CONTROL OF PORT-INTERCONNECTED DRIFTLESS SYSTEMS

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Abstract. We use a recently reported extended Matrosov’s theorem to establish sufficient and necessary conditions to stabilize a chain of driftless systems. This benchmark can be regarded as a problem of controlling a communication channel and is a generalization of the well known chained-form non holonomic systems benchmark. Roughly speaking our extended Matrosov’s theorem is an extension of La Salle’s invariance principle to the case of non autonomous systems.

1 Introduction

Many nonlinear control algorithms rely heavily on analysis tools that establish convergence to the origin for trajectories of a time-varying nonlinear system having a uniformly stable origin. For time-invariant problems, the typical analysis tool used is the Krasovski/LaSalle invariance principle. It is the key result that leads to the so-called Jurjevic/Quinn control algorithm for open-loop stable nonlinear control systems. When the closed-loop is time-varying, one tool that is often used is Barbalat’s lemma. In adaptive control, Barbalat’s lemma is frequently relied upon to establish convergence to zero of part of the state. Barbalat’s lemma has also been used to establish convergence to the origin for a class of non holonomic systems controlled by smooth time-varying feedback. For time-varying systems, another tool that has been used, but more sparingly, is Matrosov’s theorem [6, 11]. It was used e.g. in [9] to establish one of the first results on uniform global asymptotic stability (UGAS) of robot manipulators in closed loop with a tracking controller. It also appears in the context of adaptive control in [5] and output feedback control in [8].

In the recent paper [4] we presented a result which is an extension in certain directions, of Matrosov’s theorem. In particular, our result relies on the ability of finding a (non a priori fixed) number of auxiliary functions which do not have sign-definite derivatives. Also, as in [9] the bounds on these functions derivatives are allowed to be time-varying. In this paper we present a case-study which illustrates the utility of the main result in [4] as a tool to aid control design. In particular, we will address the problem of stabilizing a chain of port-interconnected and port-controlled driftless systems and establish sufficient and necessary conditions for uniform global asymptotic stability of the closed loop system. As we will see, this class of systems include as particular case, the non holonomic systems in chained form.

The rest of the paper is organized as follows. In next section we present some definitions and recall our extended Matrosov theorem from [4]. In Section 3 we present our main result. We conclude in Section 4 with some remarks.

2 Preliminaries

For two constants $\Delta \geq \delta \geq 0$ we define $B(\Delta) := \{x \in \mathbb{R}^n : |x| \leq \Delta\}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\gamma \in \mathcal{K}_\infty$ if in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{P}\mathcal{D}$ if it is continuous and positive definite. We denote by $x(\cdot, t_0, x_0)$, the solutions of the differential equation

$$\dot{x} = f(t, x)$$

with initial conditions $(t_0, x_0)$. For a function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we define its derivative in the direction $(1, f(t, x)^T)^T$ as $\dot{V}(t, x) := \frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(t, x)$. This abuse of notation is reasonable because of (1). Furthermore, with an abuse of notation we will use the same definition to express the derivative of locally Lipschitz functions. For the latter, $\dot{V}(t, x)$ is defined everywhere except on a set of measure zero where the gradient of $V$ is not defined, i.e., almost everywhere. When clear from the context and to simplify the notation we will also use $\nabla f(x)$ to denote $\frac{\partial f(x)}{\partial x}$.

Definition 1 (Uniform global stability) The origin of the system (1) is said to be uniformly globally stable (UGS) if there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ each solution $x(\cdot, t_0, x_0)$ satisfies

$$|x(t, t_0, x_0)| \leq \gamma(|x_0|) \quad \forall t \geq t_0.$$ (2)

Definition 2 (Uniform global attractivity) The origin of the system (1) is said to be uniformly globally attractive if for each $r, \sigma > 0$ there exists $T > 0$ such that

$$|x_0| \leq r \implies \|x(t, t_0, x_0)\| \leq \sigma \quad \forall t \geq t_0 + T.$$ (3)

Furthermore, we say that the (origin of the) system is uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive.

The following theorem is a generalization of the so-called Matrosov theorem which combines an auxiliary function with a Lyapunov function that establishes UGS. See [6] and the more recent expositions [11, Theorem 5.5, p.58] and [10, Theorem 2.5, p. 62]. This is the fundamental tool that we use to prove our main result.

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Theorem [4] Under the following assumptions the origin of system (1) is UGS.

Assumption 1 The origin of system (1) is UGS.

