CHATTERING REDUCTION VIA FUZZY LOGIC: APPLICATION TO STEPPER MOTOR

Bernardo Rincón Márquez, Alexander G. Loukianov and Edgar N. Sanchez

CINVESTAV IPN Unidad Guadalajara,
Apartado Postal 31-438, Guadalajara, Jal. 44550, México.
emails:brincon [louk, sanchez]@gdl.cinvestav.mx

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Abstract
In this paper, one method of robust control based on sliding mode and fuzzy logic techniques is presented. It combines hierarchical control with high gain approach for multivariable and nonlinear systems; in order to eliminate chattering in presence of disturbances. Simulation results are presented to illustrate the applicability of the approach.

1 Introduction
A simple and good technique of robust control is the sliding mode one (Utkin, 1992; Utkin, et al., 1999), since it is composed of two clear steps: selection of the sliding surface, such that the sliding mode equation on this surface is robust in presence of disturbances, and design of a discontinuous control which stabilizes the projection motion of the closed loop system on the sliding surface subspace.

It is known that the sliding mode motion is invariant with respect to disturbance, which satisfies the matching condition (Drajenovic, 1969). There are two cases for controlling systems with unmatching condition: measured disturbances, and, unmeasured ones. For the second case, the problem can be solved using high gain control, but it can produce the “chattering” (Utkin, et al., 1999) or oscillations on the sliding surface due to the imperfections in the control devices.

In this paper we propose a new control scheme using combination of the sliding mode control, block control (Luk'yanov, 1998) and fuzzy logic control (Driankov, et al., 1996) techniques to eliminate the chattering in the closed-loop system with both the matched and unmatched unknown perturbations. Note the sliding mode fuzzy logic controller was investigated for the system with matched perturbations in (Palm, et al., 1997; Alexík and Vittek, 1994; Scibile and Kouvariakias, 2001; Wong, et al., 2001; Ha, et al., 2001; Kaynak, et al., 2001)

2 Control Method
Consider a single input single output system (SISO) nonlinear system subject to a disturbances

\[ \dot{x} = f(x) + b(x)u + g(x)w \]

\[ y = h(x) \]

where \( x = (x_1, \ldots, x_n)^T \) is the state vector, \( u \) is the control input, \( y \) is the output, \( w \) represents an external disturbance which is unknown but bounded, \( f(x) \) and \( b(x) \) are sufficiently smooth and bounded functions, \( g(x) \) is an unknown but bounded function, and \( f(0)=0 \).

We assume that there exists a nonlinear transformation that reduces the system (1) to the so-called Block Controllable Form with disturbances (Luk'yanov, 1998):

\[ \dot{x}_i = f_i(x_i) + b_i(x_i)x_2 + g_i(x_i)w \]

\[ \dot{x}_j = f_j(x_j) + b_j(x_j)x_{j-1} + g_j(x_j)w, \quad i = 2, \ldots, n-1 \]

\[ \dot{\bar{x}}_n = f_n(x) + b_n(x)u + g_n(x)w \]

\[ y = x_1 \]

with \( \bar{x} = (x_1, \ldots, x_n)^T \), \( b_i \neq 0 \), \( f_i(x) \) and \( b_j(x) \) are sufficiently smooth and bounded functions.

If the disturbance \( w \) satisfies the matching condition (Drajenovic, 1969), that is, there is a scalar function \( \lambda(x) \) such that

\[ g(x) = b(x)\lambda(x) \]

then it is easy to show that \( g_i(x_i) = 0 \) and \( g_j(x_j) = 0 \), \( i = 2, \ldots, n-1 \) in (3), and therefore, sliding mode motion is invariant with respect to external disturbance. The aim of this paper is to design a discontinuous control that provides robustness, with not chattering, to the closed-loop system for the unmatched disturbances case, i.e. \( w \) does not satisfy the matching condition (5). In this case there is \( g_j(x_j) = 0 \), \( i = 2, \ldots, n-1 \). The control procedure consists of the following:

Suppose that the output \( y \) requires to follow the reference signal \( r \). Using the block control technique (Luk'yanov, 1998) we introduce the following recursive transformation:

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\[ z_1 = x_1 - r = \Phi_1(x_1, r) \quad (6a) \]
\[ z_2 = f_1(x_1) + b_1(x_1) x_2 - \dot{r} + k_1(x_1 - r) = \Phi_2(x_2, r_2) \quad (6b) \]
\[ z_3 = \tilde{b}_2 (\tilde{x}_2, t) x_3 + f_2 (\tilde{x}_2) + k_2 \Phi_2 (\tilde{x}_2, r_2) = \Phi_3 (\tilde{x}_3, r_3) \quad (6c) \]
\[ z_{i+1} = \tilde{b}_i (\tilde{x}_i) x_{i+1} + f_i (\tilde{x}_i) + k_i \Phi_i (\tilde{x}_i, r_i) = \Phi_{i+1} (\tilde{x}_{i+1}, r_{i+1}) \quad (6d) \]

where \( \tilde{x} = (x_1, \ldots, x_n)^T \) is a new variables vector, \( k_j > 0 \), \( r_2 = (r, r^{(l)}), \quad r_3 = (r, r^{(l-1)}), \quad \tilde{b}_1 = \tilde{b}_1 b_1 \). The transformation (6a)-(6d) reduces the system (3) to the following desired form:

\[
\begin{align*}
\dot{z}_1 &= -k_1 z_1 + z_2 + \tilde{g}_1 (z_1) w \\
\dot{z}_i &= -k_i z_i + z_{i+1} + \tilde{g}_i (\tilde{z}_i) w, \quad i = 2, \ldots, n-1 \\
\dot{z}_{n} &= f_{n} (\tilde{z}_{n}) + \tilde{b}_{n} (\tilde{z}_{n}) u + \tilde{g}_{n} (\tilde{z}_{n}) w
\end{align*}
\]  

\[ (7a)-(7c) \]

where \( \tilde{z}_i = (z_1, \ldots, z_i)^T \), \( \tilde{f}_{n} (\tilde{z}_{n}) \) is a continuous and bounded function, and \( \tilde{b}_n = \tilde{b}_{n-1} b_n \).

In order to generate sliding mode in (7a)-(7c) a natural choice of the switching function is \( s = z_n \) (6d). Then the desired dynamics of the closed-loop system for the case of unknown \( w \), can be selected as

\[ \dot{s} = -k_n \text{sign}(s) + \tilde{g}_n (\tilde{z}_n) w, \quad k_n > 0 \quad (8) \]

From (7c) and (8) a discontinuous control strategy can be obtained of the following form:

\[ u = -k_b \tilde{b}_n^{-1} (\tilde{z}_n) \text{sign}(s) + u_{eq} (\tilde{z}_n) \quad (9) \]

where \( u_{eq} (\tilde{z}_n) \) is the equivalent control calculated from \( \dot{s} = 0 \) in the absence of the disturbance as

\[ u_{eq} = -\tilde{b}_n^{-1} (\tilde{z}_n) \tilde{f}_n (\tilde{z}_n) \]

In order to derive the stability condition, we use a positive definite function \( V(s) = \frac{1}{2} s^2 \). Then from

\[ \dot{V} \leq -[k_n - \tilde{g}_n (\tilde{z}_n) w] |s| \]

we can obtain

\[ k_n \geq |\tilde{g}_n (\tilde{z}_n) w| \quad (10) \]

Under this condition the state converges to the surface \( s = 0 \) and the sliding mode motion occurs on this surface in a finite time. This motion is described by the following \((n-1)^{th}\) order system:

\[
\begin{align*}
\dot{z}_1 &= -k_1 z_1 + z_2 + \tilde{g}_1 (z_1) w \quad (11a) \\
\dot{z}_i &= -k_i z_i + z_{i+1} + \tilde{g}_i (\tilde{z}_i) w, \quad i = 2, \ldots, n-2 \\
\dot{z}_{n-1} &= -k_{n-1} z_{n-1} + \tilde{g}_{n-1} (\tilde{z}_{n-1}) w
\end{align*}
\]

The following assumption on the bounds of the unknown terms in (7a)-(7c), is stated:

**A1**: There exist positive constants \( q_j \) and \( d_i \) such that

\[
\begin{align*}
|\tilde{g}_1 (z_1)| &= \leq q_{11} |z_1| + d_1 \\
|\tilde{g}_2 (z_2)| &= \leq q_{22} |z_2| + k_1 q_{21} |z_1| + d_2 \\
|\tilde{g}_3 (z_3)| &= \leq q_{33} |z_3| + k_2 q_{32} |z_2| + k_1 q_{31} |z_1| + d_3 \\
|\tilde{g}_i (\tilde{z}_i)| &= \leq q_{i,i} |z_i| + \sum_{j=1}^{i-1} q_{j,i} |z_j| + d_i, \quad i = 4, \ldots, n-1.
\end{align*}
\]

