AN ITERATIVE ALGORITHM FOR THE MIXED \( H_2/H_\infty \) CONTROL PROBLEM USING \( H_2 \) NORM DECREASING CONTROLLER SETS

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Abstract

This paper is concerned with the mixed \( H_2/H_\infty \) control problem. The purpose of this paper is to give an iterative algorithm for finding a sub-optimal static state feedback controller for the mixed \( H_2/H_\infty \) control problem. The key idea of our algorithm is to construct two “controller sets”: one is a set of controllers that improve the \( H_2 \) norm of the closed loop map for a given controller and the other is a set of controllers whose elements satisfy the \( H_\infty \) norm constraint. Using two controller sets, we propose an iterative algorithm. The obtained controller is either the global optimal solution if the \( H_\infty \) norm constraint is satisfied until the \( H_2 \) norm of the objective closed loop map converges to the \( H_2 \) optimal value or a sub-optimal solution on the boundary of the \( H_\infty \) norm constraint.

1 Introduction

Recently, multiobjective control problems have received a great deal of attention [1, 2], [4]-[7], [9]-[14]. In particular, the so-called mixed \( H_2/H_\infty \) control problem for linear time invariant (LTI) systems has been studied by many researchers. In this problem, the \( H_2 \) and \( H_\infty \) norms are measures for optimal performance and robustness, respectively. The purpose of the mixed \( H_2/H_\infty \) control problem is to find a controller which minimizes the \( H_2 \) norm of one closed-loop map with an \( H_\infty \) norm constraint of another closed-loop map. That is, this problem is to find the best performance controller among the robustly stabilizing controllers. Both the \( H_2 \) and \( H_\infty \) control theories have almost been established. However the mixed \( H_2/H_\infty \) control problem have not completely been solved. This is because the mixed \( H_2/H_\infty \) control problem is quite difficult to be solved theoretically, and it is known that the order of the optimal mixed \( H_2/H_\infty \) controller is not finite in some cases. Even for a fixed order controller the problem is still very difficult, because it is a non-convex problem. For this non-convex problem, various approaches to find a sub-optimal solution have been explored. However, there is no method to obtain the global optimal solution except some special cases.

2 Problem Formulation

In this paper, consider the following LTI system:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) + B_1 w_1(t) + B_2 w_2(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t) + D_2 u(t), \\
y(t) &= x(t),
\end{align*}
\]

where \( x \) is the plant state, \( w_i (i = 1, 2) \) are any exogenous inputs, \( u \) is the control input and \( z_i (i = 1, 2) \) are the performance outputs. Throughout this paper, the following assumptions are made:

1. \((A, B)\) is controllable.
2. \((A, B_i, C_i) (i = 1, 2)\) are controllable and observable.
3. \(D_i^T D_i = I (i = 1, 2)\).
4. \(B_2\) has full column rank.
5. \[
\begin{bmatrix}
A - j\omega & B \\
C_2 & D_2
\end{bmatrix}
\]
has full column rank for all \( \omega \in \mathbb{R} \).
Let us consider the static feedback controller:

\[ u(t) = K x(t). \]  

Via the static feedback controller the closed loop system is described by

\[
\begin{align*}
\dot{x}(t) &= A_d x(t) + B_1 w_1(t) + B_2 w_2(t), \\
z_1(t) &= C_{cl1} x(t), \\
z_2(t) &= C_{cl2} x(t),
\end{align*}
\]

where

\[ A_d = A + BK_i C_{cl i} = C_i + D_i K (i = 1, 2). \]

Let \( T_{z_1 w_1}(K) \) denote the closed-loop transfer function from \( w_1 \) to \( z_1 \). For this system the mixed \( H_2/\infty \) control problem is defined as follows.

**The mixed \( H_2/\infty \) control problem (OP):** Given an achievable \( H_\infty \) norm bound \( \gamma \), find a controller that satisfies

\[
\min_K \| T_{z_2 w_2}(K) \|_2 \text{ subject to } \| T_{z_1 w_1}(K) \|_\infty < \gamma,
\]

where \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) denote the \( H_2 \) and \( \infty \) norms, respectively.