Assumption 2 There exist integers \( j, m > 0 \) and for each \( \Delta > 0 \) there exist: 1) a number \( \mu > 0, 2 \) a locally Lipschitz continuous functions \( V_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, i \in \{1, \ldots, j\} \), 3) a continuous function \( \Phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m \), 4) a continuous functions \( Y_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}, i \in \{1, \ldots, j\} \) such that, for almost all \( (t, x) \in \mathbb{R} \times \mathbb{B}(\Delta) \),

\[
\max\{\|V_i(t,x)\|, \|\Phi(t,x)\|\} \leq \mu, \quad \forall i \in \{1, \ldots, j\}
\]

\[
V_i(t,x) \leq Y_i(x, \Phi(t,x)).
\]

Assumption 3 For each integer \( k \in \{1, \ldots, j\} \)

\[
(A): \{ (z, \psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu), Y_i(z, \psi) = 0 \quad \forall i \in \{1, \ldots, k - 1\} \}
\]

implies (B): \( \{ Y_k(z, \psi) \leq 0 \} \).

Assumption 4 We have that

\[
(A): \{ (z, \psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu), Y_i(z, \psi) = 0 \quad \forall i \in \{1, \ldots, j\} \}
\]

implies (B): \( \{ z = 0 \} \).

3 Control over communication channels

We present now our main result. We will address the problem of stabilizing by smooth feedback, a series of port-interconnected driftless systems as illustrated in Figure 1. This control problem covers that of time-varying smooth feedback stabilization of chained-form nonholonomic systems (cf. [1, 7]) and we will solve it by appealing to Theorem 1. In Figure 1, each block contains a bank of integrators with nonlinearities at the input and output of the integrator. The dynamics of each block is given by

\[
\Sigma_i : \begin{cases}
\dot{x}_i = B_i(x_i)u_i \\
y_i = h_i(x_i) = B_i(x_i)^T \nabla W_i(x_i)
\end{cases}
\]

where \( i \leq n, u_i \in \mathbb{R}^m, x_i \in \mathbb{R}^{n_i} \) and the functions in the dynamics above satisfy the following conditions

\[
\nabla h(x_i)B_i(x_i) \geq c_i I, \quad c_i > 0 \quad (7)
\]

\[
|B_i(x_i)| \leq \rho_{B_i}(|x_i|), \quad \rho_{B_i} \in \mathcal{N} \quad (8)
\]

\[
W_i(x_i) \geq \alpha_i(|x_i|), \quad \alpha_i \in \mathcal{N} \quad (9)
\]

\[
\nabla W_i(x_i) \leq \rho_{W_i}(|x_i|), \quad \rho_{W_i} \in K_\infty \quad (10)
\]

\[
\theta_i(|x_i|) \geq \kappa_i(|x_i|), \quad \kappa_i \in \mathcal{P}_\mathcal{D}, \theta_i \in \mathcal{N}. \quad (11)
\]

The nonlinearity of the integrator blocks are interconnected via static, nonlinear, time-varying “communication channels”. In particular, the connection between blocks \( i \) and \( i + 1 \) is modeled by the nonlinear gain function \( g_i \) which takes values in \( \mathbb{R} \) and may depend, in general, on any of the states, time, and perhaps some additional states from outside of the network. This means that the input from the left to the \( i \)th block, denoted \( u_{i,r} \), and the input from the right to the \( i \)th block, denoted \( u_{i,t} \), are

\[
\begin{align*}
u_{i,t} &= g_{i-1} \cdot y_{i-1} \\
u_{i,r} &= g_i \cdot y_{i+1}.
\end{align*}
\]

The blocks are such that the input to the nonlinear integrator is given by

\[
u_i = u_{i,r} - u_{i,t}.
\]

We assume that the communication channel gains have the functional form

\[
\begin{align*}
\dot{z}_i &= -z_i + \bar{g}_{i,a}(t, x) \quad (12a) \\
\bar{g}_i(t, x) &= -z_i + \bar{g}_{i,a}(t, x) \quad (12b) \\
g_i(t, x, z) &= \prod_{j=i}^{n-1} \bar{g}_j(t, x, z) \quad (12c)
\end{align*}
\]

for \( i \leq n - 1 \) and where the functions \( \bar{g}_{i,a} \) are continuous and Lipschitz in \( x \) uniformly in \( t \).