To achieve the robustness property with respect to unknown but bounded uncertainty, the controller gains \( k_1, \ldots, k_{n-1} \) have to be chosen hierarchically high. Thus, since \( \tilde{g}_1 (z_1) w \) in (12a) does not depend on \( k_1 \), the value of this coefficient can be chosen so high that the term \( k_1 z_1 \) in (11a) will be dominate. By block linearization procedure, the term \( \tilde{g}_3 (z_2) w \) in (12b) depends on \( k_1 \) but not on \( k_2, \ldots, k_{n-1} \). Then for fixed \( k_1 \), the appropriate choice of \( k_2 \) value provides the domination of term \( k_2 z_2 \) in the second block of (11b), and so on.

In order to establish the required hierarchy of the control gains which ensures stability of the sliding mode motion (11a)-(11c), we choose a Lyapunov function candidate \( V \) for the system (11a)-(11c) as a sum of Lyapunov function candidates for each block of (11a)-(11c), namely

\[ V = \sum_{i=1}^{r-1} V_i, \quad V_i = \frac{1}{2} z_i^2, \quad i = 1, \ldots, n-1 \]

and let us calculate the derivatives \( \dot{V}_i, \quad i = 1, \ldots, r-1 \) step by step from the first block to the last block of (11a)-(11c).

At the first step, differentiating the Lyapunov function candidate \( V_1 = \frac{1}{2} z_1^2 \) along the trajectories of (11a) and using assumption A1, namely (12a), we get

\[
\dot{V}_1 = -k_1 z_1^2 + z_1 z_2 + z_1 \tilde{g}_1 (z_1) w \\
= -|z_1| (k_1 q_{11} |z_1| + |z_2| - d_1)
\]

which is negative in the region \( |z_1| > \frac{1}{k_1 - q_{11}} |z_2| + \frac{d_1}{k_1 - q_{11}} \).

Therefore, the state ultimately enter the domain in subspace \((z_1, z_2)\) defined by

\[ |z_1| \leq \alpha_{12} |z_2| + \beta_{12} \quad (13) \]

where the parameters \( \alpha_{12} \) and \( \beta_{12} \) defined as

\[ \alpha_{12} = (k_1 - q_{11})^{-1} \quad \text{and} \quad \beta_{12} = \alpha_{12} d_1 \]

are positive if the following condition holds:

\[ k_1 > q_{11} \quad (14) \]

At the second step, following similar lines to those taken for the first block, the derivative \( \dot{V}_2 \) of the Lyapunov function candidate \( V_2 = \frac{1}{2} z_2^2 \) calculated along the trajectories of the
second block of (11b), under conditions (12a), (12b) and (14), is given by
\[
\dot{z}_2 = -k_2 z_2^2 + z_2 [\alpha_2 (z_1, z_2, z_3) w] \\
\leq -k_2 z_2^2 + z_2 [\sum_{i=1}^{t-1} k_{i-1} (z_{i-1}) |z_{i-1}| + d_{i-1}] \
\leq -k_2 z_2^2 + z_2 [\sum_{i=1}^{t-1} k_{i-1} (z_{i-1}) |z_{i-1}| - k_{i-1} z_{i-1} - d_{i-1}]
\]
which is negative if
\[
(k_2 - k_{i-1}) z_{i-1} < 0.
\]
Hence, the state ultimately enter the domain in the subspace \((z_1, z_2, z_3)\) defined by
\[
|z_2| \leq \alpha_2 |z_3| + \beta_2
\]
and consequently
\[
|z_1| \leq \alpha_2 |z_3| + \beta_2
\]
where the parameters \(\alpha_2, \beta_2, \alpha_3, \beta_3\) are positive if the values of \(k_1\) and \(k_2\) satisfy the following inequalities
\[
k_1 > q_{11} \quad \text{and} \quad k_2 > q_{22} + k_1 q_{21} \alpha_2
\]
(15)