### 3 The global optimal solution of the mixed \( H_2/\infty \) control problem

In this section, we show the property of the global optimal solution of the mixed \( H_2/\infty \) control problem. In general, a global optimal solution of an optimization problem is either a stationary point of the objective function or a feasible solution on the boundary of a constraint. However, such property of the problem (OP) cannot be discussed, because the \( \infty \) norm constraint in (10) has no boundary. Hence, we modify the problem (OP) into the following problem.

**The modified mixed \( H_2/\infty \) control problem (MP):** Given an achievable \( H_\infty \) norm bound \( \gamma \) and sufficiently small \( \varepsilon \), find a controller that satisfies

\[
\min_K \| T_{z_2 w_2}(K) \|_2 \text{ subject to } \| T_{z_1 w_1}(K) \|_\infty \leq \gamma - \varepsilon.
\]

To state the property of the stationary point of the objective function let

\[
J(K) := \| T_{z_2 w_2}(K) \|_2^2 = \text{trace} B_2^T G B_2
\]

where \( G \) is the observability Gramian that is the solution of the following Lyapunov equation:

\[
GA_d + A_d^T G + C_{cl2}^T C_{cl2} = 0.
\]

The global optimal \( H_2 \) state feedback controller without the \( \infty \) norm constraint is given by

\[
K = K_2^* := -B^T Z_2 - D_2^T C_2
\]

where \( Z_2 \) is the stabilizing solution of the Riccati equation

\[
Z_2 (A - BD_2^T C_2) + (A - BD_2^T C_2)^T Z_2
- Z_2 B B^T Z_2 + C_2^T (I - D_2 D_2^T) C_2 = 0.
\]

**Lemma 1** [14] \( J(K) \) has the unique stationary point at \( K = K_2^* \) over all internally stabilizing controllers.

Now, we can state the property of the global solution of the modified mixed \( H_2/\infty \) control problem.

**Proposition 1** Let \( K_m^* \) be the global optimal solution of the modified mixed \( H_2/\infty \) control problem (MP). If \( \| T_{z_1 w_1}(K_m^*) \|_\infty \leq \gamma - \varepsilon \) then \( K_m^* = K_2^* \). Otherwise \( K_m^* \) exists on the boundary of the \( \infty \) norm constraint, i.e., \( \| T_{z_1 w_1}(K_m^*) \|_\infty = \gamma - \varepsilon \).

**Remark 1** From Proposition 1 it is a necessary condition for a controller \( K \) to be the global optimal controller of the problem (MP) that \( K = K_2^* \) or \( K \) is on the boundary of the \( \infty \) norm constraint.

### 4 Controller Sets

In this section, we define a “controller set” \( S_2(K_i) \) whose element achieves the better \( H_2 \) norm than \( K_i \). After then, we show that a “controller sequence” chosen from \( S_2(K_i) \) achieves a monotonically non-increasing \( H_2 \) norm which converges to the unconstrained \( H_2 \) optimal value. Similarly, we define a “controller set” \( S_\infty(K_i) \) via whose element the closed loop satisfies the \( \infty \) norm constraint.

For a given controller \( K_i \) let \( G_i = G_i^T > 0 \) be the observability Gramian, i.e., the solution of

\[
G_i A_i + A_i^T G_i + C_{2i}^T C_{2i} = 0
\]

where

\[
A_i = A + BK_i, C_{2i} = C_2 + D_2 K_i,
\]

and define a controller set \( S_2(K_i) \) as

\[
S_2(K_i) := \{ K | L_2^{G_i}(K) \leq 0 \} - \{ K_i \}
\]

where

\[
L_2^{G_i}(K) := G_i (A + BK) + (A + BK)^T G_i
+ (C_2 + D_2 K)^T (C_2 + D_2 K).
\]

This controller set \( S_2(K_i) \) has the next property.
Lemma 2 If $K_i \neq K^*_2$ then every $K \in S_2(K_i)$ is an internally stabilizing controller.