The control problem is to attach a system \( \Sigma_{n+1} \) to the right of \( \Sigma_n \), and give necessary and sufficient conditions on the communication channel gains to guarantee that the origin for the closed-loop system is uniformly globally asymptotically stable. For the controller, we will use any static “first and third sector” nonlinearity \( \sigma(\cdot) \), and the connection to \( \Sigma_n \) will be made with a reliable communication channel, e.g., \( g_n \equiv 1 \). In particular, we have

\[
\Sigma_{n+1} : \begin{cases}
y_{n+1} = \sigma(u_{n+1}) \\
u_{n+1}^T \Sigma_n \geq \rho(|u_{n+1}|) \\
u_{n+1} = y_n \\
y_{n+1} = u_n.
\end{cases}
\]

(13)

With this controller architecture and functional form for the communication channel gains indicated in Figure 1, we ask the question:

What are necessary and sufficient conditions on the communication channel gains to guarantee uniform asymptotic stability of the origin for the system (6)-(12)?

Figure 1: Communication channels.

\footnote{For the case that \( k = 1 \) one should read that \( Y_1(z, \psi) \leq 0 \) for all \( (z, \psi) \in B(\Delta) \times B(\mu) \).}
The answer will be expressed in terms of the notion of “uniform δ-persistency of excitation” (see [3]) which is defined next for completeness.

**Definition 3 (Uniform δ-persistency of excitation (Uδ-PE))**
The function \((t, \xi) \mapsto \varphi(t, \xi) \in \mathbb{R}^n\) is said to be uniformly δ persistently exciting if for each pair of strictly positive real numbers \(\delta \leq \Delta\) there exist \(T > 0\) and \(\mu > 0\) such that
\[
t \in \mathbb{R}, \quad \delta \leq |\xi| \leq \Delta \quad \implies \quad \mu \leq \int_t^{t+T} |\varphi(\xi, \tau)| d\tau.
\]

We will also make use of the following observation.

**Fact 1** For locally Lipschitz functions, uniformly in \(t\), and such that \(\phi_i(t, 0) \equiv 0\) if the product \(\prod_{i=1}^{m} \phi_i(t, x)\) is Uδ-PE then, necessarily each function \(\phi_i(t, x)\) is Uδ-PE.

Note that in the definition above, \(x\) is a constant parameter hence, the necessary and sufficient conditions for stability will be expressed in terms of the state \(x\), being constant. To that end note that when \(x\) is constant the \(z_i\) subsystems in (12) are time-invariant linear systems with time-varying inputs. To better see this, let \(x_n \equiv 0\), let \(x = \bar{x}\) with \(\bar{x}\) a constant vector and call \(\bar{x}\) the new state of the linear system (12) in this setting. Then, defining
\[
\bar{g}^c_{i,a}(t, x) := \bar{g}_i(t, x) \big|_{x_n=0}
\]
it is direct to show that the \(i\)th communication channel gain in (12) satisfies
\[
\begin{align*}
\dot{\bar{g}}^c_i(t, \bar{x}, \bar{z}) &= -\bar{z}_i + \bar{g}^c_{i,a}(t, \bar{x}) + \frac{d\bar{g}^c_{i,a}}{dt} \\
\dot{\bar{g}}^c_i(t, \bar{x}, \bar{z}) &= -\bar{g}^c_i(t, \bar{x}, \bar{z}) + \frac{d\bar{g}^c_i}{dt}
\end{align*}
\]
and we note that the steady-state solution of (16) is given by
\[
\omega_i(t, x) := \int_{-\infty}^{t} e^{-(t-\tau)} \psi_i(\tau, x) d\tau \quad i \in [1, \ldots, n-1]
\]
where
\[
\psi_i(t, x) := \frac{\partial \bar{g}^c_{i,a}(t, \bar{x})}{\partial x}.
\]
That is, \(\omega_i(t, \bar{x})\) is the steady-state value of the \(i\)th communication channel gain and correspondingly, the steady-state value of the gain of the first communication channel can be computed to be the product of all the \(\omega_i(t, \bar{x})\)'s for all \(i \leq n-1\). Based on these observations, we are now ready to present our main result.