Proceeding in the same fashion for the \(i^{th}\) block of the system (11a)-(11c), then the convergence domain in the subspace \((z_1, z_2, \ldots, z_{i-2}, z_{i-1}, z_i)\), is
\[
|z_i| \leq \alpha_{i-1} |z_{i-1}| + \beta_{i-1}
\]
(16)
where
\[
\alpha_{j,i} = \alpha_{j,i-1} \quad \text{and} \quad \beta_{j,i} = \alpha_{j,i-1} + \beta_{j,i-1}, \quad j = 1, \ldots, i - 1.
\]

At the next step, taking again the derivative of the Lyapunov function \(V_i = \frac{1}{2} z_i^2\) along the trajectories of the \(i^{th}\) block of (11a)-(11c), and using (12a)-(12d), we obtain
\[
\dot{V}_i = -k_i z_i^2 + z_i [\sum_{j=1}^{i-1} k_{j-1} (z_{j-1}) |z_{j-1}| + d_{j-1}]
\]
Using now (16), we can majorize \(\dot{V}_i\) as
\[
\dot{V}_i \leq -|z_i| [\sum_{j=1}^{i-1} k_{j-1} (z_{j-1}) |z_{j-1}| - k_{i-1} z_{i-1} - d_{i-1}].
\]
From this equation it follows that
\[
|z_i| \leq \alpha_{i-1} |z_{i-1}| + \beta_{i-1-1}
\]
(18)
where the parameters
\[
\alpha_{i,i-1} = \left(k_i - q_{i-1} \sum_{j=1}^{i-1} k_{j-1} (z_{j-1}) \alpha_{j,i-1}\right)^{-1}
\]
and
\[
\beta_{i,i-1} = \left(\sum_{j=1}^{i-1} k_{j-1} (z_{j-1}) \beta_{j,i-1} - d_{i-1}\right), \quad i = 4, \ldots, n-1
\]
are positive if the condition
\[
k_i > q_{i,i-1} + \sum_{j=1}^{i-1} k_{j-1} (z_{j-1}) \alpha_{j,i-1}
\]
holds. Substitution of (18) in (16) gives the following set of inequalities for the subspace \((z_1, z_2, \ldots, z_{i-2}, z_{i-1}, z_i)\) :
\[
|z_i| \leq \alpha_{i,i-1} |z_{i-1}| + \beta_{i,i-1}
\]
(20)

At the last step we have the domain of convergence in the subspace \((z_1, z_2, \ldots, z_{n-2}, z_{n-1})\) defined by the following inequalities:
\[
|z_i| \leq \alpha_{i,n-1} |z_{n-1}| + \beta_{i,n-1}, \quad i = 1, \ldots, n-2.
\]

These expressions are used to evaluate the derivative of the Lyapunov function candidate \(V_{n-1} = \frac{1}{2} z_{n-1}^2\) along the trajectories of (11c), that is
\[
\dot{V}_{n-1} = -k_{n-1} z_{n-1}^2 + z_{n-1} [\sum_{i=1}^{n-2} k_{i-1} (z_{i-1}) w] \\
\leq -k_{n-1} z_{n-1}^2 + |z_{n-1}| \left[ q_{n-1,n-1} + \sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) |z_{j-1}| + d_{n-1}\right] \
\leq -\left(k_{n-1} - q_{n-1,n-1} - \sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) \alpha_{j,n-1}\right) |z_{n-1}|^2 + \left(\sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) \beta_{j,n-1} + d_{n-1}\right) |z_{n-1}|
\]
If \(k_{n-1}\) is chosen such that the condition
\[
k_{n-1} > q_{n-1,n-1} + \sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) q_{n-1,j} \alpha_{j,n-1}
\]
holds, then we obtain
\[
\dot{V}_{n-1} = -2 \alpha_{n-1} V_{n-1} + \beta_{n-1} \sqrt{V_{n-1}}
\]
with positive
\[
\alpha_{n-1} = k_{n-1} - q_{n-1,n-1} - \sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) q_{n-1,j} \alpha_{j,n-1}
\]
and
\[
\beta_{n-1} = \sum_{j=1}^{n-2} k_{j-1} (z_{j-1}) q_{n-1,j} \beta_{j,n-1} + d_{n-1}.
By the Comparison Lemma (Khalil, 1996), we have
\[ |z_{n-1}(t)| \leq \gamma_{n-1} + \exp\left(\frac{\alpha_{n-1}}{2} (t - t_0)\right) |h_{n-1}| \]  
(23)

where \( \gamma_{n-1} = |z_{n-1}(t_0)| - h_{n-1} \), and \( h_{n-1} = \frac{\beta_{n-1}}{\alpha_{n-1}} \). Thus
\[ \lim_{t \to \infty} \sup |z_{n-1}(t)| \leq h_{n-1} \]  
(24)