Using the controller set $S_2(K_i)$ a controller sequence $\Pi = \{K_i, i = 0, 1, 2, \cdots \}$ is defined as follows:

**Algorithm 1:** Construction of a controller sequence $\Pi$.

**STEP 1** Give a stabilizing controller $K_0(\neq K^*_2)$ and let $i := 0$.

**STEP 2** Get $G_i > 0$ which is the solution of (16).

**STEP 3** Choose any controller from $S_2(K_i)$ and let it be $K_{i+1}$. If $K_{i+1} = K^*_2$ then exit. Otherwise $i := i + 1$ and go to STEP 2.

This controller sequence has the next properties.

**Lemma 3** Suppose $K_i \neq K^*_2$ then the following (i)-(ii) hold:

(i) The inequality $G_i \geq G_{i+1}$ holds.

(ii) The $H_2$ norm of the closed loop via the controller $K_i$ is monotonically non-increasing, i.e., $J(K_i) \geq J(K_{i+1})$.

**Proof:** From the definition of $K_i$ we have

$$G_i A_{i+1} + A^T_{i+1} G_i + C^T_{2i+1} C_{2i+1} \leq 0,$$

$$G_{i+1} A_{i+1} + A^T_{i+1} G_{i+1} + C^T_{2i+1} C_{2i+1} = 0.$$  \hspace{1cm} (21)

Subtracting (21) from (20) to get

$$(G_i - G_{i+1}) A_{i+1} + A^T_{i+1} (G_i - G_{i+1}) \leq 0.$$  \hspace{1cm} (22)

Since $A_{i+1}$ is stable it follows that $G_i - G_{i+1} \geq 0$, which implies (i) and hence

$$\text{trace} B^T_2 G_i B_2 \geq \text{trace} B^T_2 G_{i+1} B_2.$$  \hspace{1cm} (23)

Thus $J(K_i) \geq J(K_{i+1})$. \Box

To state that the controller sequence $\Pi$ converges to the unconstrained $H_2$ optimal controller $K^*_2$ we need the next lemma.

**Lemma 4** The set $S_2(K_i)$ is empty ($S_2(K_i) = \emptyset$) if and only if $K_i = K^*_2$.

**Proof:** (only if): Suppose $S_2(K_i) = \emptyset$ but $K_i \neq K^*_2$. Then let

$$K = -B^T G_i - D_i^T C_2$$

and using (16) to get

$$L_2^G(K) = -(K_i + B^T G_i + D_i^T C_2)^T (K_i + B^T G_i + D_i^T C_2).$$  \hspace{1cm} (24)

Since the RHS is negative semi-definite it follows that $K \in S_2(K_i)$. This contradicts $S_2(K_i) = \emptyset$.

(ii): Suppose $K_i = K^*_2$ and let $\tilde{K}$ be any controller such that $L_2^G(\tilde{K}) \leq 0$ and $\tilde{G}$ be the solution of (13) where $K = \tilde{K}$. Then $G_i \geq \tilde{G}$ from Lemma 3-(i). On the other hand, $G \geq Z_2(= G_i)$ for any $G$ and $K$ that satisfy (13) (see [14]). Hence, we have $G_i = \tilde{G}$, which implies $J(K^*_2) = J(K_i) = J(\tilde{K})$. From Lemma 1 $\tilde{K} = K^*_2$ and it follows $S_2(K_i) = \emptyset$. \Box

**Theorem 1** The controller sequence $\Pi$ converges to the unconstrained $H_2$ optimal controller $K^*_2$, i.e.,

$$\lim_{i \rightarrow \infty} K_i = K^*_2, (K_i \in \Pi).$$  \hspace{1cm} (25)

**Proof:** If $K_i = K^*_2$ for some $i > 0$ (25) is obvious. Hence, suppose $K_i \neq K^*_2$ for all $i \geq 0$. Since $G_i$ is monotonically non-increasing and bounded below ($G_i \geq G_{i+1} \geq Z_2 > 0$) $G_i$ converges as $i \rightarrow \infty$. Hence, from the definition $K_i$ also converges and let $K_\infty := \lim_{i \rightarrow \infty} K_i$. If $S_2(K_\infty)$ is not empty we can choose a new controller $\tilde{K}_\infty \in S_2(K_\infty)$ in STEP 3 of Algorithm 1, which contradicts the assumption that $K_\infty$ is the limit of $K_i$. Hence $S_2(K_\infty)$ is empty and $K_\infty = K^*_2$ from Lemma 4. \Box