**Theorem 2** Suppose the function \(g_i, a(\cdot, \cdot)\) is continuous and locally Lipschitz in \(x\) uniformly in \(t\). The origin of the system (6)-(12) is uniformly globally asymptotically stable (UGAS) if \(\bar{g}_i(t, 0) \equiv 0\) and the function
\[
(t, x_1, \ldots, x_{n-1}) \mapsto \prod_{i=1}^{n-1} \omega_i(t, x) \big|_{x_n=0}
\]
is Uδ-PE. Moreover, the origin is UGAS only if \(\bar{g}^c_{i,a}(t, x)\) is Uδ-PE with respect to \(\xi := \text{col}[x_1, \ldots, x_{n-1}]\). □

**Remark 1** It is worth mentioning that the system architecture above covers the so-called “skew-symmetric” systems considered in [12, 2]. We may see this if we let \(\bar{y}_i = u_i\) for all \(i\) where \(u_i\) is one of the two control inputs in those references (in particular, the controller which is required to be Uδ-PE in [2]), \(y_i = x_i\) for each \(2 \leq i \leq n\) we replace \(z_i\) by \(^3x_i\) and finally, we relate the function \(\bar{g}_{i,a}\) to the function whose second derivative in [2] is required to be Uδ-PE or, to the “heat function” in [12]. □

### 3.1 Proof of Theorem 2

We first point out the following identities.
\[
\forall i \in [1, \ldots, n-2], \quad g_i = \bar{g}_i g_{i+1}, \quad n_{i-1} = \bar{g}_{n-1}, \quad g_n > 0.
\]

Furthermore, using these identities one can show that for all \(i \in [1, \ldots, n-2]\),
\[
\begin{align*}
g_i &= \left|g_{n-1} \cdots g_i\right|^{1/n-i} \bar{g}_{n-2}^{1/n-i} \cdots \bar{g}_{i+j}^{1/n-i} \cdots \bar{y}^{n-i-1/n-i} \\
g_{n-1} &= \bar{y}_{n-1}.
\end{align*}
\]

Finally, we have that
\[
\begin{align*}
|g_i y_i| &= \left|g_{n-1} \cdots g_i\right|^{1/n-i} |y_i|^{1/n-i} |y_i|^{-1/n-i} \\
&\times \phi_i^{1/n-i} \\
&\times \left[\frac{1/n-i \cdots n-i-1/n-i}{\bar{g}_{n-2} \cdots \bar{g}_{i+j}}\right] \\
|g_{n-1} | |y_{n-1}| &= \phi_{n-1}.
\end{align*}
\]

### 3.1.1 Necessity:

It follows applying [3, Theorem 1] which states that if the origin of \(\dot{x} = F(t, x)\) with \(F(\cdot, \cdot)\) locally Lipschitz in \(x\) uniformly in \(t\), is UGAS then, necessarily \(F(\cdot, \cdot)\) is Uδ-PE. Then, defining \(F(t, x)\) as the right hand side of the closed loop system, and since \(B_i(x_i)\) is uniformly bounded for each \(\Delta > 0\) and all \(x \in \mathcal{B}(\Delta)\) we have that \(F(\cdot, \cdot)\) is Uδ-PE with respect to \(\xi\) only if \(\bar{g}_i(t, x, z)^2\) is Uδ-PE with respect to \(\xi\) for any \(i\). Hence, in view of (12) we have that each \(\bar{g}_i(t, x, z)\) is Uδ-PE with respect to \(\xi\) which in turn implies that \(\bar{g}^c_{i,a}(t, x)\) is Uδ-PE with respect to \(\xi\) for all \(i\).

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Here, the index \(i\) does not make sense because we have only one \(z\)-state, this is because all the gains \(g_i = u_i\).
Then, using (20) and (22) successively we obtain that for all 
\[ \nu > 0 \]
of the bounds (7)–(11) and show that for each \( \Delta \)
the time derivative of \( V \), we have that
\[ |x(t)| \leq \gamma(|x_0|) \quad \forall t \geq t_0, \ x_0 \in \mathbb{R}^n. \] (25)
Technically, we only have this bound on the maximal interval of definition. But with \( x \) bounded on the maximal interval of definition and the properties of \( g_i \), it follows that
\[ |z_i(t)| \leq |z_i,0| + \gamma(||x||_{\infty}) \] (26)
on the maximal interval of definition. Thus, solutions are defined for all time and, in fact, the origin is UGS.