Therefore, using the obtained upper (23) and ultimate (24) bounds on the solution \( z_{n-1}(t) \), and the inequalities (21), and going back, from the \((n-1)^{th}\) block to the first block of (11a)-(11c), we can find step-by-step upper estimations and ultimate bounds on the solutions \( z_{n-2}(t) \), \( z_{n-3}(t) \), \ldots, \( z_1(t) \).

In order to reduce the effect of the unknown disturbances action in (8), that is, ensure inequalities (15), (20) and (22) for given bounds (12a)-(12d), and, respectively, increase the region of the sliding mode stability (10), it is needed to increase the value of the controller gain \( k_n \) in (9).

This high gain, however, can produce “chattering” due to some imperfections in the control devices. To solve this problem we propose to adjust the value of the gain \( k_n \) depending on the value of \( s \) and the distance \( d_n \) defined as
\[ d_n = \left( z_1^2 + \ldots + z_{n-1}^2 \right)^{\frac{1}{2}} \]

It can be done by using the fuzzy logic scheme. For the case of a bounded control, the value of \( k_n \) begins with \( k_n = k_{n,\text{max}} \) and then, as \( s \) tends to zero, the value of \( k_n \) decreases smoothly up to \( k_n = k_{n,\text{min}} \), avoiding “chattering”.

A schematic diagram of evolution of \( s \) and \( k_n \) is shown in Figure 1. The block diagram of the closed-loop system with block transformation and sliding mode fuzzy logic controller (SMFLCBC) is presented in the Figure 2.

The diagram consists of the following parts:

**Block Control part.** This block transforms state \( x \) in the new coordinate \( z \).

\[ T_{BC}: x \rightarrow z, \text{ such that } z = T_{BC}(x) \]

where the map \( T_{BC} \) is defined by (6a)-(6d), and computes the value of the following distances:
\[ s = z_n \text{ and } d = \left( z_1^2 + \ldots + z_{n-1}^2 \right)^{\frac{1}{2}} \].

**Slope Change block:** Using fuzzy logic, the gains \( k_1, \ldots, k_{n-1} \) are modified by this block when the sliding surface \( s = 0 \) is reached. Opposed to \( k_n \) changes, the gains \( k_1, \ldots, k_{n-1} \) (slope gains) are incremented from minimal to maximal values, resulting on an increasing of the sliding mode motion rate in (11) and of the stability region.

**Fuzzy Controller block:** This block uses two inputs: \( In_1 = s \) and \( In_2 = d \), instead of all state \( x \), and it determines the gain \( k_n \) depending on the magnitude of the inputs such that to satisfy the stability condition (10). The block consists of the following parts:

**Normalization Input part** scales (normalizes) inputs such that the sliding surface is reached in a smooth and fast way.

**Fuzzification part** transforms the crisp input values in fuzzified values
\[ F: In \rightarrow Lin \text{ such that } F(In_i) = Lin(i, j) \]

where \( In_i \in In \) is a crisp input value defined on the discourse universe \( In \), and \( Lin(i, j) \) is a corresponding fuzzified input value also named as membership degree.

**Inference Mechanism part** uses the following type rule:

\[ Rule \ m: \text{ If } (Lin(1, j) \text{ and } Lin(2, k)) \text{ Then } CR(j, k) = Cout(l) \]

where \( CR(j, k) \) is the corresponding value in the rule consequent, and \( Cout(l) \) is the \( l-th \) central value of the output set.

**Defuzzification part** based on the weighted mean

Figure 2. Block Diagram
defuzzification method (Driankov, et al., 1996) produces a scalar value \( k_d \) calculated as

\[
k_d = \frac{\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \text{LAn}t(j,k) \cdot \text{CR}(j,k)}{\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \text{LAn}t(j,k)}
\]

(25)

where \( \text{LAn}t(j,k) = \min(\text{Lin}(1,j), \text{Lin}(2,k)) \) is the premise quantification of the active rule, and \( n_i \) for \( i=1,2 \), is the fuzzy set size.