Next, we construct a controller set for a given controller such that any controller in the set satisfies the $H_\infty$ norm constraint. Suppose a given controller $K_i$ satisfies the $H_\infty$ norm constraint. Then there exists $X_i = X_i^T (> 0)$ which satisfies

$$A_i X_i + X_i A_i + \gamma^{-2} X_i C_1 C_1^T X_i + B_1 B_1^T < 0$$  \hspace{1cm} (26)

where

$$A_i = A + BK_i, C_1 = C_1 + D_i K_i.$$  \hspace{1cm} (27)

and a controller set $S_\infty(K_i)$ is defined as

$$S_\infty(K_i) = \{K | L_\infty^X(K) < 0\}$$  \hspace{1cm} (28)

where

$$L_\infty^X(K) := (A + BK) X_i + X_i (A + BK)^T + \gamma^{-2} X_i (C_1 + D_i K) X_i + B_1 B_1^T.$$  \hspace{1cm} (29)

This controller set $S_\infty(K_i)$ has the next property.

**Lemma 5** Every $K \in S_\infty(K_i)$ satisfies the $H_\infty$ norm constraint, i.e., \( \|T_{z(\omega)}(K)\|_\infty < \gamma \) for $K \in S_\infty(K_i)$.

**Proof:** Obvious from the definition of $S_\infty(K_i)$. \Box

5 Iterative Algorithm

In this section, we propose an iterative algorithm for the modified mixed $H_2/H_\infty$ control problem. For a given controller $K_i$ any controller in $S_2(K_i) \cap S_\infty(K_i)$ achieves the better $H_2$ norm of the closed loop $T_{z(\omega)}(K)$ than $K_i$ while it satisfies the $H_\infty$ norm constraint. Hence the controller chosen in $S_2(K_i) \cap S_\infty(K_i)$ is a better mixed $H_2/H_\infty$ controller than $K_i$.

An iterative algorithm we propose for the modified mixed $H_2/H_\infty$ control problem (MP) is described as follows:

**Algorithm 2** : Iterative algorithm for the modified mixed $H_2/H_\infty$ control problem (MP).

**STEP 1** Take an initial stabilizing controller $K_0$ which satisfies the $H_\infty$ norm constraint and let $i := 0$. 

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STEP 2 Get $G_i$ and $X_i$ which satisfy (16) and (26), respectively.

STEP 3 Choose any controller from $S_a(K_i) \cap S_b(K_i)$ and let it be $K_{i+1}$.

STEP 4 If the $H_2$ norm is not improved (i.e., $J(K_i) = J(K_{i+1})$) or $\gamma - \|T_{z_1w_1}(K_{i+1})\|_\infty < \varepsilon$ then let $K^* = K_i$ and exit. Otherwise let $i := i + 1$ and go to STEP 2.

Remark 2 The problem to find a new controller $K_{i+1}$ in STEP 3 of Algorithm 2 is described as an LMI feasible problem that can efficiently be solved numerically.

Remark 3 The controller sequence $K_i (i = 0, 1, \cdots)$ produced by Algorithm 2 approaches to the unconstraint $H_2$ optimal controller $K_2^*$ until it encounters the boundary of the $H_\infty$ norm constraint.

Theorem 2 Let $\tilde{K} := \lim_{i \to \infty} K_i$ (30) where $K_i (i = 0, 1, \cdots)$ is the controller sequence produced by Algorithm 2. Then the following (i) and (ii) hold:

(i) If $\|T_{z_1w_1}(\tilde{K})\|_\infty < \gamma - \varepsilon$ then $\tilde{K} = K_2^*$. In this case, $\tilde{K}$ is the global optimal solution of the modified mixed $H_2/H_\infty$ control problem.

(ii) Otherwise $\tilde{K}$ exists on the boundary of the $H_\infty$ norm constraint of the modified mixed $H_2/H_\infty$ control problem, i.e., $\|T_{z_1w_1}(\tilde{K})\|_\infty = \gamma - \varepsilon$.