**Proof of UGA** (with a family of sign-definite Lyapunov functions): We will define \( 3n - 3 \) functions which may be classified in 3 groups of \( n - 1 \) functions. The first group of functions is defined as follows
\[ V_2(t, x, z) := y_n \cdot g_n \cdot y_n - 1 \] (27)
\[ V_3(t, x, z) := y_n - 1 \cdot g_n + 2 \cdot y_n - 2 \] (28)
where \( \cdot \) denotes the scalar product and for \( i = 4, \ldots, n \),
\[ V_i(t, x, z) := y_{n-i+2} \cdot g_{n-i+1} \cdot y_n^2 - g_n \cdot y_n - 1 \cdot y_n + 1. \] (29)
For clarity we recall that \( y_{n-i+2} \in \mathbb{R}^p \) and the gains \( g_k \) are scalar functions. We proceed to compute some bounds for the time derivative of \( V_i(t, x, z) \). To that end we first notice first that
\[ y_j = \nabla h_j(x_j) B_j(x_j) \{ y_j y_j + 1 - g_j y_j - 1 \}^1, \quad j \in [2, \ldots, n]. \] (30)
Long but straightforward calculations which involve the use of the bounds (7)–(11) and show that for each \( \Delta > 0 \) there exists of \( \nu > 0 \) such that for all \( (x, z) \in B(\Delta)^2 \), we have
\[ V_2(t, x, z) \leq -c_n |y_{n-1} y_n - 1|^2 + \nu (|y_n| + |\sigma(y_n)|) \] (31a)
\[ V_3(t, x, z) \leq -c_n |y_{n-2} y_n - 1|^2 + \nu (|y_n| + |g_n y_n - 1|) \] (31b)
\[ V_i(t, x, z) \leq -c_n |y_{n-i+2} y_{n-i+1} y_n|^2 + \nu (|g_n y_{n-i+2} \cdots g_n y_{n-i+3}|) \] (31c)
for all \( i \in [4, \ldots, n] \) and where the coefficients \( c_i \) come from (7).

Then, using (20) and (22) successively we obtain that for all \( i \in [2, \ldots, n-2] \),
\[ |g_i y_i| \leq |g_n y_{n-1} g_{n-i+1}|^{1/n-i-1} |y_{n-i}|^{1/n-i-1} \times \]
\[ |y_{n-i+1}^{1/n-i-1} \cdots g_n^{1/n-i-1} g_{n-i}-2 \cdots 2/n-i-1| \] (32)
so defining the following \( n+1 \) functions for all \( i \in [3, \ldots, n] \),
\[ \phi_{n-1}(t, x, z) := |g_n y_{n-1}(t, x, z)| |h_n y_{n-1}(x_{n-1})| \] (33a)
\[ \phi_{n-i+1}(t, x, z) := |g_{n-i+1} y_{n-1}(t, x, z) \cdots g_n y_{n-1}(x_{n-i+1})| \] (33b)
we can rewrite the inequalities in (31) as
\[ \hat{V}_2(t, x, z) \leq -c_n \phi_{n-1} + \nu (|y_n| + |\sigma(y_n)|) \] (34a)
\[ \hat{V}_3(t, x, z) \leq -c_n \phi_{n-i+2} + \nu (|y_n| + \phi_{n-1}) \] (34b)
\[ \hat{V}_i(t, x, z) \leq -c_n \phi_{n-i+2} \phi_{n-i+3} + \nu (\phi_{n-i+2} + \phi_{n-i+3}) \] (34c)

**Remark 2** We wish to emphasize the way the functions \( V_1 \) to \( V_n \) defined so far, are ordered. Intuitively, from \( V_1 \) we may think (following Barbalat’s lemma) that \( y_n \rightarrow 0 \) and therefore, \( x_n \rightarrow 0 \) asymptotically. To make this precise, and for the use of our Matrosov’s theorem what is important to observe is that \( V_2 \leq 0 \) on the set where the bound on \( V_1 \) is zero, that is when \( y_n \equiv 0 \). Accordingly, each of the bounds on the succeeding \( V_i \)’s contain three essential terms: the first is a negative term of \( \phi_{n-i+1} \), the next two correspond to a number \( \nu \) times \( \phi_{n-i+2} \) and \( \phi_{n-i+3} \) which appear squared and with sign '-' in the previous two derivatives \( V_{i-1} \) and \( V_{i-2} \) respectively. This hints at the idea that one should be able to recursively show that if \( y_n \rightarrow 0 \) then, so does \( \phi_{n-1} \) hence also \( \phi_{n-2} \), etc.\footnote{This reasoning is similar to the arguments employed in [12] to prove (non uniform) convergence for skew-symmetric systems.} Then, the U\$\delta$-PE assumption will be used essentially to imply that if \( \phi_1 \rightarrow 0 \), necessarily \( x_i \rightarrow 0 \). \hfill \( \square \)

