  

(37)

[5]. A comparison of (26) and (37) shows that \( \tilde{D}(z) \) is given by

\[ \tilde{D}(z) = \tilde{D}(z) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0_m \end{array} \right] + \tilde{D}_r(z) \left[ \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \]  

(38)

Using the results of [5] the quantity

\[ \tilde{y}_s(z) = \left[ \begin{array}{c} I_{m_0} \\ 0_{m_0} \end{array} \right] \tilde{D}^{-1}(z) \left[ \begin{array}{c} \tilde{D}(z) - \tilde{D}(z) \end{array} \right] \left[ \begin{array}{c} 0 \\ y_M(z) \end{array} \right] \]  

(39)

can be shown to represent the transfer behaviour of the reduced order a priori \( H_{\gamma} \) filter [6].

3.3 The a posteriori filtering case

Assumption 4. With \( \tilde{R}_r \) resulting from (33) and \( C_2 \lambda_z \) computed below, the condition \( [I_{m_0}, C_2 \lambda_z] \tilde{R}_r [I_{m_0}, C_2 \lambda_z]^\top < 0 \) holds (this is exactly (21), assuming that the a posteriori filtering problem is solvable for the chosen \( \gamma \)).

Solve again the J-spectral factorization problem (30), and compute \( \tilde{D}_r(z) \) (see (36)) and \( \tilde{D}(z) \) (see (38)). The
polynomial matrix \( \tilde{D}^+(z) \) parameterizing the \( H_o \) filter in the
\textit{a posteriori} case is related to the time domain filter
parameters by (37) for \( \Gamma_1 = A \lambda_1 \), namely
\[
\tilde{D}^+(z) = \tilde{D}(z)C(zI - A)^{-1}L_1
\]
To obtain the result (40) from (37), the gain matrix \( L_1 \) has to be
substituted by
\[
A \lambda_1 = (I - L_1 C_A A^{-1})^{-1} L_1
\]
(see (19)). This results when adding the quantity
\[
C(zI - A)^{-1}[0 \ L_1 \ 0]
\]
with
\[
L_1^+ = A \lambda_1 - L_1 = (I - L_1 C_A A^{-1})^{-1} L_1
\]
to the factorization (37). Introducing the factorization
\[
C(zI - A)^{-1} = \tilde{D}^+(z) \tilde{N}_s(z)
\]
the polynomial matrix \( \tilde{D}^+(z) \) of the reduced order \( H_o \) filter
can be computed from
\[
\tilde{D}^+(z) = \tilde{D}(z) + \tilde{N}_s(z)[0 \ L_1^+ \ 0]
\]
In order to get the (time domain) quantity (43) from the
frequency domain results consider the polynomial matrix
\[
H_1(z) = \tilde{D}_1(z) = \tilde{D}(z)[I \ 0 \ 0] = \tilde{N}_s(z)[L_1 \ \Psi_1]
\]
If one knew the state space representation of the system, one
could compute the factorization (44) and consequently also the
gain matrices \( [L_1 \ \Psi_1] \) from (46). This time domain
characterization of the system is not known, but one can assume that there is an observable canonical realization of the
system transfer matrix (4), giving rise to an \((m+s+m)\times n\) polyno-
mial matrix \( \tilde{N}_s(z) \) in (44) of the form
\[
\tilde{N}_s(z) = \text{diag}(\sigma_1^T, \ldots, \sigma_k^T)
\]
where the \( \sigma_v^T \), \( v = 1, \ldots, k \) are row vectors of the form
\[
[\delta_i^{++} \ \cdots \ \delta_i^z \ 1]
\] and the \( \delta_i^z \), \( i = 1, \ldots, m+s \) are the row
degrees \( \delta_i^z \text{[D(z)]} > 1 \). The \( \sigma_v^T \) are only defined for such \( v \),
where \( \delta_v^z \text{[D(z)]} > 1 \). For all \( i \) where \( \delta_i^z \text{[D(z)]} = 0 \) the corre-
sponding row of \( \tilde{N}_s(z) \) is a zero row.

With the above \( \tilde{N}_s(z) \), the entries in \( [L_1 \ \Psi_1] = [L_1 \ L_1 \ \Psi_1] \)
can be obtained from \( H_1(z) \) by inspection. With this result
\[
[L_1 \ 0 \ 0 \ D^{-1}(0) \tilde{N}_s(0) = -L_1 C_A A^{-1}
\]
(see (44)) can be computed which gives \( L_1^\text{DIFF} \) when substituted
in (43) and finally \( \tilde{D}^+(z) \) when substituted in (45).

Using the results of [5] for \( \tilde{D}^+(z) \) the quantity
\[
\tilde{y}_s^+(z) = [I \ 0 \ 0 \ 0] \text{[D(z)]} [\tilde{D}^+(z) - \tilde{D}(z)] + [0 C_{\lambda_1} 0] [0 \ 0 \ 0 \ 0]
\]
can be shown to represent the transfer behaviour of the
reduced order \textit{a posteriori} \( H_o \) filter [6].