Proof: This follows immediately from construction of $K_i$. \qed

6 Numerical Examples

Consider the following state-space matrices:

\[
A = \begin{bmatrix}
-0.40 & -0.04 & 0.59 \\
-0.11 & 0.37 & -0.23 \\
1.21 & 0.39 & -0.35
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1.29 & -1.10 \\
-0.02 & -1.04 \\
1.05 & -0.91
\end{bmatrix},
B_1 = \begin{bmatrix}
-0.98 & -0.90 \\
-0.68 & -0.41 \\
1.33 & -0.50
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0.80 & -0.08 \\
-0.40 & -2.00 \\
-0.75 & 1.08
\end{bmatrix},
C_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
C_2 = \begin{bmatrix}
0.00 & 0 \\
0.00 & 0
\end{bmatrix},
D_1 = D_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

Figure 2 shows the behavior of $\|T_{z_2w_2}(K_i)\|_2$ on the controller sequence $\Pi$ produced by Algorithm 1 as a function of the iteration number $i$. The common LMI solutions for $\gamma = 5$ are taken as an initial controller $K_0$. Figure 2 shows that $\|T_{z_2w_2}(K_i)\|_2$ is monotonically non-increasing on the controller sequence $\Pi$ and converges to the $H_2$ optimal value.

Figure 3 and Figure 4 show the behaviors of $\|T_{z_2w_2}(K_i)\|_2$ and $\|T_{z_1w_1}(K_i)\|_\infty$ on the controller sequence produced by Algorithm 2 as a function of the iteration number $i$ for $\gamma = 5$. The common LMI solution for $\gamma = 5$ is taken as an initial controller $K_0$. Figure 3 shows that $\|T_{z_2w_2}(K_i)\|_2$ is monotonically non-increasing. Figure 4 shows that the controller obtained by Algorithm 2 satisfies $\|T_{z_1w_1}(K_i)\|_\infty = 5 - \varepsilon$, i.e., the obtained controller exists on the boundary of the $H_\infty$ norm constraint of the modified $H_2/H_\infty$ control problem.

Figure 5 and Figure 6 show the behaviors of $\|T_{z_2w_2}(K_i^*)\|_2$ and $\|T_{z_1w_1}(K_i^*)\|_\infty$ as a function of the $H_\infty$ norm bound $\gamma$. For each $\gamma$ the common LMI solutions are taken as an initial controller $K_0$. Figure 5 and Figure 6 also show $\|T_{z_2w_2}(K_\varepsilon)\|_2$ and $\|T_{z_1w_1}(K_\varepsilon)\|_\infty$, where $K_\varepsilon$ is a controller obtained by common LMI solutions. Figure 5 shows the controllers obtained by Algorithm 2 achieve lower $H_2$ norms than the controllers obtained by common LMI solutions for all $\gamma$. Furthermore, $\|T_{z_2w_2}(K_\varepsilon^*)\|_2$ goes to the unconstraint $H_2$ optimal value as $\gamma$ increases. Let $\gamma_2^*$ be the $H_\infty$ norm of $T_{z_1w_1}(K)$ via the unconstraint $H_2$ optimal controller $K_2^*$, i.e., $\gamma_2^* := \|T_{z_1w_1}(K_2^*)\|_\infty = 6.8858$. Figure 5 shows the controllers obtained by Algorithm 2 are the global optimal solutions for $\gamma \geq \gamma_2^*$, and Figure 6 shows they are on the boundary of the $H_\infty$ norm constraint for $\gamma < \gamma_2^*$.

7 Conclusions

In this paper, we introduced two controller sets and showed that a controller sequence derived by the controller sets achieves the monotonically non-increasing $H_2$ norm that converges to the unconstraint $H_2$ optimal value. Using the two controller sets we proposed an iterative algorithm for the modified mixed $H_2/H_\infty$ control problem and show the effectiveness of our algorithm by numerical examples.

References


Figure 2: $H_2$ norm behavior on the controller sequence $\Pi$

Figure 3: $H_2$ norm vs. iteration number $i$ ($\gamma = 5$)

Figure 4: $H_\infty$ norm vs. iteration number $i$ ($\gamma = 5$)

Figure 5: $H_2$ norm as a function of $\gamma$

Figure 6: $H_\infty$ norm as a function of $\gamma$