**Denormalization part** multiplicities normalized fuzzy controller output with denormalization factor (scale), \( k_a = k_d \cdot \text{scale} \), such that the system (8) stays stable.

**Output Conditioning block:** This block verifies the constraints on the control in order to preserve stability conditions, and control limitation. Finally, the control \( u \) is obtained.

Figure 3 shows the gain \( k_d \) (25) computation for hypothetical inputs (\( I_{n1} \) and \( I_{n2} \)).

### 3 Stepper Motor Control

In this section, we apply the proposed scheme to control a permanent magnet stepper motor. Its mathematical model is given by

\[
\frac{d\theta}{dt} = \omega
\]

\[
\frac{d\omega}{dt} = \frac{1}{J}[-K_m i_a \sin(N, \theta) + K_m i_b \cos(N, \theta) - B_v \omega - \tau_f]
\]

\[
\frac{di_a}{dt} = \frac{1}{L}[-R i_a - K_m \cos(N, \theta) + u_a]
\]

\[
\frac{di_b}{dt} = \frac{1}{L}[-R i_b + K_m \cos(N, \theta) + u_b]
\]

(26)

where, \( \theta \) is the angular position; \( \omega \) is the shaft speed; \( i_a \) and \( i_b \) are the currents in phases A and B respectively; \( u_a \) and \( u_b \) are the voltages in phases A and B, respectively; \( J \) is the moment of inertia; \( R \) and \( L \) are the resistance and inductance in each of the phase windings, \( N \) is the number of rotor teeth, \( K_m \) is the motor torque constant, \( B_v \) is the viscous friction and \( \tau_f \) presents the loud torque perturbation.

Selecting the following state variables, \( x_1 = \theta \), \( x_2 = \omega \), \( x_3 = i_a \), \( x_4 = i_b \), the system (26) is represented as block controllable system consisting of tree blocks, and subject to unknown disturbance , \( \tau_f = w_1 \).

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
\frac{1}{J}[-K_m i_a \sin(N, \theta) + K_m i_b \cos(N, \theta) - B_v \omega - \tau_f] \\
\frac{1}{L}[-R i_a - K_m \cos(N, \theta) + u_a] \\
\frac{1}{L}[-R i_b + K_m \cos(N, \theta) + u_b] \\
\end{bmatrix} =
\begin{bmatrix}
-a_1 & a_2 & -a_3 & -a_4 & -d_1 & w_1 \\
-b_1 & b_2 & -b_3 & -b_4 & 0 & 0 \\
-b_5 & b_6 & -b_7 & -b_8 & 0 & 0 \\
-b_9 & b_10 & -b_11 & -b_12 & 0 & 0 \\
\end{bmatrix}
\]

(27)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} =
\begin{bmatrix}
-a_1 x_2 + b_1(x_1) x_3 - b_2(x_1) x_4 - d_1 w_1 \\
-a_3 x_3 - c_1 b_1(x_1) x_3 + b_2(x_1) x_4 \\
-a_5 x_4 + c_2 b_1(x_1) x_4 + b_3(x_1) u_1 \\
-b_1(x_1) + b_4(x_1) u_2 \\
\end{bmatrix}
\]

where \( a_2 = \frac{B}{J} \), \( a_1(x) = \frac{K_m}{J} \cos(N, \theta) \),

\[
b_2(x) = \frac{K_m}{J} \sin(N, \theta) ,
\]

\[
d_1 = \frac{1}{J} , a_3 = a_4 = \frac{R}{L} , c_1 = \frac{J}{L} .
\]

Suppose that the output \( y = x_1 \) is required to follow the reference signal \( x_{1ref} \). Following the block transformation procedure, first we define the tracking error as \( z_1 = x_1 - x_{1ref} \), and

\[
\dot{z}_1 = x_2 - \dot{x}_{1ref} .
\]

Then a desired dynamics for \( z_1 \) is introduced as

\[
\dot{z}_1 = -k_1 z_1 + z_2 .
\]

(29)

Solving (28) and (29) for \( z_2 \), we obtain

\[
z_2 = k_1 x_1 + x_2 - k_1 x_{1ref} - \dot{x}_{1ref}
\]