The quantity \( C_{\lambda_1} \) in the filter transfer behaviour of the
reduced order \textit{a posteriori} \( H_o \) filter can also be derived from the
frequency domain results. Inspection of (40) shows that
\[
[0 C_{\lambda_1} C A^{-1}\Psi_1] = -\text{D^{-1}(0)} \tilde{D}^+(0) + [I \ 0 \ 0 \ 0]
\]

\section*{4 Example}

Given a third order system with a \((3,1)\) output vector
\( y^T = [y_1 \ y_2 \ y_3] \) and a \((3,1)\) input disturbance vector \( w \).
There is one disturbed \( y_1 \) and one perfect measurement \( y_2 \)
(i.e. \( m = 2, \kappa = 1 \)). Its frequency domain representation is
\[
y(z) = \text{D^{-1}(z)[N}(z) + [0 \ 0 \ D] w(z) \text{ with}
\]
\[
\tilde{N}(z) = [0.7071 \ -2.1213 \ 0] \text{ [D(z)]}
\]
\[
\tilde{D}(z) = \begin{bmatrix}
0.4714z - 0.4125 & 0.2357z - 1.0017 & 0.2357z + 0.5893 \\
-0.4714 & -0.9428 & 1.4142z + 0.4714 \\
0.5774 & -0.5774 & -0.5774
\end{bmatrix}
\]
and
\[
\text{D} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \text{ i.e. } D_2 = [0 \ 0 \ 1]
\]
Since the measurement \( y_2 \) is not disturbed, the reduced order
\( H_o \) filter is of order \( n - \kappa = 1 \).

To save space, we consider the \textit{a priori} estimate only. The
infinite value of \( \gamma \) is \( \gamma_{\infty} = \sqrt[\gamma]{35/6} \) [8]. For \( \gamma = 2.41523 \) and
the above quantities, the right hand side of the polynomial equation (30) can be computed.

By J-spectral factorization (The first author thanks Polyx® for an α version of such a factorization program) one obtains

\[ D(z) = \begin{bmatrix} -0.583z + 0.112 & -0.908z^2 - 0.692 & -1.675z - 0.122 \\ -0.309z + 0.134 & 0.570z - 0.828 & -2.945z - 0.146 \\ 2.062z - 0.215 & -0.171z + 1.328 & -0.916z + 0.234 \end{bmatrix} \]

and

\[ J = \text{diag}(-1, 1, 1) \]

The determinant \( \text{det}(\tilde{D}(z)) \) contains two superfluous roots at \( z = 0 \) that can be extracted by

\[ V_{\text{red}}(z) = \begin{bmatrix} -1.0129z & 0.1586z & 0.0280z \\ 0.1611 & 0.9975 & 0.1760 \\ 0 & 0.1738z & -0.9848z \end{bmatrix} \]

which meets \( V_{\text{red}}(z) J V_{\text{red}}^T(z^{-1}) = J \), giving (see (31))

\[ \tilde{D}_{\text{red}}(z) = \begin{bmatrix} 0.7817 & -1.2946z - 0.6938 & 1.4913 \\ 0.3054 & -0.8297 & 2.9996 \\ -2.0357 & 1.3315 & 0.8718 \end{bmatrix} \]

The determinant of this matrix has one root at \( z = 0.1429 \), (n=κ=1) which is the eigenvalue of the fictitious filter with input y. To get \( D(z) \) parameterizing this fictitious filter, one must compute \( C_2 L_x \) via (34) and (35). With

\[ \Gamma, [\tilde{D}(z)] = \begin{bmatrix} 0.4714 & 0.2357 & 0 \\ -0.9428 & 1.4142 \\ 0.5774 & -0.5774 & 0 \end{bmatrix} \]

and

\[ \Gamma, [\tilde{D}_{\text{red}}(z)] = \begin{bmatrix} 0 & -1.2946z & 0 \\ 0.3054 & -0.8297 & 2.9996 \\ -2.0357 & 1.3315 & 0.8718 \end{bmatrix} \]

\[ C_2 L_x = [-0.1714 \ 1] \] results giving (via (36))

\[ \tilde{D}(z) = \begin{bmatrix} 0.4714z - 0.6549 & 0.2357z + 0.4125 & 0.8250 \\ -0.7138 & 0.4714 & 1.4142 \\ 0.5774 & -0.5774 & 0 \end{bmatrix} \]

With \([I_m, 0] [\tilde{D}, [I_m, 0]^T = -2.7e - 06\) the limit of condition (18) is nearly reached. The realizable filter with input y_m is parameterized by the polynomial matrix \( \tilde{D}(z) \) resulting from (38) as

\[ \tilde{D}(z) = \begin{bmatrix} 0.4714z - 0.4125 & 0.2357z + 0.4125 & 0.8250 \\ -0.4714 & 0.4714 & 1.4142 \\ 0.5774 & -0.5774 & 0 \end{bmatrix} \]

The root of \( \text{det}(\tilde{D}(z)) \) is at \( z = 0 \), which is the eigenvalue of the optimal filter [8].

5 Conclusions

Based on the time domain results a frequency domain solution has been derived for the discrete time \( H_\infty \) estimation problem for nth order plants in the presence of k perfect measurements. The \( H_\infty \) filter of order n-k is parameterized by a polynomial matrix resulting from J-spectral factorization. Also \textit{a posteriori} estimation can directly be handled in the frequency domain. The design results cover all cases between full-order and completely reduced order filters. A simple example demonstrated the design procedure.

References