The next step is to construct a group of functions to conclude on the behavior of the gains, more precisely to show that the communication channel gains converge to their steady state solution as it was briefly discussed above. To that end, we introduce the function
\[ \zeta_i(t, x, z) := z_i - \tilde{g}_{i,a}(t, x) + \omega_i(t, x), \quad i \in [1, \ldots, n-1] \] (35)
which can be regarded to some extent as the error between the \( i \)th channel communication gain and its steady state solution. Then, the next group of \( n-1 \) functions that we introduce is defined by
\[ V_{n+i}(t, x, z) := \zeta_i(t, x, z)^2. \] (36)

The aim is to bound (on compact sets of the states \( x \) and \( z \)) the total derivatives of these functions with terms of the type \(-\zeta_i(t, x, z)^2 \) plus terms involving the \(|\phi_i(t, x, z)|\)’s defined in the previous group of functions. To that end we need to introduce some useful identities and bounds on certain functions. In the sequel whenever convenient and clear from the context we will drop the arguments.

First, we immediately see that
\[ \dot{\zeta}_i = -z_i + \tilde{g}_{i,a} - \frac{\partial g_{i,a}}{\partial x} \dot{x} + \frac{\partial g_{i,a}}{\partial t} \frac{\partial \omega_i}{\partial x} \frac{\partial \omega_i}{\partial t} \] (37)
such that

\[ \tilde{\psi} \] the trajectories of (41). For this purpose we also use the
We use now this inequality to find a bound on the total
\[ \zeta \] auxiliary functions.

\[ |t| \geq 1 \nabla |t|n \sigma (y_n) - g_{n-1}y_{n-2} \]

so adding and substracting \( \tilde{g}_{i,a}(t,x) \) in (37) we get that

\[ \hat{c}_i = -\zeta_i + \tilde{y}_{i,a} - \tilde{g}_{i,a} + \left[ \int_{-\infty}^{t} e^{-(t-\tau)} \frac{\partial \tilde{\psi}_i}{\partial x}(\tau,x) d\tau - \frac{\partial \tilde{g}_{i,a}}{\partial x} \right] \cdot \hat{x}. \]

At this point we observe that the terms in brackets multiplying \( \hat{x} \) are uniformly bounded in \( t \) by a continuous function of the norm of the state and hence is bounded by a number \( \nu > 0 \) for all \( x \in B(\Delta) \). To find suitable bounds on \( \hat{x} \) we recall that

\[ \hat{x}_1 = B_1 g_1 y_2 \\
\hat{x}_j = B_j \left[ g_j y_{j+1} - g_{j-1}y_j - 1 \right], \quad j \in [2, \ldots, n-2] \\
\hat{x}_{n-1} = B_{n-1} \left[ g_{n-1}y_n - g_{n-2}y_{n-2} \right] \\
\hat{x}_n = B_n \left[ g_n \sigma (y_n) - g_{n-1}y_{n-1} \right]. \]

so we use (20), (21) and (6) and the fact that all the gains \( g_i \) and \( \tilde{g}_i \) and the functions \( B_i \) are uniformly bounded in \( t \) on compact sets \( B(\Delta) \) and proceed as we did for (30) to find that for all \( (x,z) \in B(\Delta)^2 \) and all \( i \in [2, \ldots, n-2] \),

\[ |\hat{x}| \leq \nu \left[ \phi_{i-1}^{1/n-i-1} + \phi_{i-1}^{1/i-1} + \phi_{2}^{1/n-2} + |y_n| + |\sigma (y_n)| \right. \\
\left. + \phi_{n-2}^{1/2} + \phi_{n-1} \right]. \]

We use now this inequality to find a bound on the total time derivative (at the points of existence) of (36) along the trajectories of (41). For this purpose we also use the Lipschitz property of \( \tilde{g}_{i,a}(t,\cdot) \) to see that there exists \( L > 0 \) such that \( |\tilde{g}_{i,a}(t,x) - \tilde{g}_{i,a}(t,x)| \leq L |x_n| \), and we observe that \( |\tilde{c}_i(t,x,z)| \leq \nu \) for almost all \( (t,x,z) \in R \times B(\Delta)^2 \) to obtain finally that

\[ \hat{V}_{n+1}(t,x,z) \leq -\zeta_i(t,x,z)^2 + \nu \left( |x_n| + |y_n| + \sum_{j=1}^{n-1} \phi_j^{1/n-j} \right) \]

for all \( i \in [1, \ldots, n-1] \) which is what we were seeking for.