Then

\[
\dot{z}_2 = (k_1 - a_2) x_2 + b_1(x_1) x_3 - b_2(x_1) x_4 - k_1 \dot{x}_{1ref} - \dot{x}_{1ref} - d_2 w_1
\]

(30)

From this equation and the desired dynamics

\[
\dot{z}_2 = -k_2 z_2 + z_3 - d_2 w_1
\]

(30)

we have

\[
z_3 = f_3(x_1, x_2) + b_1(x_1) x_3 - b_2(x_1) x_4 + \varphi(t)
\]

where \( z_3 \) is a new variable, \( f_3 = k_1 k_2 x_1 + (k_1 + k_2 - a_2) x_2 \),

\[
\varphi(t) = -k_1 k_2 x_{1ref} - (k_1 + k_2) \dot{x}_{1ref} - \dot{x}_{1ref} .
\]

In order to have a nonsingular transformation we introduce a new variable \( z_4 \)

\[
z_4 = -b_2(x_1) x_3 - b_1(x_1) x_4
\]

such that the matrix \( B_z = \begin{bmatrix} b_1(x_1) & -b_2(x_1) \\ -b_2(x_1) & -b_1(x_1) \end{bmatrix} \) has full rank.

In order to obtain the control action, first we define the switching functions as
Then the projection motion on the subspace \( s_1, s_2 \) is governed by

\[
\begin{bmatrix}
\dot{s}_1 \\
\dot{s}_2
\end{bmatrix} = \begin{bmatrix}
\tilde{f}_3(x) \\
\tilde{f}_4(x)
\end{bmatrix} + b_0 \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} + \begin{bmatrix}
\phi(t) \\
0
\end{bmatrix}
\]

where \( x = (x_1, x_2, x_3, x_4)^T \); \( \tilde{f}_3(x) \) and \( \tilde{f}_4(x) \) are continuous functions. The control strategy is selected of the form

\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = -B^{-1} \begin{bmatrix}k_3 \text{sign}(s_1) \\
 k_4 \text{sign}(s_2)\end{bmatrix}
\]

and the sliding mode stability conditions are

\[ b_0 k_3 > |\tilde{f}_3(x) + \phi(t)| \quad \text{and} \quad b_0 k_4 > |\tilde{f}_4(x)|. \]

Under these conditions the state converges to the sliding manifold \( s_1 = 0, s_2 = 0 \), and when this manifold is reached the sliding mode motion is described by the second order system with unknown nonvanishing perturbation

\[
\begin{align*}
\dot{z}_1 &= -k_1 z_1 + z_2 \\
\dot{z}_2 &= -k_2 z_2 + w_1.
\end{align*}
\]

In order to reduce the disturbances influence, we apply the described in the section 2 sliding mode fuzzy logic control scheme for adjusting of the controller gains \( k_1, k_2, k_3 \) and \( k_4 \) such that \( k_3 \leq u_0 \) and \( k_4 \leq u_0 \) with \( u_0 > 0 \).

### 4 Simulation Results

In this section, simulation results are presented for the Permanent Magnet Stepper Motor with parameters:

\[
\begin{align*}
L &= 10 \text{mH}, & R &= 8.4 \Omega, & J &= 3.6 \times 10^{-6} \text{Nms}^2/\text{rad}, \\
k_m &= 0.05 \text{Vs/rad}, & N_r &= 50, & B &= 1 \times 10^{-4} \text{Nms/rad}.
\end{align*}
\]

The maximal supplied voltage is \( u_0 = 2V \). In all Figures it is shown the behavior of the state \( (x_1, x_2, x_3, x_4) \) as well as new state \( (z_1, z_2, z_3, z_4) \), and the gains \( k_i \) i=1,2,3, the control \( u \), and disturbance \( w_1 = r_1 \).

Figure 4 displays Block Control Tracking (BCT) with Fuzzy Logic (FL) results using the proposed approach, where smooth variation of \( k_i \) gain is observed, and the chattering is reduced.

### 5 Conclusions

As it can be seen, the proposed scheme performance is very encouraging. Simulations assume that the disturbance cannot be measured, which is an extreme situation. However, the proposed hierarchical sliding mode control approach with the fuzzy logic control, improves the system behavior, reducing chattering and guaranteeing stability.

### 6 References