Notice that the positive terms above appear with negative signs in the bounds on the derivatives of the previous group of auxiliary functions.

Roughly speaking, from the following group of functions we will be able to get terms that allow us to conclude on the convergence of the states to zero provided that the \( \phi_i \)’s converge to zero and that the gains are persistently exciting in the specific way we imposed. So we define now

\[ V_{2n-1+i}(t,x) := -\int_t^{\infty} e^{t-\tau} |\Omega_i(\tau,x) h_i(x)|^2 d\tau \]

\[ \Omega_i(t,x) := \Pi_{j=1}^{n-1} \omega_j(t,x), \quad i \in [1, \ldots, n-1] \]

The total derivative of \( V_{2n-1+i}(t,x) \) can be easily computed to find

\[ \dot{V}_{2n-1+i}(t,x) = V_{2n-1+i}(t,x) + |\Omega_i(t,x) h_i(x)|^2 + \frac{\partial V_{2n-1+i}}{\partial x} \]

almost everywhere. Concerning the last term on the right hand side, we may proceed bounding them on compact sets of the states as we did before and use the local Lipschitz property of \( V_{2n-1+i}(t,x) \) to bound the gradient on compact sets of the state. The first two terms require more consideration.

From the results in [3] we have that defining for each \( \Omega_i(t,x) \) which is Us-P-E with respect to \( \xi = \text{col}[x_1, \ldots, x_{n-1}, 0] \), there exists \( \gamma_i \in \mathbb{K} \) such that

\[ V_{2n-1+i}(t,x) \leq -\gamma_i(|\xi|) |h_i(x_i)|^2. \]

Concerning the term \( |\Omega_i(t,x) h_i(x)|^2 \) in (46) we will derive a bound (as before on compact sets of the states) involving \( |\phi_i|, |\zeta_j| \) and \( |y_n| \) which appear with negative sign in the bounds on the previous functions’ derivatives. To that end, we use (35) and (12) to obtain a more convenient expression for \( \phi_i \) and substitute in (45) to see that

\[ |\Omega_i|^2 = \prod_{j=1}^{n-1} \left[ |\tilde{g}_j| + |\zeta_j - (\tilde{g}_j,a - \tilde{g}_j,a)|^2 \right]. \]

Then, we use the Lipschitz property of the gains \( \tilde{g}_j,a(t,\cdot) \) to obtain that for all \( (t,x,z) \in R \times B(\Delta)^2 \),

\[ \left[ \tilde{g}_j + \zeta_j - (\tilde{g}_j,a - \tilde{g}_j,a) \right]^2 \leq \tilde{g}_j^2 + \nu(|\zeta_j| + |x_n|) \]

Furthermore, using once more the uniform boundedness in \( t \) of \( \tilde{g}_j(t,\cdot) \) and arguing as above, we have that for any integer \( j \in [i, \ldots, n-1] \),

\[ \left[ \tilde{g}_j + \zeta_j - (\tilde{g}_j,a - \tilde{g}_j,a) \right]^2 \left[ \tilde{g}_{j+1} + \zeta_{j+1} - (\tilde{g}_{j+1},a - \tilde{g}_{j+1},a) \right]^2 \leq \tilde{g}_j^2 \tilde{g}_{j+1}^2 + \nu(|\zeta_j| + |\zeta_{j+1}| + |x_n|) \]

It follows that for all \( (t,x,z) \in R \times B(\Delta)^2 \),

\[ |\Omega_i h_i|^2 \leq \prod_{j=1}^{n-1} \tilde{g}_j y_j^2 + \nu \sum_{j=1}^{n-1} |\zeta_j| + |x_n| \]

where we have also used (11). Notice that from (12) the bound above is exactly the same as

\[ |\Omega_i h_i|^2 \leq \tilde{g}_j y_j^2 + \nu \sum_{j=1}^{n-1} |\zeta_j| + |x_n| \]

and using the identities (22) involving \( g_i, y_i \) we finally obtain that for all \( (t,x,z) \in R \times B(\Delta)^2 \), and all \( i \in [1, \ldots, n-1] \)

\[ |\Omega_i h_i|^2 \leq \nu( \phi_i^{2/n-i} + \sum_{j=1}^{n-1} |\zeta_j| + |x_n| ) \]

Summarizing, we have from (46), (47) and (48) that

\[ \dot{V}_{2n-1+i} \leq -\gamma_i(|\xi|) |h_i(x_i)|^2 + \nu \left( |x_n| + \phi_i^{2/n-i} + |y_n| + \sum_{j=1}^{n-1} \phi_j^{1/n-j} + |\zeta_j| \right) \]
where the last two terms come from using (42) to bound $\dot{x}_i$ in the last two terms of (46), similarly as we did to obtain (43), and from a uniform bound (for all $(t,x,z) \in \mathbb{R} \times B(\Delta)^2$) on the partial derivatives of $V_{2n+i-1}$ and $h_i$. The final group of functions is simply
\[ V_{3n-2+i}(z) = z_i^2, \quad \forall i \in \{1, \ldots, n-1\} \quad (50) \]
whose total derivative along the trajectories of the first equation in (12) yields for all $(t,x,z) \in \mathbb{R} \times B(\Delta)^2$,
\[ \dot{V}_{3n-2+i}(z) \leq -z_i^2 + \nu |x| . \quad (51) \]

With the aim at applying Theorem (1) define
\[
X := \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}, \quad \Phi(t,X) := \begin{bmatrix} \phi_1(t,x,z) \\ \vdots \\ \phi_{n-1}(t,x,z) \end{bmatrix}, \quad (52)
\]
Then, we have from (24), (31), (33), (43), (49) and (51) that for all $i \leq n-1$,
\[
\begin{align*}
\dot{V}_1 & \leq -\rho \circ \kappa_n(|X_n|) \\
\dot{V}_2 & \leq c_1 [\Phi_{n-1}]^2 + \nu [\theta_n(|X_n|) + \sigma \circ \kappa_n(|X_n|)] \\
\dot{V}_3 & \leq c_{n-1} [\Phi_{n-2}]^2 + \nu [\Phi_{n-1} + \theta_n(|X_n|)] \\
& \vdots \\
\dot{V}_{i+3} & \leq c_{n-i-1} [\Phi_{n-i-2}(t,X)]^2 + \nu [\Phi_{n-i-1}(t,X)] \\
& \vdots \\
\dot{V}_n & \leq -c_2 |\Phi_1(t,X)|^2 + \nu (|\Phi_2(t,X)| + |\Phi_3(t,X)|) \\
& \vdots \\
\dot{V}_{n+i} & \leq -|\Phi_{n+i-1}(t,X)|^2 + \nu (\theta_n(|X_n|) + |X_n| + \sum_{\ell=1}^{n-1} |\Phi_\ell(t,X)|^{1/n-\ell}) \\
& \vdots \\
\dot{V}_{2n+i-1} & \leq -\gamma_i(|X_{n-1}|) |h_i(X_i)|^2 + \\
& \nu (\theta_n(|X_n|) + |X_n| + |\Phi_i(t,X)|^{2/n-i} + \\
& \sum_{\ell=1}^{n-1} |\Phi_\ell(t,X)|^{1/n-\ell} + |\Phi_{n+i-1}(t,X)|) \\
& \vdots \\
\dot{V}_{3n-2+i} & \leq -X_{n+i}^2 + \nu |X_{1-n}|. 
\end{align*}
\]

Letting each of the bounds above be $Y_k(X,\Phi(t,X))$ with $k \in \{1, \ldots, 4n-3\}$ we see that each of these functions is bounded on compact sets (i.e. Assumption 2 holds), is non-positive on the sets where all the previous are identically zero (i.e., Assumption 3 holds). Moreover, in view of (6), in particular since $\kappa \in P\mathcal{D}$ and also $\gamma_i \in \mathcal{K}$ we see that the set where $Y_k(0,0) \equiv 0$ is the origin, $\{X = 0\}$ (i.e., Assumption 4). UGAS follows from Theorem 1.

4 Conclusions

In this paper we have addressed a case-study in control design which covers the popular example of nonholonomic systems. Our approach relies on a property known as $\delta$-persistency of excitation and a new tool for stability analysis which can be regarded as an extension of the well known Matrosov’s theorem. This tool involves the use of an arbitrary finite number of auxiliary functions whose derivatives are simultaneously zero only at the origin. Hence, it can also be regarded as a generalization to the case of non autonomous systems, of the celebrated La Salle’s invariance principle (when the latter is used to conclude stability of the origin).

References